Section 13.1  
Chemical Graph Theory  
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INTRODUCTION
Chemical graph theory (CGT) is a branch of mathematical chemistry which deals with the nontrivial applications of graph theory to solve molecular problems. In general, a graph is used to represent a molecule by considering the atoms as the vertices of the graph and the molecular bonds as the edges. Then, the main goal of CGT is to use algebraic invariants to reduce the topological structure of a molecule to a single number which characterizes either the energy of the molecule as a whole or its orbitals, its molecular branching, structural fragments, and its electronic structures, among others. These graph theoretic invariants are expected to correlate with physical observable measures by experiments in a way that theoretical predictions can be used to gain chemical insights even for not yet existing molecules. In this brief review we shall present a selection of results in some of the most relevant areas of CGT.

13.1.1 Basic Definitions

DEFINITIONS

D1: A molecular graph $G = (V, E)$ is a simple graph having $n = |V|$ nodes and $m = |E|$ edges. The nodes $v_i \in V$ represent non-hydrogen atoms and the edges $(v_i, v_j) \in E$ represent covalent bonds between the corresponding atoms. In particular, hydrocarbons are formed only by carbon and hydrogen atoms and their molecular graphs represent the carbon skeleton of the molecule.

D2: An alternant conjugated hydrocarbon is a hydrocarbon with alternant multiple (double and/or triple) and single bonds, such as the molecular graph is bipartite and the edges of the graph represent $C = C$ and $= C = C$ or $C \equiv C$ and $\equiv C - C \equiv$ bonds only.
13.1.2 Molecular Energy

**FACTS**

**F1:** In the Hückel Molecular Orbital (HMO) method for conjugated hydrocarbons the energy of the \( j \)th molecular orbital of the so-called \( \pi \)-electrons is related to the graph spectra by

\[
\lambda_j = \frac{\alpha - E_\pi(j)}{\beta},
\]

where \( \lambda_j \) is an eigenvalue of the adjacency matrix of the hydrogen-depleted graph representing the conjugated hydrocarbon and \( \alpha, \beta \) are empirical parameters [CoOlMa78, GrGuTr77, Ku06, Ya78].

**F2:** The total \( \pi \) (molecular) energy is given by

\[
E_\pi = \alpha n_e + \beta \sum_{j=1}^{n} g_j \lambda_j + \beta E,
\]

where \( n_e \) is the number of \( \pi \)-electrons in the molecule and \( g_j \) is the occupation number of the \( j \)th molecular orbital.

**F3:** For neutral conjugated systems in their ground state [Gu05],

\[
f(n) = \begin{cases} 
2 \sum_{j=1}^{n/2} \lambda_j & \text{if } n \text{ is even} \\
2 \sum_{j=1}^{(n+1)/2} \lambda_j + \lambda_{(j+1)/2} & \text{if } n \text{ is odd}
\end{cases}
\]

**REMARKS**

**R1:** In most of the conjugated molecules studied by HMO \( n \) is an even number. In such cases, \( E \) can be expressed as \( E = \sum_{j=1}^{n} \left| \lambda_j \right| \).

**R2:** The concept of graph energy is defined for any graph as \( E = \sum_{j=1}^{n} \left| \lambda_j \right| \) [Ni07]. In this case this term is not related to any “physical” energy but the index can be considered as a graph-theoretic invariant.

**R3:** \( \beta < 0 \), then in representing the energy of molecular orbitals \( \varepsilon_j = \alpha + \beta \lambda_j \) it is assumed that the largest eigenvalue represents the minimum energy, then the second largest, and so forth [CoOlMa78, GrGuTr77, Ku06, Ya78].

**R4:** Because an alternant conjugated hydrocarbon has a bipartite molecular graph: \( \lambda_j = -\lambda_{n-j+1} \) for all \( j = 1, 2, \ldots, n \).

**EXAMPLE**

**E1:** The molecule of 1,3-butadiene is a conjugated hydrocarbon whose molecular graph is the path graph with four nodes \( P_4 \). The energy of the four molecular orbitals in 1,3-butadiene, expressed by the eigenvalues of the adjacency matrix are -1.618, -0.618, 0.618, 1.618. The total energy of the molecule is -4.472.
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FACTS

F4: Let $G$ be a graph with $n$ vertices and $m$ edges. Then [Mc71],
\[ \sqrt{2m + n(n - 1)(\det A)^{n/2}} \leq E \leq \sqrt{mn}. \]

F5: Let $G$ be a graph with $m$ edges. Then, $2\sqrt{m} \leq E \leq 2m$.

F6: Let $G$ be a graph with $n$ vertices. Then, $E \geq 2\sqrt{n - 1}$, where the equality holds if $G$ is the star graph with $n$ vertices.

F7: [KoMo01] $E \leq 2m/n + \sqrt{(n - 1)(2m - 4m^2/n^2)}$ where the equality holds if and only if $G$ is $K_n$, $\frac{n}{2}K_2$, or a strongly regular graph with two eigenvalues having absolute value
\[ \sqrt{\frac{2m - (2m/n)^2}{n - 1}}. \]

F8: Let $G$ be a graph with $n$ vertices. Then [KoMo01],
\[ E \leq \frac{n}{2}(\sqrt{n} + 1), \]
where the equality holds if and only if $G$ is a strongly regular graph with parameters
\[ (n, (n + \sqrt{n})/2, (n + 2\sqrt{n})/4, (n + 2\sqrt{n})/4). \]

F9: Let $G$ be a bipartite graph with $n$ vertices and $m$ edges. Then [KoMo03],
\[ E \leq 4m/n + \sqrt{(n - 2)(2m - 8m^2/n^2)}. \]

F10: For all sufficiently large $n$, there is a graph $G$ of order $n$ such that [Ni07]
\[ E \geq \frac{n}{2}(\sqrt{n} - n^{1/10}). \]

13.1.3 Graph Nullity and Zero-Energy States

DEFINITION

D3: The nullity of a (molecular) graph, denoted by $\eta = \eta(G)$, is the algebraic multiplicity of the number zero in the spectrum of the adjacency matrix of the (molecular) graph.

REMARKS

R5: An alternant unsaturated conjugated hydrocarbon with $\eta = 0$ is predicted to have a stable, closed-shell, electron configuration. Otherwise, the respective molecule is predicted to have an unstable, open-shell, electron configuration.

R6: If $n$ is even, then $\eta$ is either zero or it is an even positive integer.
EXAMPLE

**E2:** The molecule of 1,3-cyclobutadiene is a conjugated hydrocarbon whose molecular graph is the cycle graph with four nodes $c_4$. The energy of the four molecular orbitals in 1,3-butadiene is $E_1 = \alpha - 2|\beta|$, $E_2 = \alpha + 0|\beta|$, $E_3 = \alpha + 0|\beta|$ and $E_4 = 4(\alpha + 2|\beta|)$. The nullity of this graph is $\eta = 2$ and the first orbital is occupied by a pair of electrons while the two zero-energy states have one electron each. The total $\pi$-energy is $E_\pi = 4(\alpha - |\beta|)$.

FACTS

**F11:** Let $P_n$, $C_n$ and $K_n$ be the path, cycle and complete graph with $n$ vertices, respectively. Then [BoGu09],

i) $\eta(P_n) = 0$ if $n$ is even and $\eta(P_n) = 1$ if $n$ is odd.

ii) $\eta(C_n) = 2$ if $n \equiv 0 \ (mod \ 4)$ or zero otherwise.

iii) $\eta(K_1) = 1$ and $\eta(K_{n>1}) = 0$.

**F12:** [CvGu72] Let $M = M(G)$ be the size of the maximum matching of a graph, i.e., the maximum number of mutually non-adjacent edges of $G$. Let $T$ be a tree with $n \geq 1$ vertices. Then, $\eta(T) = n - 2M$.

**F13:** [CvGuTr72] Let $G$ be a bipartite graph with $n \geq 1$ vertices and no cycle of length $4s$ ($s = 1, 2, \ldots$), then $\eta(G) = n - 2M$.

REMARK

**R7:** The nullity of benzenoid graphs, which may contain cycles of length $4s$, is also given by $\eta(G) = n - 2M$ [Gu83, FaJoSa05].

FACTS

**F14:** [Lo50] Let $G$ be a bipartite graph with incidence matrix $B$, $\eta(G) = n - 2r(B)$, where $r(B)$ is the rank of $B$.

**F15:** [ChLi07] Let $G$ be a graph with $n$ vertices and at least one cycle,

$$\eta(G) = \begin{cases} 
 n - 2g(G) + 2 & \text{if } g(G) \equiv 0 \ (mod \ 4), \\
 n - 2g(G) & \text{otherwise}
\end{cases}$$

where $g(G)$ is the girth (length of minimal cycle) of the graph.

**F16:** [ChLi07] If there is a path of length $d(p, q)$ between the vertices $p$ and $q$ of $G$

$$\eta(G) = \begin{cases} 
 n - d(p, q) & \text{if } d(p, q) \text{ is even}, \\
 n - d(p, q) - 1 & \text{otherwise}.
\end{cases}$$

**F17:** [ChLi07] Let $G$ be a simple connected graph of diameter $D$

$$\eta(G) = \begin{cases} 
 n - D & \text{if } D \text{ is even}, \\
 n - D - 1 & \text{otherwise}.
\end{cases}$$

**F18:** [ChLi07] Let $G$ be a simple connected graph on $n$ vertices having $K_p$ as a subgraph, where $2 \leq p \leq n$. Then,

$$\eta(G) \leq n - p.$$
13.1.4 Graph-Based Molecular Descriptors

DEFINITIONS

D4: A graph-based molecular descriptor, commonly known as topological index (TI), is a graph-theoretic invariant characterizing numerically the topological structure of a molecule [DeBa00].

D5: The Wiener index of a (molecular) graph is a TI defined by

\[
W = \sum_{i<j} d_{ij}
\]

where \(d_{ij}\) is the shortest-path distance between the vertices \(i\) and \(j\) [Wi47].

D6: The Hosoya index of a (molecular) graph is a TI defined by

\[
H = \sum_{i=0}^{n/2} P(G, i)
\]

where \(P(G, i)\) is the number of selections of \(i\) mutually nonadjacent edges in the graph. By definition \(P(G, 0) = 1\) and \(P(G, 1) = m\) [Ho71].

D7: The Zagreb indices of a (molecular) graph are TIs defined by [GuTr72]

\[
M_1 = \sum_{j=1}^{n} (\delta_j)^2,
\]

\[
M_2 = \sum_{i,j \in E} \delta_i \delta_j.
\]

D8: The Randić index of a (molecular) graph is a TI defined by [Ra75]

\[
\chi = \sum_{i,j \in E} (\delta_i \delta_j)^{-1/2}.
\]

D9: Let \(k = 0, 1, 2, 3, \ldots\) be the number of adjacent vertices of degrees \(\delta_i, \delta_j, \delta_l, \ldots\) in graph \(G\). Then [KiHaMuRa75, KiHa76], the Kier and Hall molecular connectivity index is defined as

\[
k_{\chi} = \sum_{i,j,l,\ldots} (\delta_i, \delta_j, \delta_l, \ldots)^{-1/2}
\]

where the summation is taken over all subgraphs of size \(k\), and the null term is the sum of all the vertex degrees (the total adjacency of \(G\)).

D10: Let \(s_i = \sum_{j=1}^{n} d_{ij}\) be the distance sum for the vertex \(i\) in a (molecular) graph. The Balaban index is a TI defined by

\[
J = \frac{m}{C+1} \sum_{i,j \in E} (s_is_j)^{-1/2},
\]

where \(C = m - n + 1\) is the cyclomatic number of the graph [Ba82].
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**D11:** The *atom-bond connectivity index* of a (molecular) graph is a TI defined by [EsToRoGu98, 7]

\[ ABC = \sum_{i,j \in E} \sqrt{\delta_i + \delta_j - 2} \delta_i \delta_j. \]

**D12:** Let \( G \) be a connected graph with adjacency matrix \( A \) and let \( D \) be a diagonal matrix of vertex degrees of \( G \). The *Laplacian matrix* of the graph is defined as \( L = D - A \).

**D13:** Let \( G \) be a connected graph with Laplacian matrix \( L \) and let \( p \) and \( q \) be two vertices of \( G \). The *resistance distance* between \( p \) and \( q \) is defined by [KlRa93]

\[ \Omega_{pq} = L^\dagger_{pp} + L^\dagger_{qq} - 2L^\dagger_{pq} \]

where \( L^\dagger_{pq} \) is the \( p, q \)-entry of the Moore-Penrose pseudo-inverse of the Laplacian matrix.

**D14:** The *Kirchhoff index* of a (molecular) graph is a TI defined by [KlRa93]

\[ Kf = \sum_{i<j} \Omega_{ij} \]

**REMARKS**

**R8:** The Wiener number has been modified to describe the basic topology of infinite polymeric macromolecules and named *Wiener infinite*, \( W_\infty \) [BoMeKa92]:

\[ W_\infty = \lim_{n \to \infty} \frac{an^3 + bn^2 + cn + d}{m \left[ \frac{n(n-1)}{2} \right]} \]

**R9:** The Randić index has been generalized to [BoErSa99]

\[ \chi^t = \sum_{i,j \in E} (\delta_i \delta_j)^t, \]

and a few mathematical results exist for the different values of \( t \) [LiSh08].

**FACTS**

**F19:** Let \( T_n \) be a tree with \( n \) vertices, then [EnJaSu76, BoTr77]

\[ W(S_n) < W(T_n) < W(P_n), \]

where \( W(S_n) = (n-1)^2 \) and \( W(P_n) = \left(\frac{n+1}{3}\right)^3 \).

**F20:** Let \( T_n \) be a tree with \( n \) vertices and let \( 0 = \mu_1 < \mu_2 \leq \ldots \leq \mu_n \) be the eigenvalues of the Laplacian matrix of the tree. Then [Me90, Mo91, DoEnGu01],

\[ W(T_n) = n \sum_{j=2}^{n} (\mu_j)^{-1}. \]
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**F21:** Let $H_k$ be a hexagonal chain with $k \geq 1$ linearly fused hexagons, then [ShLa97]

$$W(H_k) = \frac{1}{3}(16k^3 + 36k^2 + 26k + 3).$$

**F22:** Let $T$ be a tree on $n$ vertices. Let for an edge $e = (x, y)$ define $n_1(e) = |\{v \in V(T), d(v, x|T) < d(v, y|T)\}$ and $n_2(e) = |\{v \in V(T), d(v, y|T) < d(v, x|T)\}$. Then [Wi47, GuPo86, DoEnGu01],

$$W(T) = \sum_{e \in E(T)} n_1(e)n_2(e)$$

**REMARK**

R10: This is the manner in which Wiener introduced his index in 1947.

**FACTS**

**F23:** Let $T$ be a tree. Let the bipartite sets of its vertices be of cardinality $|V_a|$ and $|V_b|$. Then [BoGuPo87], $W(T)$ is odd if and only if both $|V_a|$ and $|V_b|$ are odd. If $|V_a|$ or/and $|V_b|$ is even, $W(G)$ is even.

**F24:** Let $m \geq 2$. Let $T_1, T_2, \ldots, T_m$ be trees with disjoint vertex sets and orders $n_1, n_2, \ldots, n_m$. Let $i = 1, 2, \ldots, m, w_i \in V(T_i)$. Let $T$ be a tree on $n \geq 3$ vertices, obtained by joining a new vertex $u$ to each of the vertices $w_1, w_2, \ldots, w_m$. Then [CaRoRo85, DoEnGu01],

$$W(T) = m \sum_{i=1}^{m}[W(T_i) + (n - n_i)d(w_i|T_i) - n_i^2] + n(n - 1)$$

**F25:** Let $T$ be a tree on $n$ vertices. Let $v$ and $u$ be vertices on a pendant edge. Then [DoGu94],

$$W(T) = \frac{1}{4}[n^2(n-1) - \sum_{(u,v) \in E(T)} [d(v|T) - d(u|T)]^2].$$

**F26:** Let $T$ be a tree on $n$ vertices. Let $\deg(v)$ be the degree of vertex $v$. Then [KIMiPiTr92, DoGu94, Gu94],

$$W(T) = \frac{1}{4}[n(n-1) + \sum_{v \in V(T)} \deg(v)d(v|T)]$$

**F27:** Let $T$ be a tree on $n$ vertices and $u$ branching points. Then [DoGr77],

$$W(T) = \left(\frac{n+1}{3}\right) - \sum_{u} \sum_{1 \leq i < j < k \leq m} n_in_jn_k.$$
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**F29:** Let $W_\infty$ be the Wiener infinite index, $N_1$ and $C_1$ the number of atoms and cycles in the monomeric unit, and $d$ the distance between two neighboring monomeric units in the polymer graph. Then [BaBaBo01],

$$W_\infty = \frac{d}{3(N_1 + C_1)}.$$

**F30:** Let $N$, $R_g^2$, and $W$ be the number of atoms of a polymer whose macromolecule contains no atomic rings, the mean-square radius of gyration of the polymer, and the Wiener number of the polymer graph. Let also $b$ be the length of the covalent bond connecting two monomeric units, let $c$ be the number of polymer chains in a unit volume, and let $\xi$ be the friction coefficient. Then [BoMaDe02],

$$R_g^2 = \frac{b^2}{6N^2}W; \quad \eta_0 = \frac{cb^2\xi}{6N^2}W.$$

**F31:** Let $g$ be the Zimm–Stockmayer branching ratio of a branched macromolecule containing no atomic rings. Let also $W$, $W_{lin}$, and $R_g^2, R_{g,lin}^2$ be the Wiener indices and the mean-square radius of gyration of the branched and linear polymer graph with the same molecular weight. Then [BoMaDe02],

$$g = \frac{R_g^2}{R_{g,lin}^2} = \frac{w}{w_{lin}}.$$

**F32:** Let subgraphs $G_i$ cover upon a vertex $u'$. Let also $d(u|G)$ and $d(u, v|G_i)$ be the distance numbers of the common vertex $u$ in graph $G$ and its $i^{th}$ component $G_i$. Then [PoBo86],

$$W(G) = \sum_i W(G_i) + nd(u \in G) - \sum_i n_i d(u \in G_i).$$

**F33:** Let $I$ be the number of isomorphic components $G'$, which cover to form graph $H$, and let each of the $G'$s have $n'$ vertices. Let also $W(G')$, $W(H)$, and $d(u|G')$ be the Wiener number of $G'$ and $H$, and the distance number of vertex $u$ in $G'$. Then [PoBo90],

$$W(H) = I \cdot W(G') + (n' - 1) \cdot I (I - 1) \cdot d(u|G').$$

**F34:** Let graphs $G_1$ and $G_2$ have $n_1$ and $n_2$ vertices, and let the graphs be linked by a bridge $\{uv\}$. Then,

$$W(H) = W(G_1) + W(G_2) + n_1 n_2 + n_2 d(u|G_1) + n_1 d(v|G_2).$$

**F35:** Let an edge $\{uv\}$ be divided by an inserted vertex $x$. Let also the total distance of vertex $x$ in the graph $H$ obtained by $d(xH)$, and the number of geodesics containing vertex $s$, which are enlarged due to the division of the edge be $b(s)$. Then,

$$W(H) = W(G) + d(x|H) + \sum_{s \in G} b(s)]/2.$$

**F36:** Let the edge considered in Fact F35 be a bridge. Let also the number of vertices in the two subgraphs $G_1$ and $G_2$ be $n_1$ and $n_2$, and let $uG_1$ and $vG_2$. Then,

$$W(H) = W(G) + n_1 n_2 + [d(u|G) + d(v|G) + n_1 + n_2]/2.$$
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**F37:** Let a subgraph of $n_1$ vertices be transferred from a terminal vertex $u$ to another terminal vertex $v$. Let also the distance numbers of $u$ and $v$ be denoted by $d(u|G)$ and $d(v|G)$. Then [PoBo86],

$$\Delta W = n_1[d(u|G) - d(v|G)].$$

**F38:** Let a subgraph be transferred from a terminal vertex $u$ to another terminal vertex $v$. Let also the length of the shortest path $uv$ be $L$, the position of the branches located between $u$ and $v$ be $i$, and the number of vertices in these intermediate branches $i$, located symmetrically with respect to $u$ and $v$, be $n_{v,i}$ and $n_{u,i}$. Then [PoBo90],

$$\Delta W = \sum_i i[(L - 2i)(n_{u,i} - n_{v,i})].$$

**F39:** Let $T_n$ be a tree with $n$ vertices and let $F_n$ be the $n$th Fibonacci number. Then, $n \leq Z(T_n) \leq F_n + 1$ where the lower bound is obtained for $S_n$ and the upper bound is obtained for $P_n$ [Gu77].

**F40:** Let $G$ be a graph with $k$ components $G_1, G_2, \ldots, g_k$. Then [GuPo86],

$$Z(G) = \prod_{i=1}^{k} Z(G_i).$$

**F41:** Let $G$ be a graph, let $pq \in E$ be an edge, and $p \in V$ be a vertex of $G$ [GuPo86]. Then,

i) $Z(G) = Z(G - pq) + Z(G - \{p, q\}).$

ii) $Z(G) = Z(G - p) + \sum_{p, q \in E} Z(G - \{p, q\}).$

**F42:** Let $G$ be a graph and let $pq \in E$. Then [WaYeYa10],

$$Z(G) \geq Z(G - pq).$$

**F43:** Let $G$ be a graph with $|P_i|$ paths of length $i$, $|P_i| = m$, and $|C_3|$ triangles. Then [BrKeMeRu05],

i) $M_1 = 2m + 2|P_2|.$

ii) $M_2 = m + 2|P_2| + |P_3| + 3|C_3|.$

**F44:** Let $G$ be a connected graph with $n$ vertices and $m$ edges. Then [De98],

$$M_1 \leq m \left(\frac{2m}{n-1} + n - 2\right),$$

with equality if and only if the graph is $S_n$ or $K_n$. 
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F45: [DaGu04] Let $G$ be a graph with $n$ vertices. Then

$$0 \leq M_2 \leq \frac{1}{2}n(n-1)^3,$$

where the upper bound is obtained for the complete graph and the lower one for the empty graph.

F46: [DaGu04] Let $G$ be a connected graph with $n$ vertices, $m$ edges, and minimum degree $\delta_{\text{min}}$. Then

$$M_2 \leq 2m^2 - (n-1)m\delta_{\text{min}} + \frac{1}{2}(\delta_{\text{min}} - 1)m \left( \frac{2m}{n-1} + n - 2 \right),$$

with equality if and only if the graph is $S_n$ or $K_n$.

F47: The Randić index is bounded as [CaGuHaPa03, LiSh08]

$$\sqrt{n-1} \leq \chi \leq \frac{n}{2},$$

where the lower bound is reached for the star $S_n$ and the upper bound is attained for any regular network with $n$ nodes indistinct of its degree.

F48: Let $T_n$ be a chemical tree ($\delta_{\text{min}} \leq 4$) with $n$ vertices and $n_1 \geq 3$ pendant vertices. Then,

$$\chi(T_n) \leq \frac{n}{2} + n_1 \left( \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{6}} - \frac{7}{6} \right),$$

with equality if and only if the tree is $T(3,2)$ [HaMe03].

F49: [Es10] Let $k = \left[ \delta_{1}^{-1/2} \delta_{2}^{-1/2} \ldots \delta_{n}^{-1/2} \right]^T$. Then

$$\chi = \frac{1}{2}(n - k^TLk).$$

F50: Let $G$ be a connected graph with $n$ vertices, $m$ edges and let $\lambda_1$ be the largest eigenvalue of the adjacency matrix of $G$. Then [FaMaSa93, CaHa04],

i) $\lambda_1 \geq \frac{m}{\chi},$

ii) $\chi + \lambda_1 \geq 2\sqrt{n-1} (n \geq 3),$

iii) $\chi\lambda_1 \geq n-1 (n \geq 3).$

F51: Let $G$ be a connected graph with $n$ vertices. Then [DoGu10]

$$J(P_n) \leq J(G) \leq J(K_n),$$

where

$$J(P_n) = (n-1)\sum_{i=1}^{n-1}(s_is_{i+1})^{-1/2}, \ s_i = \frac{(n-i+1)(i-1)i}{2} + \frac{(i-1)i}{2}$$

and

$$J(K_n) = \frac{n^2(n-1)}{2(n^2 - 3n + 4)}.$$
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F52: [Da10] Let $G$ be a connected graph with $m$ edges and let $\delta_{\text{max}}$ be the maximum vertex degree. Then,

$$ABC \geq \frac{2^{7/4}m\sqrt{\delta_{\text{max}}}}{\sqrt[3]{4}\delta_{\text{max}} \left(\sqrt{\delta_{\text{max}}} + \sqrt{2}\right)}$$

when equality is attained for the path graph with $n$ vertices.

F53: [ChGu11, DaGuFu11] Among graphs with $n$ vertices the complete graph has the greatest ABC index and this maximal-ABC graph is unique.

F54: [ChGu11, DaGuFu11] The smallest ABC index for a connected graph with $n$ vertices must be a tree and this minimal-ABC tree need not be unique.

F55: [FuGrVu09] Among trees with $n$ vertices, the star has the greatest ABC index and this maximal-ABC tree is unique.

REMARK R11: The trees with vertices for which the ABC index is minimum are not known.

FACTS

F56: Let $L(G-u)$ be the matrix resulting from removing the $u^{th}$ row and column of the Laplacian and let $L(G-u-v)$ be the matrix resulting from removing both the $u^{th}$ and $v^{th}$ rows and columns of $L$. The resistance distance can be calculated as [BaGuXi03]:

$$\Omega(u,v) = \frac{\text{det} \ L(G-u-v)}{\text{det} \ L(G-u)}.$$ 

F57: Let $U_k(u)$ be the $u^{th}$ entry of the $k^{th}$ orthonormal eigenvector associated to the Laplacian eigenvalue $\mu_k$, which has been ordered as $0 = \mu_1 < \mu_2 \leq \ldots \leq \mu_n$. Then [XiGu03],

$$\Omega(u,v) = \sum_{k=2}^{n} \frac{1}{\mu_k} [U_k(u) - U_k(v)]^2.$$ 

F58: Let the resistance matrix $\Omega$ be the matrix containing the resistance distance between every pair of vertices in a graph. Then [GoBoSa08],

$$\Omega = |1\rangle \text{diag}\{(L + (1/n)J)^{-1}\}^T + \text{diag}\{L + (1/n)J\}^{-1} |1\rangle - 2(L + (1/n)J)^{-1}$$

where $J = |1\rangle \langle 1|$ is an all-ones matrix.

F59: Let $G$ be a connected graph with $n$ vertices; the Kirchhoff index is given by

$$Kf(G) = nTr \int_0^{\infty} \left( e^{-tL} - \frac{1}{n} 11^T \right) dt,$$

where $1$ is an all-ones column vector [GoBoSa08].

F60: Let $G$ be a connected graph with $n \geq 3$ vertices, $m$ edges, and let $\delta_{\text{max}}$ be the maximum vertex degree. Then [ZhTr08],

$$Kf(G) \geq \frac{n}{1 + \delta_{\text{max}}} + \frac{n(n-2)^2}{2m - 1 - \delta_{\text{max}}}.$$
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**F61:** Let $G$ be a connected graph with $n \geq 2$ vertices, $m$ edges, and let $\delta_{\text{min}}$ and $\delta_{\text{max}}$ be the minimum and maximum vertex degree, respectively. Let $0 = \mu_1 < \mu_2 \leq \ldots \leq \mu_n$ be the eigenvalues of the Laplacian matrix. Then \cite{ZhTr09},

$$\frac{n}{\delta_{\text{max}}} \sum_{j=2}^{n} \frac{1}{\mu_j} \leq K_f(G) \leq \frac{n}{\delta_{\text{min}}} \sum_{j=2}^{n} \frac{1}{\mu_j},$$

with equalities at both sides if and only if it is regular.

**F62:** Let $G$ be a connected bipartite graph with $n \geq 2$ vertices, and let $\delta_{\text{max}}$ be the maximum vertex degree. Then \cite{ZhTr09},

$$K_f(G) \geq n (2n - 3) \delta_{\text{max}},$$

with equality if and only if $G$ is $K_{2,n/2}^2$.

13.1.5 Walk-Based Molecular Parameters

**DEFINITIONS**

**D15:** A walk of length $k$ is a sequence of (not necessarily distinct) nodes $v_0, v_1, \ldots, v_{k-1}, v_k$ such that for each $i = 1, 2, \ldots, k$ there is a link from $v_{i-1}$ to $v_i$. A walk is a closed walk if $v_0 = v_k$. The number of edges in the walk is called the length of the walk.

**D16:** The vector $w = [\mu_1, \mu_2, \ldots, \mu_k]$, where $\mu_j$ is the number of closed walks of length $j$ or $j^\text{th}$ spectral moment of the adjacency matrix in the graph and $k < \infty$, represents a molecular descriptor, such as a molecular property $A$ can be expressed as

$$A = \sum_j b_j \mu_j + \alpha,$$

where $b_j$ and $\alpha$ are empirical coefficients \cite{GuTr72, JiTaHo84, BoKi92, Es08b}.

**REMARK**

**R12:** Every $\mu_j$ can be expressed in terms of subgraphs, which allows us to express a molecular property as a combination of fragmental molecular contributions.

**DEFINITIONS**

**D17:** The weighted sum of all closed walks starting at a given node represents an atomic descriptor, subgraph centrality, for the corresponding atom in a molecule \cite{Es00, EsRo05},

$$EE_p = \sum_{k=0}^{\infty} \frac{(A^k)_{pp}}{k!} = (e^A)_{pp},$$

where $e^A$ is a matrix function that can be defined using the following Taylor series:

$$e^A = I + A + \frac{A^2}{2!} + \frac{A^3}{3!} + \ldots + \frac{A^k}{k!} + \ldots$$
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D18: The sum of subgraph centralities of all atoms in a molecule is a molecular descriptor called the Estrada index of the graph [Es00, EsRo05, DeGuRa07],

$$EE(G) = \sum_{p=1}^{n} EE_p.$$  

D19: The subgraph centrality and Estrada index have the following spectral representations [Es00, EsRo05, DeGuRa07]:

$$EE_p = \sum_{j=1}^{n} [\varphi_j(p)]^2 e^{\lambda j},$$  \hspace{1cm} (13.1.1)

$$EE(G) = \sum_{j=1}^{n} e^{\lambda j}. \hspace{1cm} (13.1.2)$$

REMARK

R13: The Estrada index of a molecular graph in which every edge is weighted by the parameter $\beta = (kT)^{-1}$, where $T$ is the temperature and $k$ is the Boltzmann constant, represents the electronic partition function of a molecule as defined by $Z_e = \sum_{j=1}^{n} e^{-\beta \epsilon_j}$ [EsHa07].

DEFINITIONS

D20: The probability that the system is found in a particular state can be obtained by considering a Maxwell–Boltzmann distribution [EsHa07]:

$$p_j = \frac{e^{\beta \lambda_j}}{\sum_j e^{\beta \lambda_j}} = \frac{e^{\beta \lambda_j}}{EE(G, \beta)}.$$

D21: The enthalpy $H(G)$ and Helmholtz free energy $F(G)$ of the graph are, respectively, [EsHa07]

$$H(G, \beta) = -\sum_{j=1}^{n} \lambda_j p_j,$$

$$F(G, \beta) = \beta^{-1} \ln EE.$$  \hspace{1cm} (13.1.4)

FACTS

F63: [EsHi10] The Estrada index can be obtained as $EE = tr(e^{\beta A})$, where $tr$ is the trace and

$$exp(A) = \sum_{k=0}^{\infty} \frac{A^k}{k!}.$$  

F64: [DeGuRa07] The Estrada index of a network $G$ of size $n$ is bounded as

$$n < EE(G) < e^{n-1} + \frac{n-1}{e},$$

where the lower bound is obtained for the graph having $n$ nodes and no links and the upper bound is attained for the complete graph $K_n$.  

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**F65:** Let $T_n$ be a tree with $n$ vertices, then [De09, DeRaGu09]

$$EE(s_n) > EE(T_n) > EE(P_n),$$

where $EE(S_n) = n - 2 + 2 \cosh(\sqrt{n - 1})$, and $EE(P_n) = \sum_{r=1}^{n} e^{2 \cos(2r\pi/(n+1))}$.

**F66:** [DeRaGu09] Let $G$ be a graph with $n$ vertices and $m$ edges. Then

$$\sqrt{n^2 + 4m} \leq EE(G_n) \leq n - 1 + e^{\sqrt{2m}}.$$

**F67:** [BeBo10] Let $G$ be a graph with $n$ vertices. Let $\delta_j$ be the degree of the $j^{th}$ vertex and let $a, b \in \mathbb{R}$ be such that the spectrum of $A$ is contained in $[a, b]$. Then,

$$\sum_{j=1}^{n} \frac{b^2 e^{\delta_j} + \delta_j e^{-b}}{b^2 + \delta_j} \leq EE \leq \sum_{j=1}^{n} \frac{a^2 e^{\delta_j} + \delta_j e^{-a}}{a^2 + \delta_j}.$$

**F68:** [BeBo10] Let $G$ be a graph with $n$ vertices. Let $\delta_j$ be the degree of the $j^{th}$ vertex and let $a = 1 - n$ and $b = n - 1$. Then

$$\frac{(n-1)^2 e^{\frac{n}{1-n}} + e^{1-n}}{n-1} \leq EE \leq \frac{n-1}{e} \cdot \frac{n - 1 + e^n}{n - 2}.$$

**F69:** [EjFiLuZo07] Let $G$ be a regular graph with $n$ nodes of degree $d = q + 1$. Then,

$$EE(G, \beta) = n \left[ \frac{q + 1}{2\pi} \int_{-2\sqrt{q}}^{2\sqrt{q}} e^{\beta s} \sqrt{4q - s^2} ds + \frac{1}{n} \sum_{\gamma} \sum_{k=1}^{\infty} \frac{i(\gamma)}{2l(\gamma)/2} I_{kl(\gamma)}(2\sqrt{q}\beta) \right],$$

where $\gamma$ runs over all (oriented) primitive geodesics in the network, $l(\gamma)$ is the length of $\gamma$, and $I_m(z)$ is the Bessel function of the first kind

$$I_m(z) = \sum_{r=0}^{\infty} \frac{(z/2)^{n+2r}}{r!(n+r)!}.$$

**F70:** [EsHa07] The electronic parameters are bounded as:

i) $0 \leq S(G, \beta) \leq \beta \ln n$,

ii) $- \beta(n-1) \leq H(G, \beta) \leq 0$,

iii) $- \beta(n-1) \leq F(G, \beta) \leq -\beta \ln n$.

The lower bounds are obtained for the complete graph as $n \to \infty$ and the upper bounds are reached for the null graph with $n$ nodes.
13.1.6 Vibrational Analysis of Graphs

DEFINITIONS

D22: A ball-spring graph is a graph in which every node is a ball of mass $m$ and every link is a spring with the spring constant $m\omega^2$ connecting two balls. The ball-spring graph is submerged into a thermal bath at the temperature $T$, such that the balls oscillate under thermal disturbances.

D23: The coordinates chosen to describe a configuration of the system are $x_i, i = 1, 2, \ldots, n$, each of which indicates the fluctuation of the ball $i$ from its equilibrium point $x_i = 0$.

D24: The ball-spring graphs are described by any of the following Hamiltonians:

$$H_A = \sum_i \left( \frac{p_i^2}{2m} + \frac{Km\omega^2}{2}x_i^2 \right) - \frac{m\omega^2}{2} \sum_{i,j} x_i A_{ij} x_j,$$

$$H_L = \sum_i \frac{p_i^2}{2m} + \frac{m\omega^2}{2} \sum_{i,j} x_i L_{ij} x_j,$$

where $p_i$ is the momentum of the node $i$, $K$ is a constant satisfying $K \geq \max_i k_i$, and $k_i$ is the degree of the node $i$ [EsHaBe12].

D25: A classical vibrational scenario is one in which the momenta $p_i$ and the coordinates $x_i$ are independent variables. A quantum vibrational scenario is one in which the momenta $p_j$ and the coordinates $x_i$ are not independent variables but they are operators that satisfy the commutation relation $[x_i, p_j] = i\hbar \delta_{ij}$, where $i = \sqrt{-1}$, $\hbar$ is the Dirac constant, and $\delta_{ij}$ is the Dirac delta function [EsHaBe12].

FACT

F71: The mean displacement of node $i$ in the classical vibrational scenario is given by any of the following expressions depending on the Hamiltonian selected [EsHaBe12]:

$$\Delta x_i = \sqrt{\langle x_i^2 \rangle} = \frac{1}{\beta m K \omega^2} \left[ (I - A/K)^{-1} \right]_{ii},$$

$$\Delta x_i = \sqrt{\langle x_i^2 \rangle} = \frac{1}{\beta m \omega^2} \left[ (L^\dagger) \right]_{ii},$$

where $L^\dagger$ is the Moore–Penrose generalized inverse of the Laplacian.

REMARK

R14: By obviating the physical constants the mean atomic displacements in the classical picture are given by the diagonal entries of the resolvent of the adjacency matrix or of the pseudoinverse of the Laplacian, respectively. The last expression was also investigated in [BaAtEr97, EsHa07].
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FACT

**F72**: The mean displacement of node $i$ in the quantum vibrational scenario is given by any of the following expressions depending on the Hamiltonian selected [EsHaBe12]:

\[ \Delta x_i = \sqrt{\langle x_i^2 \rangle} = e^{-\beta \Omega} \left( \exp \left( \frac{\beta \hbar \omega^2}{2 \Omega} A \right) \right)_{ii}, \]

\[ \Delta x_i = \sqrt{\langle x_i^2 \rangle} = \lim_{\Omega \to 0} \left( \exp \left( \frac{\beta \hbar \omega^2}{2 \Omega} L \right) \right)_{pq}, \]

\[ = 1 + \lim_{\Omega \to 0} O_{2p} O_{2q} \exp \left[ \frac{\beta \hbar \omega^2}{2 \Omega} \mu_2 \right], \]

where $\mu_2$ is the second eigenvalue of the Laplacian matrix and $\Omega = \sqrt{K/m\omega}$.

REMARK

**R15**: The displacement correlation between a pair of nodes $\langle x_i x_j \rangle$ is given by the $(i,j)$-entry of the corresponding matrix [EsHaBe12].

FACTS

**F73**: The resistance distance between a pair of nodes in a graph can be expressed in terms of the node displacements due to small vibrations/oscillations as follows [EsHa07]:

\[ \Omega_{ij} = [(\Delta x_i)^2 + (\Delta x_j)^2 - \langle x_i x_j \rangle - \langle x_j x_i \rangle] = \langle (x_i - x_j)^2 \rangle. \]

**F74**: The sum of resistance distances for a given node in a graph,

\[ R_i = \sum_{j=1}^{n} (L^\dagger_{ii} + L^\dagger_{jj} - 2 L^\dagger_{ij}), \]

is related to the node displacements as [EsHa07]

\[ R_i = n(\Delta x_i)^2 + \sum_{i=1}^{n} (\Delta x_i)^2 = n \left[ (\Delta x_i)^2 + \langle \Delta x \rangle^2 \right]. \]

**F75**: The potential energy of the vibrations in a graph is given by [EsHa07]

\[ \langle V(x) \rangle = \frac{1}{2n} \sum_{i=1}^{n} k_i R_i - \frac{1}{2n} \sum_{i,j \in E} (R_i + R_j - n\Omega_{ij}). \]
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