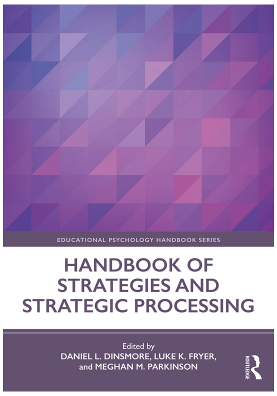

Handbook of Strategies and Strategic Processing

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Mathematics Strategy Interventions

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MATHEMATICS STRATEGY INTERVENTIONS

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In mathematics, a strategy can be defined as "a general approach for accomplishing a task or solving a problem that may include sequences of steps to be executed, as well as the rationale behind the use and effectiveness of these steps" (Star et al., 2015, p. 26). For example, a student may always approach a quadratic equation by using the quadratic formula (which involves a particular sequence of steps), even though some quadratics can be solved in other ways (e.g., by factoring). Some experts prefer to first "clear the denominator" when they encounter a linear equation with a fraction in front of parentheses (e.g., $\frac{1}{3}(x+6)=4$ becomes $x+6=12$), whereas a student struggling to learn algebra may generally prefer to use the distributive property whenever parentheses are involved, because it is familiar and reliable (Newton, Star, & Lynch, 2010; Star & Newton, 2009).

When students hold misconceptions, their general approach may lead to errors. For example, a student may believe that common denominators are always necessary for operating with fractions, leading him or her to use that strategy to multiply two fractions. Studies identifying common errors can help illuminate misconceptions and poor strategies that need to be targeted as part of an intervention. Therefore, in the first part of this chapter I overview major errors documented in the literature, then I present some interventions designed to target these errors.

As illustrated above, sometimes there are multiple valid strategies for solving mathematics problems. Therefore, in the second part of the chapter I focus on interventions that promote flexible problem solving. "Procedural flexibility includes identifying and implementing multiple methods to solve algebra problems, as well as choosing the most appropriate method" (Star et al., 2015, p. 2). As with the strategy of clearing the denominator described above, employing alternative strategies can make problems easier to solve in some way (e.g., fewer steps). Experts demonstrate this kind of flexibility (Star & Newton, 2009), but researchers have found that it is slow to develop in students (Newton, Lange, & Booth, 2019).

Both erroneous and flexible strategy use can be viewed through Siegler's Overlapping Waves Theory (OWT), which helps to explain how strategy use changes over time. Children typically know multiple ways to solve a given problem; with practice, they begin to use more accurate and efficient strategies (Fazio, DeWolf, & Siegler, 2016; Opfer & Siegler, 2007). The development of accurate, efficient strategies is not smooth or linear. For example, to find the total number of objects given two sets, a child might sometimes count all of the objects, whereas other times the child might use a memorized addition fact to find the total. With practice and feedback, correct strategies replace incorrect ones, and more efficient strategies replace less efficient ones. This chapter focuses on both erroneous and alternative strategies, specifically for fractions and early algebra, and it overviews interventions for addressing errors and promoting flexibility. Algebra has been identified as a gatekeeper and a major concern of researchers and educators interested in mathematics learning (National Mathematics Advisory Panel, 2008). Fractions represent critical prerequisite knowledge for algebra learning that cause difficulties for many children, therefore receiving increased attention in the literature in recent years (Siegler, Fazio, Bailey, & Zhou, 2013). Although whole number knowledge and basic arithmetic are foundational to both fractions and algebra, these topics have received significant attention in the strategy literature. Interested readers are encouraged to seek out that work (e.g., Baroody & Dowker, 2003; Carpenter, Fennema, & Franke, 1996; Shrager & Siegler, 1998; Vanbinst, Ghesquière, & De Smedt, 2012; Verschaffel & De Corte, 1993; Verschaffel, Luwel, Torbeyns, & Van Dooren, 2009). Given space limitations, the current chapter is by no means an exhaustive review of the literature on strategies and strategy interventions related to fractions and algebra.

INCORRECT STRATEGIES

Errors with Fractions

The challenge posed by fractions, along with the important role of fractions for learning algebra, has resulted in increased attention for this topic (Siegler et al., 2013). Because fractions do not have all of the same properties as whole numbers, children must adjust their thinking to accommodate this "new" kind of number. If they do not, it may lead to errors based on misconceptions about fractions, even as they try to integrate information from instruction (Siegler, Thompson, & Schneider, 2011; Stafylidou & Vosniadou, 2004).

Knowledge of fraction magnitudes, or relative sizes of fractions, is especially critical for later mathematics learning. It predicts overall mathematics achievement as well as learning in algebra (Booth, Newton, & Twiss-Garrity, 2014; Siegler et al., 2011). In their study of 200 students, ages 10–16, Stafylidou and Vosniadou (2004) analyzed responses and justifications to ordering tasks and identified different categories of misconceptions students held as they learned about fractions. For example, students may believe that $\frac{4}{3}$ is smaller than $\frac{5}{6}$ because 3 and 4 are smaller than 5 and 6. In this case, they are treating the numbers as if they are all whole numbers. As students learn about the size of fractional parts relative to the denominator, they may erroneously claim that $\frac{5}{4}$ is smaller than $\frac{4}{3}$ because the larger numbers indicate smaller fractions. Fazio et al. (2016) found that low performing students at a community college used the "bigger denominator" strategy frequently for comparing fractions, resulting in many errors. On the other hand, Malone and Fuchs (2017) found the whole number error to be especially pervasive among a group of at-risk fourth graders.

Students also make a variety of errors with fraction arithmetic, often following predictable patterns (Braithwaite, Pyke, & Siegler, 2017; Newton, Willard, & Teufel, 2014; Siegler & Pyke, 2013). As with comparing fractions, they make errors by overgeneralizing their knowledge of whole numbers. A well-documented example is treating fraction addition as two separate whole number addition problems, erroneously adding across numerators and denominators (e.g., $\frac{1}{2} + \frac{2}{3} = \frac{3}{5}$). Students also make errors by overgeneralizing rules for fraction arithmetic. A student using the incorrect strategy above might instead be thinking of fraction multiplication, for which you multiply across numerators and denominators. Another common error in this category would be to find (or keep) a common denominator when it is not appropriate to do so, such as with fraction multiplication (e.g., $\frac{2}{5} \times \frac{3}{5} = \frac{6}{5}$) or fraction division (e.g., $\frac{6}{5} \div \frac{3}{5} = \frac{2}{5}$). Students making this error have most likely learned how to add fractions, in which case a common denominator is needed.

Interventions

The Institute of Education Sciences (IES) practice guide (Siegler et al., 2010) for effective fraction instruction offers several recommendations for targeting the erroneous strategies described above. The two recommendations with moderate (as opposed to minimal) research-based evidence are as follows. First, students need to "recognize that fractions are numbers and that they expand the number system beyond whole numbers" (p. 19). Interventions that make use of number lines in particular are recommended, given they can assist students in understanding the place of fractions within the number system (Siegler et al., 2010; Wu, 2009). The Common Core State Standards for mathematics recommends introducing fractions using number lines to students as early as third grade (NGA Center & CCSSO, 2010). Second, students need to "understand why procedures for computations make sense" (Siegler et al., 2010, p. 26).

Number Lines. Wu (2009) posited that making use of number lines as a teaching tool provides coherence to the study of numbers. "In particular, regardless of whether a number is a whole number, a fraction, a rational number, or an irrational number, it takes up its natural place on this line" (p. 8). Number lines can support students' understanding of magnitude and equivalence, as well as computations with fractions. Unfortunately, research focused on number line interventions has been scarce. Shin and Bryant (2015) synthesized the fraction intervention research that targeted students struggling to learn mathematics. Across 17 studies, most interventions included concrete and visual representations. However, none of them included number lines.

In a recent study, Hamdan and Gunderson (2017) found a causal effect for number line training. Second and third grade students learning about fractions using a number line out-performed students who were introduced to fractions with an area model on a transfer task that involved fraction comparison. Only the students using a number line representation were able to use what they had learned to compare two fractions.

Fuchs and colleagues (Fuchs, Malone, Schumacher, Namkung, & Wang, 2017) recently reported on the impact of a multi-year program to target at-risk students' fraction knowledge by emphasizing fraction magnitude, including the use of number lines. In a series of studies, they compared performance for students involved in the intervention to a business-as-usual condition, where the primary focus was on a part-whole interpretation of fractions. Students in the intervention group outperformed the control each year on the number line estimation task, which asked students to place fractions and mixed numbers on a number line marked 0 and 2 at the endpoints. Students in the intervention group also outperformed control on addition and subtraction of fractions and mixed numbers. Finally, the intervention group outperformed control on a set of released items from the National Assessment of Educational Progress (NAEP) that emphasized magnitude and part-whole equally.

Over several years of the project, Fuchs and colleagues (Fuchs et al., 2017) tested the effect of additional supports beyond the emphasis on magnitude. For example, in year 2 they varied the type of practice students experienced. One group was focused on fluency, while the other was focused on explaining their thinking. Although no significant difference was found for the three fraction measures, they did find a moderating effect for working memory. In particular, students with very low working memory benefited more from the explaining condition, while students with high working memory benefited more from the fluency practice. However, given that a larger proportion of students benefited from fluency practice, this form of practice was included in the program for subsequent years.

In year 3, additive and multiplicative word problem conditions were included using schema-based instruction, where students were taught to recognize the problem type and to represent the type with a number sentence or visual display. A control group focused on identifying key words in the word problems. The multiplicative group outperformed the additive and control groups on multiplicative word problems and outperformed the control group on additive word problems. The additive group outperformed the other two groups on additive problems but performed similarly to the control group on multiplicative word problems (Fuchs et al., 2017).

In year 4, the researchers compared the added effect of *supported* self-explanations compared to training in solving multiplicative word problems (Fuchs et al., 2017, 2016). Rather than requiring at-risk students to produce their own explanations, self-explanations were modeled and the students analyzed, discussed, and elaborated on them. On word problems, students in the word problem group tended to outperform the explanation group. However, for both magnitude comparison and quality of explanations, students in the supported self-explanation group tended to outperform the word problem group. Moreover, students with both high and low working memory performed similarly in this condition.

The work by Fuchs and colleagues (Fuchs et al., 2017, 2016) described above underscores the importance of emphasizing fraction magnitudes, including the use of number lines, to promote early fraction knowledge. As noted by the authors, their program targeted fourth grade standards, and so additional research is needed "to address the challenges associated with multiplying and dividing fractions as well as other complex mathematics curricular targets" (p. 638).

Understanding Why Procedures Work. Mathematics educators have long held an interest in promoting deep understanding of fraction procedures, suggesting that simply knowing how to compute with fractions is insufficient in many ways. Highlighting this distinction, Skemp (1976) referred to knowing "how" as having an *instrumental understanding*, while knowing both "how and why" suggests a *relational understanding*. For adding fractions, this distinction would suggest, on the one hand, knowing that you need to find common denominators to add ½ and ⅕ versus, on the other, knowing that you need to find common denominators so that the fractions are renamed using same-sized parts, making it easy to find out the total number of those parts.

More broadly, Hiebert and Lefevre (1986) characterized *conceptual knowledge* "as knowledge that is rich in relationships" (p. 3). This characterization of knowledge could include the latter example above, as well as other important relationships. For example, you might know that ½ and ⅕ can also be renamed as decimals and then added, and that the sum is equivalent to the one obtained using fractions. You might also be able to estimate the sum, knowing it must be slightly larger than ½, the first addend. Unfortunately, even when students can accurately identify which fraction is the larger one or estimate the magnitude of each fraction, many are unsuccessful at estimating the sum of the two fractions (Braithwaite, Tian, & Siegler, 2018; Cramer & Wyberg, 2007). This lack of understanding, as well as errors such as the overgeneralizations described above, has prompted researchers and educators to search for ways to support students in making sense of fraction computation.

Visual Representations. As indicted above, the number line is an important representation for understanding fractions, and Fuchs and colleagues (2017) have reported good success at integrating this representation into a program designed to strengthen children's understanding of fraction magnitudes. Research is needed to extend this work to fraction computation. On the other hand, extensive research has focused on concrete manipulatives for visually representing fractions, in order to help students make sense of them. Research on the use of manipulatives in mathematics and science generally supports fading from concrete to abstract representations, as suggested by Bruner's modes of representation (Fyfe, McNeil, Son, & Goldstone, 2014). According to Bruner, the use of concrete and pictorial models should precede work with symbols only. Students need support linking these representations, especially the pictorial and symbolic ones. A meta-analysis focused on fraction skills found that a concrete to abstract progression showed promise for students with disabilities, although more research is needed (Ennis & Losinksi, 2019).

Word Problems. Building on extensive research by Carpenter and colleagues that focused on children's strategies for solving whole number problems (Carpenter et al., 1996), Empson and Levy (2011) emphasized word problems as a critical tool for sense-making with fractions. They recommended a progression of problem types with instructional support to help formalize student thinking about these problems. For example, when asked an equal sharing problem about four children sharing three cookies, a student may draw a picture of a cookie, split it into four parts, and distribute each part to a person. Repeating this process for each cookie results in three one-fourths for each person. In response to this strategy, a teacher might provide a number sentence to match the drawing, such as ¼ + ¼ + ¼ = ¾. These researchers emphasized that a key aspect of this approach involves making mathematical ideas and properties explicit by linking students' strategies directly to the mathematical sentences that represent them.

Summary

There is a long history of educators and psychologists promoting sense-making in mathematics. According to the IES practice guide for effective fraction instruction (Siegler et al., 2010), two general areas of sense-making for fractions are well-supported by research. First, students need to understand that fractions are numbers. Second, students need to understand why fraction procedures work the way they do. Without these understandings, students tend to overgeneralize their prior knowledge in ways that are not appropriate. Interventions targeting these errors highlight *representations* as key to fostering understanding and sense-making. Representing fractions on the number line, with concrete manipulatives and within word problems are all important for fostering deep understanding and appropriate strategy use.

Errors with Algebra

Booth and colleagues (Booth, Barbieri, Eyer, & Paré-Blagoev, 2014) analyzed the errors of Algebra I students across five school districts in four states, in order to understand not only which errors were most common but also which ones were most detrimental to algebra achievement as measured by standardized test items. Based on the extant literature, they coded for conceptual errors with variables, fractions, negatives, operations, mathematical properties, and equality/inequality. For comparison, they also coded for errors with arithmetic.

Across the year, the most prevalent errors were those involving negatives, arithmetic, variables, and the equal sign. A more nuanced analysis revealed a time sensitive aspect for the contribution of particular errors to the end of year assessment scores. For example, while negative number errors were prevalent throughout the year, making those errors at the end of the year was predictive of lower scores on the end of year assessment items. A similar result was found for arithmetic. Errors with equal/inequality became increasingly prevalent throughout the year, and making these errors in the middle or end of the year was predictive of lower end of year scores. Making errors with variables, such as combining unlike terms, in the first part of the year also predicted lower end of year scores.

Some errors that were not among the most common ones were still predictive of end of year scores. In particular, students who began the year making errors with operations (e.g., treating $5 + x$ as $5x$) and with properties (e.g., treating $5 - x$ as $x - 5$) were more likely to have lower end of year algebra scores.

Interventions

The IES practice guide for improving algebra knowledge outlines three research-based recommendations (Star et al., 2015). The first recommendation is to use solved problems, also known as worked examples, to help students analyze and reason about algebraic strategies. The second recommendation is to help students to notice and make use of underlying structures, including the presence and placement of variables and symbols, in algebra problems so that they can see similarities across problem types. The final recommendation is for students to learn and choose from multiple strategies when solving algebra problems. These first two, worked examples and noticing structure, are addressed in this section. The recommendation to learn and use multiple strategies is addressed in the following section, which is focused on procedural flexibility.

Worked Examples. The use of worked examples, which includes a problem presented along with solution method, has a long history in the cognitive and educational psychology literature, particularly for learning mathematics (Atkinson, Derry, Renkl, & Wortham, 2000). Research generally supports the use of worked examples early in the skill acquisition process, along with prompts for self-explanations so that the learner is asked to elaborate on what is presented in the example. For optimal learning, worked examples are interleaved with problems for the learners to solve.

Worked examples with self-explanation prompts have been used along with practice problems to effectively support learning in algebra classrooms. Incorrect examples may be especially important for supporting conceptual knowledge of algebra (Booth, Lange, Koedinger, & Newton, 2013), given they are designed to target common errors/misconceptions. Students with low prior knowledge seem to benefit most from either incorrect worked examples (Barbieri & Booth, 2016) or a combination of correct and incorrect examples (Booth et al., 2015). In a recent study, Barbieri, Miller-Cotto, and Booth (2019) found that eighth graders prone to making conceptual errors as they practiced graphing-related skills benefited more from studying correct and incorrect examples with visual signaling cues than from worked examples with self-explanations. These visual cues were designed to signal important conceptual features in the problems, such as identifying slope and y-intercept within the worked example. Students who made fewer errors during practice problems benefited similarly from worked examples with self-explanations and with visual signaling cues.

Noticing and Using Structure. To mathematicians, noticing and using structure is a fundamental aspect of doing mathematics, so fundamental that mathematics can be described as "the science of structure" (Newton & Sword, 2018, p. 33). In algebra, structure can be thought of as "the underlying mathematical features and relationships of an expression, representation, or equation" (Star et al., 2015, p. 6). As noted above, feature knowledge such as knowledge of the equal sign, negatives, and variables is critical to success in algebra. Noticing structure means noticing how these features are arranged in an algebraic representation and being able to see meaning in those arrangements. For example, a student who notices structure can clearly understand that $2+3+x=10$ and $2+3=x+10$ have quite different meanings, despite the surface similarities.

Experts including mathematicians assert that using precise language can support an understanding of structure (Newton & Sword, 2018; National Governors Association Center for Best Practices & the Council of Chief State School Officers, 2010; Star et al., 2015). For example, they caution against imprecise language such as stating, "two negatives make a positive," because although a negative times a negative results in a positive number, a negative plus a negative does not. Students sometimes overgeneralize these kinds of imprecise statements, making errors such as thinking −5a − 4a is 9a (Vlassis, 2004).

One way to capture students' noticing of structure is to measure errors while encoding equations (McNeil & Alibali, 2004). Booth and Davenport (2013) measured feature encoding of algebra by displaying an algebraic equation on a screen for 5 seconds and then asking students to recreate the equation on their paper after it disappears from the screen. Their findings confirm that feature knowledge of algebra supports feature encoding and predicts the ability to solve algebraic equations.

Varying the Format. The equal sign is one feature that has received significant attention in the literature. As noted in the example above, the placement of the equal sign matters. Students who view the equal sign as an indication to find an answer may just add all of the constants together no matter their placement in the equation (e.g., erroneously finding 15 as the value of x for $2+3=x+10$). In a study by McNeil and Alibali (2004), fourth graders making this error frequently encoded non-conventional equations as if they were conventional ones, with all to-be-added terms on the left-hand side of the equal sign (e.g., encoding 4 + 5 = 2 + __ as 4 + 5 + 2 = __). Although in late elementary school most students have developed a basic relational understanding of the equal sign, where they recognize it as an indication that two expressions have the same value, a robust understanding of equivalence continues to develop through middle school (Fyfe, Matthews, Amsel, McEldoon, & McNeil, 2018; Rittle-Johnson, Matthews, Taylor, & McEldoon, 2011).

A simple, yet effective intervention for addressing equal sign errors is to more frequently pose unconventional arithmetic problems so that students are not overly accustomed to all terms being on the left of the equal sign, with only a blank for the answer on the right. Under the CCSS, students are expected to be presented with simple, unconventional problems as early as first grade (National Governors Association Center for Best Practices & the Council of Chief State School Officers, 2010). McNeil, Fyfe, and Dunwiddie (2015) provided this kind of modified practice as part of an intervention with second graders. Modified practice workbooks in their study included unconventional problems (e.g., __ = 3 + 5 and 8 = __ + 5), whereas students in a control group received conventional problems to practice (e.g., 3 + 5 = __). To support a relational understanding of equivalence, the equal sign was sometimes replaced with words that conveyed its meaning, such as "is the same amount as". Finally, the problems were organized such that students encountered several in a row with the same sum (e.g., __ = 3 + 5 and __ = 2 + 6). Students in the modified workbook condition outperformed students in the control, with lasting effects.

Although the modified practice workbooks (McNeil et al., 2015) resulted in positive, long-term effects compared to traditional practice, not all students achieved a robust understanding of the equal sign. Building on that work, McNeil and colleagues designed and tested a more comprehensive intervention for targeting equal sign knowledge (McNeil, Hornburg, Brletic-Shipley, & Matthews, 2019). In their study of 142 second grade students, an active control group was compared to a comprehensive intervention group. The active control group received practice problems similar to the intervention in McNeil et al. (2015). The comprehensive intervention group included three additional elements: (1) introduction of the equal sign in non-arithmetic situations (e.g., 5 = 5); (2) concreteness fading, where representations became more abstract over time; and (3) prompts for comparing problem formats and strategies (including incorrect ones). Students randomly assigned to the comprehensive intervention improved significantly more in their understanding of equivalence than students assigned to the active control classrooms, with large effect.

Given that students are introduced to the equal sign very early, much of the research on this topic has been conducted with elementary school children. However, as noted by Booth et al. (2014), students taking algebra make errors related to equal sign knowledge, and these errors predict lower end of year algebra scores. Therefore, more research is needed on how to remediate this knowledge for students learning algebra.

Representations. Asking students to discuss, compare, and move between different algebraic representations, such as equations, graphs, word problems, and diagrams, can support student understanding of underlying algebraic structures (National Council of Teachers of Mathematics, 2000; Star et al., 2015). Additionally, some researchers and educators recommend physical models to support student learning. A balance scale, for example, provides students with a model for understanding that the equal sign indicates that two expressions have the same value. Related, it also provides insight into the idea that the same operation is performed to both sides in order to maintain that equivalency. In a study with 40 eighth graders, Vlassis (2002) found that the model was limited, however, in its ability to help students understand negatives within the equation.

As with fractions, number lines are one way to represent negatives. Tsang, Blair, Bofferding, and Schwartz (2015) compared three instructional conditions for fourth graders learning about negatives, and all three conditions incorporated the number line as the primary representation. The Jumping condition included a small figure that moved along the number line in a way that corresponded to an addition problem (e.g., for $5 + -3$, starting at 5 and "jumping" to the left three spaces). The Stacking condition included magnetic manipulatives with different colors to represent positives and negatives. For $5 + -3$, five blue blocks are placed on the number line, between 0 and 5. Three red blocks are then stacked on top of the blue ones, starting at 5. Each red and blue pair "cancel" each other, so two blue blocks are left. The Folding condition involved place the 5 blue blocks between 0 and 5, and the 3 red blocks between 0 and -3. Then, the blocks were folded toward each other, with three of them "canceling" each other out. The physical movement of folding in this condition was designed to emphasize the underlying symmetrical structure inherent in the number line. Students in the Folding condition outperformed students in the other two groups on items not yet taught, such as negative fractions and algebra readiness problems (e.g., $2 + -3 = \_\_ + -1$).

Comparing. The use of worked examples as an effective instructional tool is not limited to single examples paired with prompts for self-explanation. Another approach supported by research is to use two worked examples, presented side-by-side, in order help students to make comparisons and draw important conclusions. In a study that incorporated pairs of side-by-side worked examples into year-long algebra classrooms, greater use of the comparison materials was associated with greater gains in procedural knowledge (Star et al., 2015). These materials included four types of comparisons, often involving one problem worked in two ways. For example, a correct method of solving a problem might be compared to an incorrect method. Alternatively, one method might highlight why the other one works. Or, a general and alternative method might be compared in order to promote flexible problem solving. Other times, two different problems might be compared. For example, two variations of a problem might be compared in order to highlight a particular idea (e.g., slope), or two visually similar problems might be compared in order to highlight an important difference (e.g., $x^3 \cdot x^4$ and $(x^3)^4$).

Ziegler and Stern (2016) studied this last type of comparison in particular and found long-term benefits for sixth graders with little or no prior knowledge of algebra. Rather than studying worked out examples, these students were exposed to direct instruction as a means to compare and contrast problems with similar features (e.g., $3x + 3x + 3x$ and $3x \cdot 3x \cdot 3x$). The contrast group was presented with addition and multiplication problems simultaneously, whereas the sequential group was presented with addition problems followed by multiplication problems. Although the sequential group outperformed the contrast group on immediate learning, the contrast group outperformed the sequential group on follow-up measures at three time points.

Comparison of this kind holds promise for alleviating some of the errors identified by Booth et al. (2014). Specifically, comparing expressions with similar features but different operations (e.g., $5 + x$ and $5x$) or different ordering of terms (e.g., $5 - x$ and $x - 5$) may help students understand and remember important ideas that are critical for success in algebra. It may also be helpful for addressing errors with negative numbers. During interviews with students about integer arithmetic, Bishop, Lamb, Philipp, Whitacre, and Schappelle (2016) found that some students naturally used these kinds of comparisons when they explained their reasoning about integers. In their comparisons, students varied the sign of the numbers (e.g., $6 - 2$ and $6 - (-2)$), the operations (e.g., $-8 + 3$ and $-8 - 3$), and features such as the order of the addends (e.g., $-5 + 2$ and $5 + -2$) in order to logically reason about what must be true or not true for a given problem. For example, a student reasoned that $6 - (-2)$ cannot be 4 because $6 - 2$ is 4.

Although these kinds of comparisons were not prevalent in the Bishop et al. (2016) study, they occurred across a variety of grade levels. The researchers therefore suggested that it may be fruitful to purposefully incorporate comparisons of this sort into instruction, along with prompts to support students' reasoning about them. They also suggested that students have more opportunities to make conjectures about how operations work, to encourage students to think more about the underlying mathematical structures. One teacher who regularly did this found her students were able to correctly conjecture that dividing across numerators and denominators is a valid approach to fraction division (Newton et al., 2014), a fact that is often obscured through traditional instruction. Systematic research is needed to understand the effects of this kind of intervention.

Summary

As with fraction knowledge, knowledge of algebra requires sense-making. Students need to understand algebraic symbols and the meanings that are conveyed by them and the way they are organized. These understandings can support deep and accurate strategy use in algebra. The IES practice guide for algebra learning recommends the use of worked examples to help students understand algebraic strategies.

Incorrect worked examples can be especially helpful for targeting errors based on common misconceptions. A second recommendation includes helping students notice and use structure. Similar to fractions, representations such as number lines, concrete manipulatives, and word problems, can help. Graphs and equations are also critically important representations for understanding algebraic structures. Varying the format of the equations can highlight important structures such as the equal sign. Comparison also supports noticing and using structure, as it can highlight important similarities or differences between representations that might otherwise be obscured.

ALTERNATIVE CORRECT STRATEGIES

Flexibility with Fractions

Research on flexibility with fractions is scarce, but some research suggests that competent students and experts in mathematics are quite flexible with fractions. Smith (1995) interviewed students at the elementary, middle, and secondary levels about several ordering and equivalence tasks, to better understand their strategies and reasoning about fractions. His findings suggested that students skilled with fractions used a variety of strategies for solving problems, not just the traditional ones learned in school. In fact, they tended to use the most general strategy as a last resort, opting for strategies specific to the problem at hand. Their preferred strategies were often ones that made the problem easier in some way (e.g., required less computation).

Newton (2008) found a similar pattern for experts such as mathematicians, engineers, and high school mathematics teachers that took a fraction assessment designed for a study of preservice teachers. When asked to solve fraction computation items, they generally opted for strategies that made the problem easier for themselves. Out of 11 possible points for flexibility, they scored 8.44 points on average, compared to 2.27 for preservice teachers. Although not published as part of the study, one example expert strategy was to solve $6\,\frac{2}{5} - 2\,\frac{4}{5}$ by subtracting the whole numbers and then subtracting the fractions. Doing so resulted in a new subtraction problem, $4 - \frac{2}{5}$, which can be mentally calculated to be $3 - \frac{3}{5}$ based on the fact that two-fifths and three-fifths is one whole. A more common strategy was to rename $6 - \frac{2}{5}$ as $5 - \frac{7}{5}$ and then subtract like parts. In contrast, a strategy used frequently by preservice teachers was to rename both mixed numbers as improper fractions and then subtract numerators, keeping the denominator the same. The preservice teachers in this study generally used the strategy of changing mixed numbers to improper fractions whenever mixed numbers were involved, even when it required more steps.

A case study of a small special education classroom revealed that non-experts are capable of learning and using a variety of strategies for solving fraction problems (Newton et al., 2014). Interestingly, for some problems the more general approach led to errors for these students, whereas an efficient, alternative method led to success. For example, on an end-of-unit fraction assessment, 6 out of 11 students correctly solved $\frac{9}{14} \div \frac{1}{7}$ by dividing the numerators and dividing the denominators. On the other hand, three students attempted to solve the problem by multiplying by the reciprocal (i.e., $\frac{9}{14} \times \frac{7}{1}$), but they all made errors converting the product, $\frac{63}{14}$, to a mixed number. On the other hand, seven students correctly multiplied by the reciprocal to solve $2/9 \div 3/8$, which did not involve simplifying.

Of note is that four students used both methods successfully during the assessment. However, as mentioned previously the teacher in this classroom regularly introduced a new topic by asking students to make conjectures based on their prior knowledge. This atypical practice lends itself to the use of multiple strategies, since not all students will make the same conjectures each time. Based on the work of Bishop et al. (2016), it seems likely that the requests for conjectures about how things might work helped these students to focus on underlying structures and patterns. In the case of dividing across numerators and denominators, students may reason that since division is related to multiplication and fraction multiplication works by multiplying numerators and denominators, then fraction division should work similarly. Research is needed to better understand the role of conjecturing in the development of flexibility.

Representational flexibility with fractions, or an ability to move smoothly between different fraction representations (e.g., circle, rectangle, and number line), has been a focus for some researchers. Deliyianni, Gagatsis, Elia, and Panaoura (2016) examined the relationship between representational flexibility, problem solving with novel tasks, and understanding of fraction addition using confirmatory factor analysis. Findings suggest that problem solving and representational flexibility constitute major components of fraction understanding. Further, representational flexibility with fraction addition is better supported by the number line and rectangle representations than by circles. These results are consistent with standards documents for mathematics education. In particular, the National Council of Teachers of Mathematics (2000) forwarded problem solving and movement between representations as two important processes that students should experience in mathematics classrooms. And as noted above, the Common Core State Standards (National Governors Association Center for Best Practices & the Council of Chief State School Officers, 2010) recently asserted that the number line is a critical representation for learning fractions.

Flexibility with Algebra

Similar to fractions, flexibility in algebraic thinking is also a characteristic of experts. Star and Newton (2009) interviewed experts about their strategy choices when solving algebra problems, and they had a tendency to value and use cognitively efficient strategies. They typically justified strategy choices by saying a particular method was "easier." When prompted to elaborate, "easier" usually involved fewer steps but, more importantly, it referred to less need for written computation.

On the other hand, students struggling to learn algebra may be more concerned with methods that they can successfully use (Newton et al., 2010). This focus on accuracy may or may not involve the most efficient method. For example, a student might prefer clearing the denominator for problems such as $1/3(x+6)=4$ not because it is the method with the fewest steps, but because he or she is not confident about distributing the fraction without errors. This same student may use the distributive property for a problem such as $3(x+6)=15$ because it is most familiar and the student may be confident in using it successfully. These students are more likely to value efficiency when they are equally confident in different problem-solving strategies.

Flexibility in algebra is predicted by both procedural and conceptual knowledge (Schneider, Rittle-Johnson, & Star, 2011). However, students with both low and high prior algebra knowledge seem to appreciate efficiency in problem solving, even when they do not use more efficient methods (Newton et al., 2019). The third recommendation described above, learning and choosing from multiple strategies, is particularly relevant for promoting procedural flexibility with algebra (Star et al., 2015). By comparing and using different ways to solve the same problem, students are encouraged to notice problem features that make a particular strategy more efficient in some cases.

Interventions

Comparing. As noted above, comparing two worked examples presented side by side can help students draw conclusions about important ideas or strategies (Star et al., 2015). In particular, comparing a general strategy to a more efficient alternative can highlight when one strategy might be more efficient than the other one.

Relative to studying examples sequentially, comparing two examples side by side led to improved procedural knowledge and flexibility in a study involving seventh graders learning to solve equations (Rittle-Johnson & Star, 2007). The intervention in this study was specifically designed to help students attend to structural features that might make one method more efficient in some cases. For example, when solving $3(x + 5) + 4(x + 5) = 14$, students might notice the parentheses and then distribute as a first step to solving the equation. An alternative approach would rely on noticing that each set of parentheses contains the same expression; therefore, a first step can to be combine the two like expressions. The compare condition included prompts to help students notice these features and consider when the alternative method might be a good problem-solving choice. Students in another condition studied the same worked examples, but sequentially (on separate pages) and with prompts that avoided comparisons. The fact that both groups were exposed to both multiple solution methods highlights comparison as an effective intervention for accurate and efficient equation-solving strategies during early algebra learning.

In a follow-up study of seventh and eighth graders, Rittle-Johnson and Star (2009) explored the effects of different types of comparisons in algebra. In one condition, students compared different solution methods for the same problem. Students in another condition compared different problem types solved using the same solution method. The third condition involved comparing equivalent problems using the same method. These researchers found that comparing different methods for solving the same problem was more effective for conceptual knowledge and flexibility than the other two types of comparisons.

Some interventions have targeted *representational flexibility* in the context of algebra. Nistal, Van Dooren, and Verschaffel (2014) tested an intervention targeted at improving students' representational flexibility, conceptualized as the ability to make adaptive choices when choosing representations (e.g., table, graph, formula), to solve linear function problems. The intervention was individualized, in that it involved providing students with feedback on their accuracy using different representations to solve problems at pretest. It also included asking students to reflect on their representational choices and compare them to other (example) students' choices and reasons for making those choices. It was emphasized that the best choice may be depend on both the problem as well as the student (given not everyone has the same proficiency with different representations). Students in the intervention group learned and used representations to a similar extent as students in a control group. However, students who received the intervention solved problems with more speed and accuracy at posttest. Further analyses indicate that this improvement was a result of an improved representational flexibility. In other words, students chose representations for themselves that led to better accuracy at posttest compared to pretest.

Generating More than One Strategy. Comparing two worked examples, presented side by side, can lead to increased flexibility with algebra by providing opportunities to learn about and evaluate different solution strategies (Star et al., 2015a). However, "a specific learner's flexibility in using different methods … depends upon the familiarity of the specific problem at hand" (Atkinson et al., 2000, p. 185). One recommendation to help students gain fluency with a particular strategy is to interleave worked examples with practice problems. However, even when students have knowledge of more efficient strategies, they do not always use them regularly (Newton et al., 2019). The ability to generate multiple solution methods to a single algebra problem seems to be a key skill that helps explain the gap between *knowledge* (e.g., being able to recognize valid alternatives) and *use* of efficient strategies for solving algebra problems.

Asking students to generate a new way to solve an equation led to increased flexibility in a study of sixth grade students with no prior experience in algebra, compared to a control group that was asked to solve a new problem in the same way (Star & Seifert, 2006). Despite solving fewer problems overall, the two groups had similar levels of equation-solving accuracy at posttest. In this case, students were generating alternative solution strategies through invention. Another possibility is to ask students to generate more than one way to solve a problem by recalling methods they learned through studying worked examples. Research is needed to confirm whether this kind of activity can effectively support students learning to flexibly use a variety of problem-solving strategies.

LIMITATIONS AND FUTURE DIRECTIONS

Despite the many advances reported here, some gaps remain in our understanding of effective strategy interventions for fractions and algebra. For example, our knowledge of negatives is limited compared to their positive counterparts. Yet, errors with negatives prevent many students from being successful in algebra (Booth et al., 2014). Further, as noted above, more research is needed on ways to promote regular use of flexible strategies in algebra, such as asking students to solve the same problem in two ways once they have learned multiple strategies. Although there has been renewed interest in fraction knowledge recently, much of this research has focused on identifying challenges to learning fractions and finding effective interventions for faulty strategies (e.g., Fuchs et al., 2017; Siegler et al., 2013). Given the difficulties that children and adults have with fractions, this focus is a reasonable and laudable one. Yet, research on flexibility with fractions is scarce and so future directions should include ways to promote flexibility with fractions.

Much of the research reported here has focused on conceptual and procedural knowledge of fractions and algebra, such as the ability to order fractions or to solve linear equations. While this knowledge is clearly important, additional research is needed to understand and support problem solving with fractions and algebra. "Problem solving" sometimes refers to a student calculating an answer, even to a routine exercise, as opposed to studying a worked example (see Newton & Sword, 2018). I am instead referring to opportunities for students to apply knowledge to a new situation, often presented in the form of word problems. Jitendra, Harwell, Dupuis, and Karl (2017) have conducted extensive research on this kind of problem solving, with a focus on proportional reasoning. Proportions and proportional reasoning represent a bridge between fractions and algebra, making it a critical mathematical milestone in late middle school. Their works suggests that schema-based instruction, where students are taught to identify and represent the underlying structure of a problem, has been effective with general and special education students. The work of researchers like Empson and Levy (2011) builds on the premise that problem solving is important as a *means* for understanding fractions (not simply as an application of fraction knowledge). Future research should extend work presented in this chapter by considering the role of problem solving in learning fractions and algebra. For example, is problem solving compatible with the use of worked examples? If so, how can they work together to support learning?

Related, increased efforts to cross the boundaries of psychology and education are needed. Obersteiner, Dresler, Bieck, and Moeller (2019) recently reported on findings in these fields and in neuroscience, leading to recommendations for fraction instruction. Moreover, research in these fields should come together to support development and understanding of effective comprehensive interventions, such as those described by Fuchs et al. (2017) and McNeil et al. (2019). Interventions that are successful in short laboratory studies or controlled classroom studies sometimes fall short in the context of an ongoing and complicated classroom, for a variety of reason, including lack of implementation by teachers (Star et al., 2015). Attention, prior knowledge, fidelity, and many other factors can prevent even empirically supported practices from making the same impact in the classroom. Yet, for research on fractions and algebra to make a difference with students, the findings must be integrated into regular instruction.

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