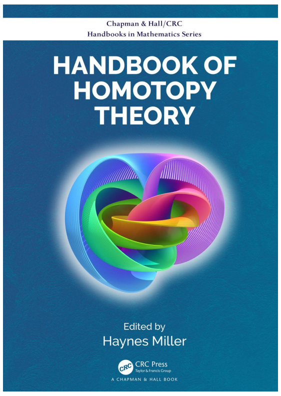


# Handbook of Homotopy Theory

Haynes Miller

# Algebraic models in the homotopy theory of classifying spaces

Publication details
https://www.routledgehandbooks.com/doi/10.1201/9781351251624-9
Natàlia Castellana
**Published online on: 18 Dec 2019**

# 9

# *Algebraic models in the homotopy theory of classifying spaces*

***Natàlia Castellana***

Classically, algebraic topology refers to the study of the homotopy type of topological spaces by assigning algebraic invariants in a functorial way, but in the last decades homotopical methods have influenced other areas of mathematics. In this context, some topological spaces are more suitable for those kinds of interactions since they already encode algebraic information. This is the case of classifying spaces for groups. The classification of geometrical structures reduces to the understanding the homotopy type of maps into classifying spaces.

Homological algebra is an example of a bidirectional interaction between algebra and homotopy theory, especially group cohomology since it can be computed from the category of modules over the group ring or just as singular cohomology of the classifying space of the group. Choosing different coefficients will reflect different data and properties from the group. The other way around, to what extent this data will determine or classify the space, after a suitable localization or completion functor?

The guiding example taken in this chapter is the Stable Elements Theorem in mod $p$ cohomology of a finite group which clearly expresses how $p$-subgroups and their conjugacy relations determine the mod $p$ cohomology of a group. This information is encoded in a category, the fusion category of the group. In the 1990s Puig [107] introduced the notion of a saturated fusion system on a finite $p$-group abstracting the properties of the fusion category of a finite group. It was possible to describe conjugacy patterns among $p$-subgroups without referring to the presence of an ambient group, providing an axiomatized and uniform framework for dealing with conjugacy problems.

This algebraic nature of classifying spaces produced spectacular results in comparing algebraic and homotopical constructions. For example, for discrete groups there is a bijective correspondence between homotopy classes of pointed maps and group homomorphisms, and conjugacy relations correspond to homotopy equivalences. Starting in the 1980s, a series of developments took the homotopy theory of classifying spaces to the next level: Miller's theorem proving the Sullivan's conjecture [93], Carlsson proof of the Segal conjecture [40] and Lannes's T-functor technology to describe mapping spaces between classifying spaces [84]. It was then that the term homotopical group theory started to be used to refer to the development of the homotopy theory of classifying spaces by describing homotopical analogues of the group theoretic constructions.

The use of Bousfield-Kan $p$-completion functor [22] to isolate properties at a prime led to the search of homotopical analogues for classifying spaces of connected Lie groups. Compact

*Mathematics Subject Classification.* 55R35, 20J05, 20R35, 55P42.

*Key words and phrases.* classifying space, finite group, Lie group, fusion, homotopy theory.

Lie groups are finite loop spaces, but only requiring a finite loop space structure didn't give a good homotopical extension of the notion of a Lie group. But after $p$-completion, most of the group structure could be well determined in terms of homotopy theoretical notions. In this program, Dwyer-Wilkerson defined the notion of a $p$-compact group in [62]. The development of homotopical group theory in this context by many authors led to the classification of $p$-compact groups by work of Andersen-Grodal-Moller-Viruel [4] and Andersen-Grodal [5] by algebraic root data obtained from the loop space and similar to the classification of compact Lie groups.

A new breakthrough happened in 2003 when Broto-Levi-Oliver [26] formalized the notion of a classifying space for a saturated fusion system in a categorical way motivated by their study of self-homotopy equivalences of classifying spaces of finite groups. By work of Chermak [43] and Oliver [103], there is a unique classifying space associated to a saturated fusion system, up to homotopy. That was the essence of the Martino-Priddy conjecture proved by Oliver [101, 100]: does the fusion system associated to a finite group uniquely determine the homotopy type of the $p$-completion of the classifying space? From this modern point of view, results in both unstable and stable homotopy theory can be reformulated and proved by local methods only.

The goal of Sections 9.1 and 9.2 is to serve as a guiding example for the rest of the chapter. Sections 9.3 and 9.4 introduce saturated fusions systems and their classifying spaces, describing relevant results in comparison to the examples in previous sections. The notion of saturated fusion system on a $p$-discrete toral group is described in Section 9.5. This theory models compact Lie groups and finite loop spaces in general, among other examples. Finally, in Section 9.6 we briefly present recent instances of the interaction between homotopy theory and modular representation theory in the context of fusion systems.

This exposition in not exhaustive, but the goal is to motivate and to give a taste of some of the latest developments in the theory. An exhaustive reference which covers group theory, homotopy theory and representation theory is the book by Aschbacher, Oliver and Kessar [10]. Another complementary reference is the book by Craven [49].

**Notation:** We denote by $\mathrm{Syl}_p(G)$ the set of Sylow $p$-subgroups of a finite group $G$. Given $g, x \in G$, $c_g(x) = gxg^{-1}$. Given $P, Q \leq G$, the transporter set is

$$N_G(P,Q) = \{g \in G \mid gPg^{-1} = c_g(P) \leq Q\}.$$

The subgroup $O^p(G) \leq G$ is the smallest normal subgroup of $G$ of index a power of $p$. The subgroup $O_p(G)$ is the largest normal $p$-subgroup of $G$. Analogously, the subgroup $O^{p'}(G) \leq G$ is the smallest normal subgroup of $G$ of index prime to $p$, and $O_{p'}(G)$ is the largest normal subgroup of $G$ of order prime to $p$. We will also use $\mathrm{Out}_{\mathcal{F}}(P) = \mathrm{Aut}_{\mathcal{F}}(P)/\mathrm{Inn}(P)$ and $\mathrm{Rep}(P,Q) = \mathrm{Inn}(Q)\backslash \mathrm{Hom}(P,Q)$. We will denote $H^*(-;\mathbb{F}_p)$ by $H^*(-)$.

## Acknowledgements

The author was partially supported by FEDER-MEC grant MTM2016-80439-P and acknowledges financial support from the Spanish Ministry of Economy and Competitiveness through the "María de Maeztu" Programme for Units of Excellence in R&D (MDM-2014-0445). The author would furthermore like to thank the Isaac Newton Institute for Mathematical Sciences for support and hospitality during the programme *Homotopy Harnessing Higher Structures*, where work on this paper was undertaken. This work was supported by EPSRC grant no EP/K032208/1.



## 9.1 Key words in finite group cohomology

Let $G$ be a finite group, $R$ a commutative ring with unit. If $M$ is an $RG$-module where $RG$ is the group ring, the cohomology of $G$ with coefficients in $M$ is

$$H^*(G; M) = \mathrm{Ext}^*_{RG}(R, M).$$

The versatility of group cohomology comes from the fact that as a functor it factors through the homotopy category of topological spaces via the classifying space of $G$, denoted by $BG$. Then, $H^*(G; M) \cong H^*(BG; M)$ where the right-hand side is just singular cohomology with twisted coefficients.

If $M = R$ with the trivial action, $H^*(G; R)$ is a finitely generated graded commutative ring. Golod [70] proved this fact when $G$ is a finite $p$-group and $R = \mathbb{Z}, \mathbb{Z}/p^a$, and Venkov [124] for a general $G$ when $R = \mathbb{Z}/p$. Independently, Evens [65] showed that if $R$ is Noetherian, $H^*(G; R)$ is a finitely generated algebra over $R$; even more, if $M$ is an $RG$-module that is Noetherian as an $R$-module, then $H^*(G; M)$ is a Noetherian module over $H^*(G; R)$. Another important fact from Evens work is that if $H \leq G$, then $H^*(H; R)$ is a finitely generated $H^*(G)$-module via the induced morphism by the inclusion $\mathrm{Res}^G_H \colon H^*(H; R) \to H^*(G; R)$.

Let me sketch two different ways of proving the finite generation of $H^*(G; R)$. We fix $R = \mathbb{F}_p$ from now on unless specified, since this is the situation we will be mostly interested in the rest of the paper.

The first strategy we sketch (used by Venkov) considers a *faithful complex representation* $\rho\colon G \to U(n)$. For example, one can take the regular representation. A morphism induces a map between classifying spaces $B\rho\colon BG \to BU(n)$. The homotopy fiber of $B\rho$ is $U(n)/G$ which has the homotopy type of a finite $CW$-complex. The cohomology of $BU(n)$ is known to be polynomial on the Chern classes $c_1, \ldots, c_n$ in degree 2, therefore finitely generated. The Serre spectral sequence for this fibration is a bounded spectral sequence of $H^*(BU(n))$-modules whose $E_2$-term $H^*(BU(n)) \otimes H^*(U(n)/G)$ is a finitely generated $H^*(BU(n))$-module. It follows then that the same is true for $E_\infty$, and this fact will imply that $H^*(BG)$ is a finitely generated ring.

The second strategy (used by Evens) uses a *transfer homomorphism* argument which allows us to reduce the proof to $p$-groups. Let $H \leq G$, $H^*(H)$ is a $H^*(G)$-module via the homomorphism induced in cohomology $\mathrm{Res}^G_H \colon H^*(G) \to H^*(H)$. There is a transfer homomorphism in group cohomology (defined at the level of cochains)

$$\mathrm{Tr}^H_G \colon H^*(H) \longrightarrow H^*(G),$$

which has the property that is a morphism of $H^*(G)$-modules (by Frobenius reciprocity formula).

If $S \in \mathrm{Syl}_p(G)$ is a Sylow $p$-subgroup, the properties of the transfer homomorphism show that $\mathrm{Res}^G_S \colon H^*(G) \to H^*(S)$ admits then an $H^*(G)$-module retract $\mathrm{Tr}^S_G$ since $\mathrm{Tr}^S_G \circ \mathrm{Res}^G_H$ is multiplication by $[G : S]$ which is of order prime to $p$. Then, applying [62, Lemma 2.4] (see also [12, Lemma 2.4]), $H^*(G)$ is finitely generated if $H^*(S)$ is so.

The proof that $H^*(G)$ is a finitely generated ring when $G$ is a $p$-group uses induction on the order of the group. The cohomology of the cyclic group of order $p$ can be computed directly from the definition. To work out the induction step one uses that the center of a $p$-group is always non-trivial, and then applies a spectral sequence argument for a central extension of groups.

The properties of the transfer show that $\mathrm{Res}^G_S \colon H^*(G) \to H^*(S)$ is injective. Can the image of $\mathrm{Res}^G_S$ be identified? If $c_g$ is the morphism induced by conjugation by $g \in G$, then $c_g^* \colon H^*(G) \to H^*(G)$ is the identity, but not when restricted to subgroups of $G$.

**Definition 9.1.1.** Let $R$ be a commutative ring and $M$ and $RG$-module. An element $x \in H^*(S, M)$ is *stable* if, for every $g \in G$, we have

$$\mathrm{Res}^S_{S\cap g^{-1}Sg}(x) = (c_g^*) \circ \mathrm{Res}^S_{gSg^{-1}\cap S}(x),$$

where $c_g^*$ is induced by the pair $(c_g, g^{-1}) \colon (G, M) \to (G, M)$.

The elements in the image of $\mathrm{Res}^G_S$ must satisfy the set of equations described in Definition 9.1.1. We can encode all these relations in the following category, called the transporter category.

**Definition 9.1.2.** The *transporter category* $\mathcal{T}_S(G)$ is the category whose objects are subgroups $P \le S$ and a morphism from $P$ to $Q$ is given by those elements $g \in G$ such $gPg^{-1} = c_g(P) \le Q$, that is, $\mathrm{Mor}_{\mathcal{T}_S(G)}(P, Q) = N_G(P, Q)$. Composition is group multiplication.

Then, the submodule of stable elements in $H^*(S)$ is precisely the inverse limit $\lim_{P\in\mathcal{T}_S(G)} H^*(P; M)$. And, $\mathrm{Im}(\mathrm{Res}^G_S) \subset \lim_{P\in\mathcal{T}_S(G)} H^*(P; M)$.

A classical result in group cohomology is the *stable elements theorem* which goes back to the work of Cartan-Eilenberg [41]. It uses mainly the properties of the transfer, especially the double coset formula which describes $\mathrm{Res}^G_S \circ \mathrm{Tr}^S_G$.

**Theorem 9.1.3.** *Let $G$ be a finite group, $S \in \mathrm{Syl}_p(G)$ and $M$ a $\mathbb{Z}_{(p)}G$- module. The morphism $\mathrm{Res}^G_S \colon H^*(G; M) \to H^*(S; M)$ is injective and*

$$\mathrm{Im}(\mathrm{Res}^G_S) = \lim_{P\in\mathcal{T}_S(G)} H^*(P; M).$$

Theorem 9.1.3 describes $H^*(G)$ as the set of solutions of a system of equations involving the cohomology of finite $p$-groups, and conjugacy relations among subgroups.

We go back to the situation $M = R = \mathbb{F}_p$. In that case, all the information needed in Theorem 9.1.3 to compute $H^*(G)$, or the submodule of the stable elements, is described by the group homomorphisms $c_g$ for any $g \in G$ (it is not necessary then to keep track of the element $g \in G$ that induces $c_g$).

**Definition 9.1.4.** Let $S \in \mathrm{Syl}_p(G)$. The fusion category $\mathcal{F}_S(G)$ or *fusion system* of $G$ on $S$ is the subcategory of the category of groups whose objects are $p$-subgroups $P \leq S$ and a morphism from $P$ to $Q$ is given by a group homomorphism induced by an element $g \in G$ such $gPg^{-1} = c_g(P) \leq Q$, that is

$$\mathrm{Mor}_{\mathcal{F}_S(G)}(P, Q) = \mathrm{Hom}_G(P, Q) = N_G(P, Q)/C_G(P)$$

where $C_G(P)$ is the centralizer of $P$ in $G$.

In that case,

$$H^*(G) \cong \lim_{P \in \mathcal{F}_S(G)} H^*(P).$$

In a course in group cohomology, the first computation obtained directly from the definition is that of a cyclic group of order $p$. Applying the Kunneth formula we know explicitly the cohomology of any *elementary abelian $p$-group* $V = (\mathbb{Z}/p)^n$. If $p = 2$, then it is polynomial $\mathbb{F}_2[x_1, \ldots, x_n]$ and if $p$ is odd then it is a tensor product of an exterior algebra and a polynomial algebra, $E(x_1, \ldots, x_n) \otimes \mathbb{F}_p[y_1, \ldots, y_n]$, where $|x_i| = 1$ and $|y_i| = 2$.

How much information about $H^*(G)$ is captured by the collection of images of restrictions to elementary abelian subgroups?

**Definition 9.1.5.** Let $\mathcal{F}_S^e(G)$ be the full subcategory of $\mathcal{F}_S(G)$ whose objects are elementary abelian $p$-subgroups of $S$.

Quillen and Venkov [111, 110] studied the product of restriction homomorphisms

$$\prod \mathrm{Res}_V^G \colon H^*(G) \longrightarrow \lim_{V \in \mathcal{F}_S^e(G)} H^*(V) \subset \prod H^*(V).$$

**Definition 9.1.6.** Let $f \colon R \to S$ a morphism of (graded) commutative $\mathbb{F}_p$-algebras. We say that $f$ is an *$F$-monomorphism* if every element in $\mathrm{Ker}(f)$ is nilpotent, *$F$-epimorphism* if for every $s \in S$, there exists a natural number $n$ such that $s^{p^n} \in \mathrm{Im}(f)$. We say that $f$ is an *$F$-isomorphism* if it is both an $F$-monomorphism and an $F$-epimorphism.

**Theorem 9.1.7.** *The morphism $\prod \mathrm{Res}_V^G$ has nilpotent kernel, and*

$$H^*(G) \longrightarrow \lim_{V \in \mathcal{F}_S^e(G)} H^*(V)$$

*is an $F$-isomorphism.*

Identifying $H^*(G)$ up to nilpotent elements is good enough to obtain certain type of information. For example, Quillen [110] proved that the Krull dimension of $H^*(G)$ is the maximal rank of an elementary abelian $p$-subgroup (using the fact that a prime ideal must contain all nilpotent elements).

Theorem 9.1.7 and some commutative algebra are the key ingredients in Quillen's strong *stratification* theorem in [110]. Denote by $\mathrm{Spec}_G^h$ the homogeneous prime ideal spectrum of $H^*(G)$, and $\mathrm{res}_G^H \colon \mathrm{Spec}_H^h \to \mathrm{Spec}_G^h$ induced by $H \leq G$.

For an elementary abelian $p$-subgroup $E \leq G$, let $\mathrm{Spec}_E^+ = \mathrm{Spec}_E^h \setminus \bigcup_{E' < E} \mathrm{res}_E^{E'}(\mathrm{Spec}_{E'}^h)$. Finally, we write $\mathrm{Spec}_{G,E}^+ = \mathrm{res}_G^E \mathrm{Spec}(E)^+$.

**Theorem 9.1.8.** *Let $G$ be a finite $p$-group. The spectrum of homogeneous prime ideals $\mathrm{Spec}^h_G$ admits a decomposition as a disjoint union*

$$\mathrm{Spec}^h_G \cong \amalg_{E \in \mathcal{E}(G)} \mathrm{Spec}^+_{G,E}$$

*where $\mathcal{E}(G)$ is a set if representatives of $G$-conjugacy classes of elementary abelian $p$-subgroups of $S$.*

From this point, we go back to the challenging task of computing group cohomology using the description given by the Stable Elements Theorem 9.1.3. One way is by simplifying the set of equations by, as we do in linear algebra, eliminating redundancies: understanding how $p$-subgroups are conjugated among each other. This is the concept of *fusion* in group theory which has been studied since the beginning of last century. Let us state the following theorem due to Burnside as an example (see [32, Page 155]).

**Theorem 9.1.9** (Burnside Fusion Theorem)**.** *Let $G$ be a finite group with an abelian $p$-Sylow subgroup $S$. If $x, y \in S$ are conjugate by an element in $G$, then they are conjugate by an element in $N_G(S)$.*

**Remark 9.1.10.** Theorem 9.1.9 also holds for subgroups. That is, if $P, Q \leq S$ are conjugate in $G$ then they are also conjugate in $N_G(S)$.

We can translate this statement in terms of the corresponding fusion systems. Note that if $H \leq G$ with $T \in \mathrm{Syl}_p(H)$ and $S \in \mathrm{Syl}_p(G)$ with $T \leq S$ then $\mathcal{F}_T(H) \subset \mathcal{F}_S(G)$ as a subcategory. Then Theorem 9.1.9 says that if $S$ is abelian then $\mathcal{F}_S(N_G(S)) = \mathcal{F}_S(G)$.

**Definition 9.1.11.** Let $G$ be a finite group and $S \in \mathrm{Syl}_p(G)$. A subgroup $S \leq K \leq G$ is said to *control fusion* in $G$ (or control $G$-fusion) if, whenever two subgroups $P, Q \leq G$ are conjugate in $G$, then they are conjugate in $K$.

**Corollary 9.1.12.** *Let $S \in \mathrm{Syl}_p(G)$. If $K \leq G$ controls $G$-fusion then $\mathcal{F}_S(K) = \mathcal{F}_S(G)$ and $\mathrm{Res}^G_K \colon H^*(G) \to H^*(K)$ is an isomorphism.*

A converse holds due to the following result of Mislin [94].

**Theorem 9.1.13.** *Let $f \colon H \to G$ be a group morphism of finite groups. Fix $S \in \mathrm{Syl}_p(G)$ and $T \in \mathrm{Syl}_p(H)$ with $f(T) \leq S$. Then $f^* \colon H^*(G) \to H^*(H)$ is an isomorphism iff the induced functor $\mathcal{F}_T(H) \to \mathcal{F}_S(G)$ is an equivalence of categories.*

An example is given by Burnside Theorem, Theorem 9.1.9: $N_G(S)$ controls fusion in $G$ if $S \in \mathrm{Syl}_p(G)$ is abelian. It is worth noticing that Burnside fusion theorem can also be interpreted in the following way: any conjugation among $p$-subgroups of $S$ is the restriction of an automorphism of $S$.

Normalizers of non-trivial $p$-subgroups are called *$p$-local subgroups* of $G$. In general one cannot find a single $p$-local subgroup controlling $G$-fusion, but a weaker statement can still hold by considering a set of $p$-local subgroups.

A *collection* is a set of subgroups of a group closed under conjugation. Let $\mathcal{H}$ be a collection of subgroups of $S$, we denote by $\mathcal{F}^{\mathcal{H}}_S(G)$ the full subcategory of $\mathcal{F}_S(G)$ with object set $\mathrm{Ob}(\mathcal{F}^{\mathcal{H}}_S(G)) = \mathcal{H}$.

**Definition 9.1.14.** Let $G$ be a finite group and $S \in \mathrm{Syl}_p(G)$. Let $\mathcal{H}$ be a collection of subgroups of $S$. We say that $\mathcal{F}_S(G)$ is *$\mathcal{H}$-generated* if every morphism in $\mathcal{F}_S(G)$ is a composite of restrictions of automorphisms in $\mathcal{F}_S(G)$ of subgroups in $\mathcal{H}$. That is, for each isomorphism $P \to P'$ in $\mathcal{F}_S(G)$, there exist sequences of subgroups of $S$,

$$P = P_0, P_1, \dots, P_k = P' \qquad \text{and} \qquad Q_1, Q_2, \dots, Q_k$$

and morphisms $\varphi_i \in \mathrm{Aut}_{\mathcal{F}_S(G)}(Q_i)$ such that the following hold:

1. $Q_i$ is in $\mathcal{H}$ for each $i$.
2. $P_{i-1}, P_i \leq Q_i$ and $\varphi_i(P_{i-1}) = P_i$ for each $i$.
3. $\varphi = \varphi_k \circ \varphi_{k-1} \circ \cdots \circ \varphi_1$.

**Remark 9.1.15.** In general, given a collection $\mathcal{H}$ there is a monomorphism

$$\lim_{P \in \mathcal{F}_S(G)} H^*(P) \hookrightarrow \lim_{P \in \mathcal{F}_S^{\mathcal{H}}(G)} H^*(P),$$

which is an isomorphism if $\mathcal{F}_S(G)$ is $\mathcal{H}$-generated.

Alperin's fusion theorem [2] is about such collections $\mathcal{H}$ of subgroups for which $\mathcal{F}_S(G)$ is $\mathcal{H}$-generated. Alperin considers certain intersections of Sylow $p$-sugroups (tame intersections), and Goldschmidt [69] and Puig [108] consider the collection of essential proper subgroups. We introduce one of the collections which will be used in the next section.

**Definition 9.1.16.** Let $G$ be a finite group. A subgroup $P \leq S$, where $S \in \mathrm{Syl}_P(G)$, is *$p$-centric* if $C_G(P) \cong Z(P) \times O^p(C_G(P))$ with $O^p(C_G(P))$ of order prime to $p$ (i.e. $Z(P) \in \mathrm{Syl}_p(C_G(P))$).

**Theorem 9.1.17** (Alperin's fusion theorem)**.** *Let $G$ be a finite group and $S \in \mathrm{Syl}_p(G)$. If $\mathcal{H}$ is the collection of non-trivial $p$-centric subgroups of $G$, then $\mathcal{F}_S(G)$ is $\mathcal{H}$-generated.*

We finish with an example which combines the key words appearing in this section. We say that a finite group $G$ is $p$-nilpotent if there is $H \trianglelefteq G$ of order prime to $p$ and $p$-power index (i.e. $G \cong S \ltimes H$ where $S \in \mathrm{Syl}_p(G)$). The following two theorems characterize $p$-nilpotent groups in terms of fusion and mod $p$ cohomology.

**Theorem 9.1.18** (Frobenius. See 39.4 in [7])**.** *Let $G$ be a finite group and $S \in \mathrm{Syl}_p(G)$. $G$ is $p$-nilpotent iff $S$ controls fusion in $G$.*

**Theorem 9.1.19.** (Quillen, [109]) *Let $G$ be a finite group and $S \in \mathrm{Syl}_p(G)$. If $\mathrm{Res}^G_S \colon H^*(G) \to H^*(S)$ is an $F$-isomorphism, then $S$ controls fusion in $G$ if $p$ is odd.*

Recent work of Benson-Grodal-Henke [16] have generalized Quillen's result 9.1.19 on the control of fusion by elementary abelian $p$-subgroups to subgroups of index prime to $p$.

**Theorem 9.1.20.** *Let $G$ be a finite group and $H \leq G$ be an inclusion of finite groups of index prime to $p$, $p$ and odd prime. If $\mathrm{Res}^G_H \colon H^*(G) \to H^*(H)$ is an $F$-isomorphism then $H$ controls fusion in $G$.*

These results express to what extent fusion on elementary abelian subgroups controls the fusion of the group.

## 9.2 Decomposing classifying spaces

Let $G$ be a finite group and $S \in \mathrm{Syl}_p(G)$. Group cohomology with coefficients in a commutative ring $R$ can also be defined as (singular) cohomology of the classifying space for G, $H^*(BG; R)$. As in Section 9.1 we fix $R = \mathbb{F}_p$. The main idea we should keep is that mod $p$ cohomology retains and isolates $p$-local information of the group $G$ which is precisely encoded in the fusion system $\mathcal{F}_S(G)$ and the collection of $p$-subgroups. More precisely, if $\mathcal{H}$ is a collection of $p$-subgroups that generates $\mathcal{F}_S(G)$, then we only need $\mathcal{F}_S^{\mathcal{H}}(G)$ and $\{P\}_{P\in\mathcal{H}}$ for computing $H^*(G)$.

In homotopy theory, there is a functor that isolates the information of a topological space that is reflected in mod $p$ cohomology: *Bousfield-Kan p-completion* functor (see [22] and also [10, Section III.1.4]).

It is a functor $(-)^\wedge_p\colon$ Top $\to$ Top with a natural transformation from the identity functor $\phi_X\colon X \to X^\wedge_p$ with the following property: if $f\colon X \to Y$ induces an isomorphism in mod $p$ cohomology iff $f^\wedge_p\colon X^\wedge_p \to Y^\wedge_p$ is a weak homotopy equivalence ([22, I.5.5]).

The natural transformation $\phi_X$ does not need to induce an isomorphism in mod $p$ cohomology. If it does, $\phi_X^*$ is a isomorphism, we say that $X$ is $p$-good. By [22], if $\pi_1(X)$ is finite then $X$ is $p$-good (see [22, VII.5.1]). In particular our motivating example $BG$ is $p$-good, therefore $H^*(BG) \cong H^*(BG^\wedge_p)$. We say that $X$ is $p$-complete if $X^\wedge_p \simeq X$. If $G$ is a $p$-group then $BG$ is already $p$-complete ([10, Proposition 1.10]). In general, $BG^\wedge_p$ is not an Eilenberg-MacLane space anymore, but the fundamental group can still be computed in terms of $G$, $\pi_1(BG^\wedge_p) \cong G/O^p(G)$ (see [22, VII.5.1] or [10, Proposition 1.11]).

A model for $BG$ is the nerve of the category $\mathcal{B}G$: it is a category with a single object $\bullet$ and $\mathrm{Hom}_{\mathcal{B}G}(\bullet, \bullet) = G$ where composition is given by group multiplication. We can also consider the nerve of the transporter category $\mathcal{T}_S(G)$ as suggested by group cohomology. Note that $\mathrm{Aut}_{\mathcal{T}_S(G)}(1) = G$. By [10, Page 134], the inclusion $\mathcal{B}G \subset \mathcal{T}_S(G)$ induces a homotopy equivalence on nerves $|\mathcal{B}G| \simeq |\mathcal{T}_S(G)|$, providing then a model for $BG$.

**Definition 9.2.1.** Let $G$ be a finite group and $S \in \mathrm{Syl}_p(G)$. The *orbit category* $\mathcal{O}_S(G)$ is the category whose objects are $p$-subgroups $P \le S$, and given $R, T \le S$, $\mathrm{Mor}_{\mathcal{O}_S(G)}(R, T) = T\backslash N_G(R, T)$.

Analogously, if $\mathcal{H}$ is a collection of subgroups of $S$, we denote by $\mathcal{O}_S^{\mathcal{H}}(G) \subset \mathcal{O}_S(G)$ the full subcategory generated by the object set $\mathcal{H}$. Let $\mathcal{T}_S^{\mathcal{H}}(G) \subset \mathcal{T}_S(G)$ be the full subcategory of the transporter category with $\mathrm{Ob}(\mathcal{T}_S^{\mathcal{H}}(G)) = \mathcal{H}$.

Following the strategy in Section 9.1, we consider a collection $\mathcal{H}$ of subgroups of $S$. In [57] Dwyer studied systematically the homotopy type of $|\mathcal{T}_S^{\mathcal{H}}(G)|$ and $|\mathcal{T}_S^{\mathcal{H}}(G)|^\wedge_p$ for several choices of $\mathcal{H}$ in his work on *homology decompositions* of classifying spaces of finite groups. He described $|\mathcal{T}_S^{\mathcal{H}}(G)|$ in terms of the quotient categories $\mathcal{O}_S^{\mathcal{H}}(G)$ and $\mathcal{F}_S^{\mathcal{H}}(G)$.

There are functors

$$p_{\mathcal{O}}\colon \mathcal{T}_S^{\mathcal{H}}(G) \to \mathcal{O}_S^{\mathcal{H}}(G)$$

$$p_{\mathcal{F}}\colon \mathcal{T}_S^{\mathcal{H}}(G) \to \mathcal{F}_S^{\mathcal{H}}(G)$$

that are the identity on objects and quotient by the action of $P \leq N_G(P)$ (resp. $C_G(P) \leq N_G(P)$) on morphisms.

**Theorem 9.2.2.** [57] *Let $G$ be a finite group, $S \in \mathrm{Syl}_p(G)$ and $\mathcal{H}$ a collection of $p$-subgroups of $S$. There are functors*

$$\alpha \colon (\mathcal{F}_S^{\mathcal{H}}(G))^{op} \to \mathrm{Top}$$

*and*

$$\beta \colon \mathcal{O}_S^{\mathcal{H}}(G) \to \mathrm{Top}$$

*such that*

$$\operatorname*{hocolim}_{\mathcal{F}_S^{\mathcal{H}}(G)^{op}} \alpha \simeq |\mathcal{T}_S^{\mathcal{H}}(G)| \simeq \operatorname*{hocolim}_{\mathcal{O}_S^{\mathcal{H}}(G)} \beta$$

*and for each $P \in \mathcal{H}$, $\alpha(P) \simeq BC_G(P)$ and $\beta(P) \simeq BP$.*

**Remark 9.2.3.** The decomposition provided by $\alpha$ (resp. $\beta$) is called a centralizer decomposition (resp. subgroup decomposition). In [57], Dwyer also considers a third type of decomposition for $|\mathcal{T}_S^{\mathcal{H}}(G)|$, the normalizer decomposition, with the property that the indexing category is a poset.

**Remark 9.2.4.** The functor $\beta$ is easily described from the orbit category, $\beta(P) = EG \times_G G/P$.

For which collections $\mathcal{H}$ do we have an equivalence $|\mathcal{T}_S^{\mathcal{H}}(G)|_p^\wedge \simeq BG_p^\wedge$? Recall that there is a functor

$$\iota \colon \mathcal{T}_S^{\mathcal{H}}(G) \to \mathcal{B}G,$$

which sends every morphism set $N_G(P,Q)$ into $G$. A collection $\mathcal{H}$ is *ample* if $\iota_p^\wedge \colon |\mathcal{T}_S^{\mathcal{H}}(G)|_p^\wedge \simeq BG_p^\wedge$. For example, the collection of non-trivial $p$-subgroups (see [79]) or the collection of elementary abelian $p$-subgroups (see [78], [56]). We introduced the notion of $p$-centric subgroup in Definition 9.1.16. The collection of $p$-centric subgroups is also ample ([57]). More information about ample collections can be found in the work of Grodal [75].

Restricting to the collection of $p$-centric subgroups allows us to consider an intermediate quotient when constructing the fusion category from the transporter category by first letting $O^p(C_G(P))$ act.

**Definition 9.2.5.** Let $G$ be a finite group and $S \in \mathrm{Syl}_p(G)$. The *centric linking system* $\mathcal{L}_S^c(G)$ is the category whose objects are $p$-centric subgroups $P \leq S$, and

$$\mathrm{Hom}_{\mathcal{L}_S^c(G)}(R,T) = N_G(R,T)/O^p(C_G(R)).$$

This category was introduced by Broto-Levi-Oliver in [25] where they prove the following equivalence.

**Theorem 9.2.6.** *Let $G$ be a finite group and $S \in \mathrm{Syl}_p(G)$. There is an equivalence $|\mathcal{L}_S^c(G)|_p^\wedge \simeq BG_p^\wedge$.*

**Remark 9.2.7.** The key point is that the functor $\mathcal{T}_S^c(G) \to \mathcal{L}_S^c(G)$ induces a mod $p$ equivalence on nerves, since $O^p(C_G(P))$ is of order prime to $p$. This inspires the following definition: $P \leq S$ is $p$-quasicentric if $O^p(C_G(P))$ is of order prime to $p$. Let $\mathcal{L}_S^q(G)$ be the category whose objects are $p$-quasicentric subgroups $P \leq S$, and

$$\mathrm{Hom}_{\mathcal{L}_S^q(G)}(R,T) = N_G(R,T)/O^p(C_G(R)).$$

The inclusion of categories $\mathcal{L}^c \subset \mathcal{L}^q$ also induces a homotopy equivalence on nerves ([23, Theorem 3.5]).

The next theorem shows that the centric linking category $\mathcal{L}_S(G)^c$ is an algebraic model that completely determines the homotopy type of $BG_p^\wedge$.

**Theorem 9.2.8.** [25] *For any prime $p$ and any pair of finite groups $G_1$ and $G_2$, the $p$-completed classifying spaces $(BG_1)_p^\wedge$ and $(BG_2)_p^\wedge$ are homotopy equivalent if and only if the centric linking systems are equivalent.*

One direction is clear since an equivalence of categories induces a homotopy equivalences on nerves. The other direction depends on the description of homotopy classes of maps between classifying spaces in terms of groups morphisms which allows the authors to reconstruct the centric linking system from the homotopy type of $BG_p^\wedge$.

**Theorem 9.2.9.** *Let $G$ be a finite group, and $P$ a finite $p$-group. The classifying space functor induces a bijection*

$$\mathrm{Rep}(P,G) \cong [BP, BG_p^\wedge].$$

*Moreover, for each $\rho \in \mathrm{Rep}(P,G)$, there is a homotopy equivalence*

$$BC_G(\rho(P)) \xrightarrow{\simeq} \mathrm{Map}(BP, BG_p^\wedge)_{B\rho}.$$

The first part is a theorem of Mislin [94] and the second part is by Dwyer-Zabrodsky [64].

In order to prove Theorem 9.2.8, Broto-Levi-Oliver (see [10, Section 3.2]) define the fusion system and centric linking system of a topological space $X$ with respect to a map $f\colon BS \to X$ where $S$ is a finite $p$-group. We illustrate by describing the simpler fusion category with respect to $f$, $\mathcal{F}_f(X)$. Objects is given by the set of groups $P \leq S$ and $\mathrm{Hom}_{\mathcal{F}_f(X)}(P,Q)$ are group monomorphisms $\varphi\colon P \to Q$ such that $f|_{BQ} \circ B\varphi \simeq f|_{BP}$. If $f\colon BS \to BG$ is induced by the inclusion of $S \in \mathrm{Syl}_p(G)$ then they show that $\mathcal{F}_f(BG) \simeq \mathcal{F}_{f_p^\wedge}(BG_p^\wedge) \simeq \mathcal{F}_S(G)$.

But, is the fusion system enough to determine the homotopy type of $BG_p^\wedge$? This is the statement of the Martino-Priddy conjecture proved by Oliver.

**Definition 9.2.10.** For $i = 1, 2$, let $S_i \in \mathrm{Syl}_p(G_i)$ for finite groups $G_i$. A *fusion preserving isomorphism* is an isomorphism $\varphi\colon S_1 \to S_2$ that induces an isomorphism of categories from $\mathcal{F}_{S_1}(G_1)$ to $\mathcal{F}_{S_2}(G_2)$, by sending $P \leq S_1$ to $\varphi(P) \leq S_2$.

**Theorem 9.2.11** (Martino-Priddy conjecture, [100], [101])**.** *For any prime $p$, and any pair of finite groups $G_1$ and $G_2$, $(BG_1)_p^\wedge \simeq (BG_2)_p^\wedge$ iff there is a fusion preserving isomorphim $\varphi\colon S_1 \to S_2$.*

There are two strong and beautiful applications of the explicit description of the homotopy type of $BG^\wedge_p$ from the centric linking system.

First, Broto-Levi-Oliver [25] describe the topological monoid of self-homotopy equivalences $\mathrm{Aut}(BG^\wedge_p)$ in terms of a subgroupoid of the groupoid of self-equivalences of the category $\mathcal{L}^c_S(G)$ and natural isomorphisms of functors, those called isotypical and which respect certain strucuture. A similar statement holds for $\mathrm{Out}(BG^\wedge_p)$, the group of homotopy classes of self-homotopy equivalences of $BG^\wedge_p$. In [25], the authors prove that $\mathrm{Out}(BG^\wedge_p)$ is isomorphic to the group of natural isomorphism classes of isotypical selfequivalences of $\mathcal{L}^c_S(G)$. Understanding the relation of these groups with the actual $\mathrm{Out}(G)$ is crucial in the study of fibrations and extensions of the fusion system and whether they can be realized by a group.

Second, Broto-Moller-Oliver [30] produce a beautiful application of techniques in homotopy theory to finite group theory. They prove equivalences between fusion systems of finite groups of Lie type by showing that their $p$-completed classifying spaces are equivalent. In this way they obtain equivalences that where unknown to group theorists before.

## 9.3 (Saturated) Fusion systems

The theory of fusion systems is a new way to solve questions in finite group theory and homotopy theory involving conjugacy relations.

In the 1990s, Lluís Puig [107] defined the notion of a saturated fusion system (Frobenius category) on a $p$-group $S$ inspired by the properties of the fusion system of a finite group, but motivated by work in modular representation theory where similar categories could be constructed from the defect group of a block. His definition focuses on the properties of the morphisms and hides the role of the group $G$, abstracting the concept of conjugacy relations among subgroups. Next, we give the axiomatic definition introduced by Broto-Levi-Oliver in [26] which is equivalent to the original one due to Puig. We start with the basic notion of a fusion system.

**Definition 9.3.1.** A *fusion system* $\mathcal{F}$ on a finite $p$–group $S$ is a subcategory of the category of groups whose objects are the subgroups $P \leq S$ and such that the set of morphisms $\mathrm{Hom}_{\mathcal{F}}(P,Q)$ between two subgroups $P$ and $Q$ satisfies the following conditions:

(a) $\mathrm{Hom}_S(P,Q) \subseteq \mathrm{Hom}_{\mathcal{F}}(P,Q) \subseteq \mathrm{Inj}(P,Q)$ for all $P, Q \leq S$.

(b) Every morphism in $\mathcal{F}$ factors as an isomorphism in $\mathcal{F}$ followed by an inclusion.

This concept is too general for the purpose of modeling the structure of $p$-subgroups of a given finite group $G$ and the following axioms are the essence of the definition of a saturated fusion system.

**Definition 9.3.2.** Let $\mathcal{F}$ be a fusion system on a $p$–group $S$.

- We say that two subgroups $P, Q \leq S$ are $\mathcal{F}$*–conjugate* if they are isomorphic in $\mathcal{F}$. We denote by $\{P\}^{\mathcal{F}}$ the set of subgroups $P' \leq S$ that are $\mathcal{F}$-conjugate to $P \leq S$.

- A subgroup $P \leq S$ is *fully centralized* in $\mathcal{F}$ if $|C_S(P)| \geq |C_S(P')|$ for all $P' \in \{P\}^{\mathcal{F}}$.
- A subgroup $P \leq S$ is *fully normalized* in $\mathcal{F}$ if $|N_S(P)| \geq |N_S(P')|$ for all $P' \in \{P\}^{\mathcal{F}}$.
- $\mathcal{F}$ is a *saturated fusion system* if the following conditions hold:
  (I) (Sylow axiom) Each fully normalized subgroup $P \leq S$ is fully centralized and the group $\mathrm{Aut}_S(P)$ is a $p$–Sylow subgroup of $\mathrm{Aut}_{\mathcal{F}}(P)$.
  (II) (Extension axiom) If $P \leq S$ and $\varphi \in \mathrm{Hom}_{\mathcal{F}}(P, S)$ are such that $\varphi P$ is fully centralized, and if we set
  $$N_\varphi = \{g \in N_S(P) \mid \varphi c_g \varphi^{-1} \in \mathrm{Aut}_S(\varphi P)\},$$
  then there is $\overline{\varphi} \in \mathrm{Hom}_{\mathcal{F}}(N_\varphi, S)$ such that $\overline{\varphi}_{|P} = \varphi$.

The first axiom is called the Sylow axiom since it is intended to model the fact that $S$ must play the role of a Sylow $p$-subgroup in this structure. The second axiom is defined to model morphisms to behave like morphisms induced by conjugation.

**Remark 9.3.3.** The axioms of a saturated fusion system have other different equivalent formulations due to Roberts and Shpectorov [119] and to Stancu (see [90, Page 387], [83]).

The motivating example is the fusion system of a finite group $G$ with a fixed Sylow $p$–subgroup $S \in \mathrm{Syl}_p(G)$ in Definition 9.1.4: $\mathcal{F}_S(G)$ is a saturated fusion system ([26], [10, Theorem 2.3]).

**Definition 9.3.4.** A saturated fusion system $\mathcal{F}$ on a finite $p$-group $S$ is *realizable* if there exists a finite group $G$ with $S \in \mathrm{Syl}_p(G)$ such that $\mathcal{F} \cong \mathcal{F}_S(G)$. If $\mathcal{F}$ is not realizable, we say that it is exotic.

Note that a realizable saturated fusion system can be realized by more than a finite group $G$. For example, if $N \trianglelefteq G$ with $N$ of order prime to $p$, then $G$ and $G/N$ have isomorphic fusion systems.

In the original paper, Broto-Levi-Oliver [26] already described exotic examples at odd primes. The following are some examples of exotic fusion systems in the literature.

**Example 9.3.5.** 1. The only known examples of "simple" exotic fusions system at $p = 2$ fit in the family $\mathrm{Sol}(q)$, due to Solomon [123] but formalized by Levi-Oliver [87] (and [9]). The Sylow 2-subgroup is $S \in \mathrm{Syl}_2(\mathrm{Spin}_7(q))$ where $q$ is an odd power of prime $p$. If $q \equiv \pm 3 \bmod 8$ then $\mathrm{Sol}(q)$ is exotic.

2. If $p$ is odd, Ruiz-Viruel [122] classified all possible saturated fusion systems on the extra-special $p$-group of order $p^3$ and exponent $p$. If $p = 7$, they describe three exotic saturated fusion systems. Later Díaz-Ruiz-Viruel [55] work out the classification when $S$ is a $p$-group of rank 2, $p$ odd. In this case, there is a family of exotic examples for $p = 3$. Ruiz [121] describes exotic examples for $p \geq 5$ of large $p$-rank.

3. Other exotic examples can be found in [47] by Clelland-Parker, in [27] by Broto-Levi-Oliver, and by Andersen-Oliver-Ventura [6].

**Remark 9.3.6.** It is worth pointing out that the way to show that the known exotic saturated fusion systems are so is by using the classification of finite simple groups, except for the Solomon example $\mathrm{Sol}(q)$. This example came out from the project of the classification of finite simple groups. This structure showed up at $p = 2$ in [123]: there is no finite group with the same Sylow 2-subgroup as $\mathrm{Spin}_7(3)$ such that $\mathcal{F}_S(\mathrm{Spin}_7(3)) \subset \mathcal{F}_S(G)$ and such that all involutions in $S$ are $G$-conjugate. This example was introduced into homotopy theory by Benson (unpublished notes, [13]).

**Remark 9.3.7.** Leary-Stancu [85] and Robinson [120] independently showed that, given a saturated fusion system $\mathcal{F}$ over $S$, there is always an infinite group $G$ with $S$ as a maximal $p$-subgroup that defines $\mathcal{F}$ by conjugacy. Park [105] proved that there is always a finite group $G$ such that $\mathcal{F}$ is isomorphic to $\mathcal{F}_S(G)$ but $S$ is not a Sylow $p$-subgroup of $G$.

One of the main issues when constructing new saturated fusion systems is to check the saturation axioms, which have to be satisfied for subgroups and morphisms. Some results simplify this process by reducing to certain collection of subgroups.

**Definition 9.3.8.** Let $\mathcal{F}$ be a fusion systems on a finite $p$-group $S$, and let $\mathcal{H}$ be a collection of subgroups of $S$ (i.e. closed under $\mathcal{F}$-conjugacy).

- $\mathcal{F}^{\mathcal{H}}$ is the full subcategory of $\mathcal{F}$ with object set $\mathcal{H}$.
- We say that $\mathcal{F}$ is $\mathcal{H}$-*generated* if any morphism in $\mathcal{F}$ is the composite of restrictions of morphisms between subgroups in $\mathcal{H}$.
- We say that $\mathcal{F}$ is $\mathcal{H}$-*saturated* if the saturation axioms are satisfied in $\mathcal{F}_{\mathcal{H}}$.

**Definition 9.3.9.** Let be $\mathcal{F}$ a fusion system on a finite $p$–group $S$.

- A subgroup $P \leq S$ is $\mathcal{F}$*–centric* if $P$ and all its $\mathcal{F}$–conjugates contain their $S$–centralizers.
- A subgroup $P \leq S$ is $\mathcal{F}$*–radical* if $\mathrm{Out}_{\mathcal{F}}(P)$ is $p$–reduced, that is, if $\mathrm{Out}_{\mathcal{F}}(P) = \mathrm{Aut}_{\mathcal{F}}(P)/\mathrm{Inn}(P)$ has no proper normal $p$–subgroups.

We will use $\mathcal{F}^c$ to denote the full subcategory of $\mathcal{F}$ whose objects are the $\mathcal{F}$–centric subgroups and $\mathcal{F}^{cr}$ for the full subcategory of $\mathcal{F}$–centric, $\mathcal{F}$–radical subgroups.

The following theorem is a version of Alperin's fusion theorem for saturated fusion systems (Theorem A.10 in [25]). There are other versions of Alperin's fusion theorem for saturated fusion systems involving other sets of subgroups (essential subgroups), see [10, Theorem 3.5].

**Theorem 9.3.10.** [26] *Let $\mathcal{F}$ be a saturated fusion system on a finite $p$-group $S$. Let $\mathcal{H}$ be the set of fully normalized $\mathcal{F}$-centric $\mathcal{F}$-radical subgroups of $S$. Then $\mathcal{F}$ is $\mathcal{H}$-generated.*

The following theorem reduces the task of checking saturation axioms to a collection of subgroups satisfying certain hypothesis.

**Theorem 9.3.11.** [23] *Let $\mathcal{F}$ be a fusion system on a finite $p$-group $S$. Let $\mathcal{H}$ be a set of subgroups of $S$ closed under $\mathcal{F}$-conjugacy and such that $\mathcal{F}$ is $\mathcal{H}$-generated and $\mathcal{H}$-saturated.*

*Assume that each $\mathcal{F}$-centric subgroup not in $\mathcal{H}$ is $\mathcal{F}$-conjugate to some subgroup $P \leq S$ such that*

$$\mathrm{Out}_S(P) \cap O_p(\mathrm{Out}_{\mathcal{F}}(P)) \neq 1.$$

*Then $\mathcal{F}$ is saturated.*

**Remark 9.3.12.** In Theorem 9.3.11 the hypothesis on $\mathcal{H}$ imply that $\mathcal{H}$ must contain all $\mathcal{F}$-centric $\mathcal{F}$-radical subgroups.

Both Theorem 9.3.11 and Theorem 9.3.10 allow one to construct saturated fusion systems from situations where morphisms are only explicitly described in a set of subgroups of $S$, by considering the fusion system generated by those morphisms.

The interest in understanding how exotic examples of saturated fusion systems arise in the theory of fusion systems is related to the classification of finite simple groups and to get a better understanding of it. In this context, one is lead to local finite group theory: developing concepts in analogy to the theory of finite groups but finding local analogues. For example, basic notions are that of the normalizer and centralizer of a subgroup.

**Definition 9.3.13.** Let $\mathcal{F}$ be a saturated fusion system on a finite $p$-group $S$ and $P \leq S$.

- $N_{\mathcal{F}}(P)$ is the fusion system over $N_S(P)$ where for $R, S \leq N_S(P)$, $\varphi \in \mathrm{Hom}_{N_{\mathcal{F}}(P)}(R, S)$ if there exists $\varphi' \in \mathrm{Hom}_{\mathcal{F}}(RP, SP)$ such that $\varphi'|_R = \varphi$ and $\varphi|_P \in \mathrm{Aut}(P)$.
- $C_{\mathcal{F}}(P)$ is the fusion system over $C_S(P)$ where for $R, S \leq C_S(P)$, $\varphi \in \mathrm{Hom}_{C_{\mathcal{F}}(P)}(R, S)$ if there exists $\varphi' \in \mathrm{Hom}_{\mathcal{F}}(RP, SP)$ such that $\varphi'|_R = \varphi$ and $\varphi|_P = \mathrm{id}_P$.

**Proposition 9.3.14.** [26] *Let $\mathcal{F}$ be a saturated fusion system on a finite p-group $S$ and $P \leq S$. If $P$ is fully normalized (resp. centralized) then $N_{\mathcal{F}}(P)$ (resp. $C_{\mathcal{F}}(P)$) is a saturated fusion system. Moreover, if $\mathcal{F} = \mathcal{F}_S(G)$ then $N_{\mathcal{F}}(P) \cong \mathcal{F}_{N_S(P)}(N_G(P))$ and $C_{\mathcal{F}}(P) \cong \mathcal{F}_{C_S(P)}(C_G(P))$.*

With Definition 9.3.13 at hand, one can make sense of the notion of a normal subgroup of a saturated fusion system (if $N_{\mathcal{F}}(P) \cong \mathcal{F}$) or the center of a fusion system (if $C_{\mathcal{F}}(Z) \cong \mathcal{F}$).

The existence of $\mathcal{F}$-centric $\mathcal{F}$-normal subgroups in a saturated fusion system implies that $\mathcal{F}$ is realizable by a unique group $G$ satisfying certain properties. This situation has been described in [23].

**Definition 9.3.15.** A saturated fusion system $\mathcal{F}$ over a finite $p$-group $S$ is *constrained* if it has an $\mathcal{F}$-normal $\mathcal{F}$-centric subgroup.

**Theorem 9.3.16.** *Let $\mathcal{F}$ be a constrained saturated fusion system on a finite p-group $S$. Then there is a finite group $G$ with $O_{p'}(G) = 1$ and $C_G(O_p(G)) \leq O_p(G)$ such that $\mathcal{F} \cong \mathcal{F}_S(G)$, and is unique satisfying this property.*

**Remark 9.3.17.** We can apply Theorem 9.3.16 to the case where $S$ is abelian. Then if $\mathcal{F} = \mathcal{F}_S(G)$ for a finite group $G$, we recover the Frobenius Theorem 9.1.18 on the control of fusion by $N_G(S) \leq G$.

In Remark 9.2.7, we introduced the notion of a $p$-quasicentric subgroup in a finite group $G$ . There is an analogue definition in the context of fusion systems.

**Definition 9.3.18.** Let $\mathcal{F}$ be a saturated fusion system on a finite $p$-group $S$. We say that $P \leq S$ is $\mathcal{F}$-*quasicentric* if for each $P' \in P^{\mathcal{F}}$ that is fully centralized, $C_{\mathcal{F}}(P') \cong \mathcal{F}_{C_S(P')}(C_S(P'))$.

If $\mathcal{F}^q \subset \mathcal{F}$ is the full subcategory generated by $\mathcal{F}$-quasicentric groups, then $\mathcal{F}^c \subset \mathcal{F}^q$. Recently, Ellen Henke [77] has introduced another collection of subgroups, called subcentric subgroups. When considering subsystems and quotients it is relevant to have large collections with properties that are preserved by quotients for example, making more accessible their study.

**Definition 9.3.19.** Let $\mathcal{F}$ a saturated fusion system on a finite $p$-group $S$. We say that $P \leq S$ is *subcentric* if, for any fully $\mathcal{F}$-normalized $P' \in \{P\}^{\mathcal{F}}$, $N_{\mathcal{F}}(P)$ is constrained.

If $\mathcal{F}^s \subset \mathcal{F}$ is the full subcategory of subcentric subgroups, we have a chain of inclusions

$$\mathcal{F}^{cr} \subset \mathcal{F}^c \subset \mathcal{F}^q \subset \mathcal{F}^s.$$

An extension theory of fusion systems is developed in several papers, for example, in [24] (where fusion subsystems of $p$-power index and of index prime to $p$ where classified as well as central extensions), and [102].

There is also a theory of normal fusion subsystems developed by Aschbacher [8] and also studied by Craven in [48]. Then, a simple saturated fusion system is a saturated fusion system with no non-trivial normal fusion subsystems. One of the main goals in the theory is the understanding of simple saturated fusion systems at the prime 2. Linckelmann [90] have shown that $\mathrm{Sol}(q)$ for $q$ an odd prime power $q \equiv \pm 3 \mod 8$ is simple. Then, only a single family of simple exotic examples is known for $p = 2$ , $\mathrm{Sol}(q)$.

To consider a category of saturated fusion systems, one needs a definition for a morphism of fusions systems, which will need to preserve some structure from the subgroups.

**Definition 9.3.20.** Let $\mathcal{F}$ and $\mathcal{F}'$ be two fusion systems over $S$ and $S'$ respectively. A *morphism of fusion systems* is a pair $(\alpha, \Psi)$ where $\alpha\colon S \to S'$ is a morphism and $\Psi\colon \mathcal{F} \to \mathcal{F}'$ a functor with $\alpha(P) = \Psi(P)$ for any $P \leq S$ and $\Psi(\varphi) \circ \alpha = \alpha \circ \varphi$ for any $\varphi \in \mathrm{Mor}(\mathcal{F})$.

Essentially, a morphism of fusion systems is a group homomorphism between Sylow subgroups that is compatible with the fusion/conjugacy relations.

**Remark 9.3.21.** In analogy to Section 9.1, Theorem 9.1.20 holds also for fusion systems ([16]). That is, if $\mathcal{F}_0 \subset \mathcal{F}$ are saturated fusion systems on the same finite $p$-group isomorphic on the collection of elementary abelian $p$-subgroups, then $\mathcal{F}_0 \cong \mathcal{F}$, if $p$ is odd. For $p = 2$ the authors proved the result on control of fusion by enlarging the collection to include abelian subgroups of exponent at most 4.

## 9.4 Homotopy theory of fusion systems

The homotopy theory of fusion systems does not refer to the study of the nerve of $\mathcal{F}$, but rather the nerve of an associated "linking system" which plays the role of the classifying space of an ambient finite group $G$ in case $\mathcal{F} = \mathcal{F}_S(G)$ for $S \in \mathrm{Syl}_p(G)$. We start by explaining what is the classifying space of a saturated fusion system in the context of unstable homotopy theory. Then, we will describe how the stable homotopy theory of the classifying space is algebraically modelled.

### 9.4.1 The unstable homotopy theory

Given a saturated fusion system $\mathcal{F}$, we define the stable elements of the mod $p$ cohomology to be

$$H^*(\mathcal{F}) := \lim_{P \in \mathcal{F}} H^*(P) \subset H^*(S).$$

If $\mathcal{F} = \mathcal{F}_S(G)$ for a finite group $G$ with $S \in \mathrm{Syl}_p(G)$, then the centric linking system (Definition 9.2.5) satisfies that $H^*(|\mathcal{L}_S(G)|_p^\wedge) \cong H^*(G) \cong H^*(\mathcal{F}_S(G))$.

Broto-Levi-Oliver [26, Definition 1.7] abstracted the main properties of $\mathcal{L}_S(G)$ in the definition of a linking system, which is the extra structure needed to obtain a classifying space that behaves like $BG_p^\wedge$ for a finite group $G$. The definition we present is more general allowing bigger collections of subgroups (see [23, Definition 3.3] and [102, Definition 3]).

**Definition 9.4.1.** Let $\mathcal{F}$ be a fusion system on a finite$p$–group $S$. A *linking system* associated to $\mathcal{F}$ is a finite category $\mathcal{L}$ together with functors

$$\mathcal{T}_S^{\mathrm{Ob}(\mathcal{L})}(S) \xrightarrow{\delta} \mathcal{L} \xrightarrow{\pi} \mathcal{F}^c$$

that satisfy the following conditions.

(A1) $\mathrm{Ob}(\mathcal{L}) \subseteq \mathrm{Ob}(\mathcal{F})$ is a set of subgroups $P \leq S$ closed under $\mathcal{F}$-conjugacy and over-groups that contains $\mathcal{F}$-centric and $\mathcal{F}$-radical subgroups. Each object $P \leq S$ in $\mathcal{L}$ is isomorphic to one which is fully $\mathcal{F}$-centralized.

(A2) $\delta$ is the identity on objects and $\pi$ is the inclusion on objects. For each pair of objects $P$, $Q$ in $\mathcal{L}$ such that $P$ is fully $\mathcal{F}$-centralized, $C_S(P)$ acts freely on $\mathrm{Mor}_{\mathcal{L}}(P,Q)$ by right composition via $\delta$ and $\pi$ induces a bijection

$$\mathrm{Mor}_{\mathcal{L}}(P,Q)/Z(P) \xrightarrow{\cong} \mathrm{Hom}_{\mathcal{F}}(P,Q).$$

(B) For each $\mathcal{F}$–centric subgroup $P \leq S$ and each $g \in P$, the functor $\pi$ sends $\delta_P(g)$ to $c_g \in \mathrm{Aut}_{\mathcal{F}}(P)$.

(C) For each $f \in \mathrm{Mor}_{\mathcal{L}}(P,Q)$ and each $g \in \mathcal{T}_S(P,Q)$, the following square commutes in $\mathcal{L}$

$$\begin{array}{ccc} P & \xrightarrow{f} & Q \\ {\scriptstyle \delta_P(g)}\downarrow & & \downarrow{\scriptstyle \delta_Q(\pi(f)(g))} \\ P & \xrightarrow[f]{} & Q. \end{array}$$

If $\mathrm{Ob}(\mathcal{L})$ is the set of $\mathcal{F}$-centric subgroups, then $\mathcal{L}$ is called a *centric linking system*. The axioms are defined in a way that mimick the properties of morphisms in the centric linking system associated to $\mathcal{F}_S(G)$ and its relation with the fusion system, $\mathcal{L}_S^c(G) \to \mathcal{F}_S^c(G)$.

**Remark 9.4.2.** From Definition 9.4.1, the functor $\delta\colon \mathcal{T}_S(S) \to \mathcal{L}$ induces a map $BS \to |\mathcal{L}|^\wedge_p$ which models the inclusion of a Sylow $p$-subgroup and which is used to define restriction to subgroups $P \leq S$, $BP \to |\mathcal{L}|^\wedge_p$.

**Example 9.4.3.** [26] Let $G$ be a finite group and $S \in \mathrm{Syl}_p(G)$. Then $\mathcal{L}^c_S(G)$ is a centric linking system associated to $G$.

What can we say about the collection $\mathrm{Ob}(\mathcal{L})$ for a given linking system $\mathcal{L}$? In [102, Proposition 4], Oliver shows that all objects in $\mathcal{L}$ are $\mathcal{F}$-quasicentric subgroups. Another important property is that if we have an inclusion of collections $\mathcal{H} \subset \mathcal{H}'$, then given a linking system $\mathcal{L}$ with object set $\mathcal{H}$ is contained in a linking system $\mathcal{L}'$ with object $\mathcal{H}'$ ([10, Propostion III.4.8]). But in fact, the homotopy type of the nerve of a linking system does not depend on the collection $\mathcal{H}$ ([23, Theorem 3.5]): $|\mathcal{L}| \simeq |\mathcal{L}'|$.

Next theorem shows that the structure defined provides a classifying space for a fusion system in the sense that topologically realizes $H^*(\mathcal{F})$.

**Theorem 9.4.4.** [26] *Let $\mathcal{F}$ be a saturated fusion system on a finite $p$-group $S$. Assume there exists a linking system $\mathcal{L}$ associated to $\mathcal{F}$. Then*

$$H^*(|\mathcal{L}^c|^\wedge_p) \cong H^*(\mathcal{F}).$$

*Moreover, $H^*(\mathcal{F})$ is Noetherian.*

**Remark 9.4.5.** A stable elements formula for cohomology with twisted coefficients has been investigated by Molinier ([95], [96]) and Levi-Ragnarsson [88].

**Question 9.4.6.** *Given a saturated fusion system $\mathcal{F}$, is there a linking system associated to $\mathcal{F}$?*

From the definition, there is no reason to believe that this is the case. Broto-Levi-Oliver [26] developed an obstruction theory for the existence and uniqueness of a centric linking system which is related to a question in homotopy theory. Recall from Definition 9.3.8 that $\mathcal{F}^{\mathcal{H}} \subset \mathcal{F}$ is the full subcategory generated by the objects in the collection $\mathcal{H}$.

**Definition 9.4.7.** Let $\mathcal{F}$ be a saturated fusion system and $\mathcal{H}$ a collection of subgroups of $S$. The *orbit category* of $\mathcal{F}^{\mathcal{H}}$ is the category $\mathcal{O}(\mathcal{F}^{\mathcal{H}})$ where $\mathrm{Ob}(\mathcal{O}(\mathcal{F}^{\mathcal{H}})) = \mathrm{Ob}(\mathcal{F}^{\mathcal{H}})$ and

$$\mathrm{Mor}_{\mathcal{O}(\mathcal{F}^{\mathcal{H}})}(P,Q) = \mathrm{Inn}(Q)\backslash \mathrm{Mor}_{\mathcal{F}^{\mathcal{H}}}(P,Q) = \mathrm{Rep}_{\mathcal{F}^{\mathcal{H}}}(P,Q).$$

Assuming there exists a centric linking system $\mathcal{L}^c$, one can consider the functor $\pi\colon \mathcal{L}^c \to \mathcal{F}^c \to \mathcal{O}(\mathcal{F}^c)$. The left Kan extension $L_\pi(*)$ of the constant functor to a point $*\colon \mathcal{O}(\mathcal{F}^c) \to \mathrm{Top}$ gives a homotopy equivalence

$$\operatorname*{hocolim}_{\mathcal{O}(\mathcal{F}^c)} L_\pi(*) \simeq |\mathcal{L}^c|,$$

by Segal's homotopy push-down theorem, with $L_\pi(*)(P) \simeq BP$, for any $P \in \mathcal{F}^c$ [26, Proposition 2.2]. The converse is true: given a subgroup decomposition by a functor $\beta\colon \mathcal{O}(\mathcal{F}^c) \to \mathrm{Top}$, with $\beta(P) \simeq BP$, one can recover a centric linking system by considering homotopy classes of maps from classifying spaces of finite $p$-groups into $(\mathrm{hocolim}_{\mathcal{O}(\mathcal{F}^c)}\, \beta)^\wedge_p$.

The key observation is that there is a bijection between centric linking systems associated to a saturated fusion systems and lifts of the classifying space functor $B\colon \mathcal{O}(\mathcal{F}^c) \to \mathrm{HoTop}$ to Top ([10, Proposition 5.31]).

Broto-Levi-Oliver stated the obstruction theory for Question 9.4.6 in [26, Proposition 3.1] which depends upon a functor $\mathcal{Z}\colon \mathcal{O}(\mathcal{F}^c) \to \mathrm{Ab}$ with $\mathcal{Z}(P) = Z(P)$. The obstructions to the existence lie in $\lim^3_{\mathcal{O}(\mathcal{F}^c)} \mathcal{Z}$, and $\lim^2_{\mathcal{O}(\mathcal{F}^c)} \mathcal{Z}$ acts freely and transitively on the set of isomorphic classes of centric linking systems if it is nonempty.

**Remark 9.4.8.** The proof of the Martino-Priddy conjecture by Oliver, Theorem 9.2.11, goes through a systematic computation of the obstruction groups using the classification of finite simple groups.

**Remark 9.4.9.** At this point, Broto-Levi-Oliver defined the notion of a *$p$–local finite group*. It is a triple $(S, \mathcal{F}, \mathcal{L})$, where $\mathcal{F}$ is a saturated fusion system on a finite $p$–group $S$ and $\mathcal{L}$ is a centric linking system associated to $\mathcal{F}$. The classifying space of the $p$–local finite group $(S, \mathcal{F}, \mathcal{L})$ is the space $|\mathcal{L}|^\wedge_p$.

A positive answer to Question 9.4.6 was obtained by Chermak in [43] using a direct construction method.

**Theorem 9.4.10.** *Each saturated fusion system on a finite $p$-group $S$ has an associated centric linking system, which is unique up to isomorphism.*

Chermak introduced the notion of a partial group, inspired by properties of the set $\mathrm{Mor}(\mathcal{L})$ and that of a locality in [43]. Later, Oliver [103] developed another proof, inspired by Chermak's work, by showing that the relevant obstruction groups vanish. It is worth mentioning that both proofs use the classification of finite simple groups. Finally, Glauberman-Lynd [68] provided a proof which does not depend on the classification of finite simple groups.

**Remark 9.4.11.** Partial groups and localities can be described in terms of simplicial sets satisfying certain properties. From this point of view, they are being studied by homotopy theorists. We mention work of Molinier [97] who proved Alperin's fusion theorem in this context. Chermak-González [44] described a unified setting to include many structures as localities, and González developed an extension theory of partial groups and localities [71].

**Remark 9.4.12.** Henke [77] has enlarged the collections in which a linking system and a transporter system can be defined by considering the collection of subcentric subgroups and proving existence and uniqueness of such systems.

Now, the only information needed to construct a classifying space is the saturated fusion system itself.

**Definition 9.4.13.** Let $\mathcal{F}$ be a saturated fusion system on a finite $p$-group $S$. The *classifying space $B\mathcal{F}$* is $|\mathcal{L}|^\wedge_p$ where $\mathcal{L}$ is a centric linking system associated to $\mathcal{F}$.

Most of the homotopical group theory describing homotopic constructions, like mapping spaces, in terms of fusion data can be developed in this context.

**Definition 9.4.14.** Let $\mathcal{F}$ be a saturated fusion system, and $P$ a finite $p$-group. We define $\mathrm{Rep}(P,\mathcal{F}) = \mathrm{Hom}(P,S)/\sim$ with $f \sim g$ if there is $\alpha \in \mathrm{Iso}_F(f(P), g(P))$ such that $\alpha \circ f = g$.

**Theorem 9.4.15.** [26] *For any finite $p$-group $P$ and any saturated fusion system, the classifying space functor and composing with $\iota : BS \to B\mathcal{F}$ induces a bijection*

$$\mathrm{Rep}(P,\mathcal{F}) \xrightarrow{\cong} [BP, B\mathcal{F}].$$

*Moreover, given $f \in \mathrm{Rep}(P,\mathcal{F})$ such that $f(P)$ is fully centralized then*

$$\mathrm{Map}(BP, B\mathcal{F})_f \simeq BC_{\mathcal{F}}(P).$$

The proof of Theorem 9.4.15 uses the methods and $T$-functor technology developed by Lannes [84] and which have been applied very successfully in studying the homotopy type of $p$-completed classifying spaces. The description of $\mathrm{Out}(B\mathcal{F})$ in terms of equivalences of the centric linking system is also obtained in [26].

The $F$-isomorphism theorem also holds for classifying spaces of saturated fusion systems. We denote by $\mathcal{F}^e \subset \mathcal{F}$ the full subcategory generated by elementary abelian $p$-subgroups. The $F$-isomorphism theorem was proven by Broto-Levi-Oliver [26] and the strong stratification result by Barthel-Castellana-Heard-Valenzuela [12], and Linckelmann [91].

**Theorem 9.4.16.** [26] *The morphism*

$$\prod \mathrm{Res}_V^{\mathcal{F}} \colon H^*(B\mathcal{F}) \longrightarrow \lim_{\mathcal{F}^e} H^*(V)$$

*is an $F$-isomorphism.*

Denote by $\mathrm{Spec}_{\mathcal{F}}^h$ the homogeneous prime ideal spectrum of $H^*(\mathcal{F})$. If $E \subset S$ is an elementary abelian $p$-subgroup, let $\mathrm{Spec}_{\mathcal{F},E}^+ = \mathrm{res}_{\mathcal{F}}^E \mathrm{Spec}(E)^+$ where $\mathrm{res}_{\mathcal{F}}^E \colon \mathrm{Spec}_E^h \to \mathrm{Spec}_{\mathcal{F}}^h$. Recall that $\mathrm{Spec}_E^+ = \mathrm{Spec}_E^h \setminus \bigcup_{E'<E} \mathrm{res}_E^{E'}(\mathrm{Spec}_{E'}^h)$.

**Theorem 9.4.17.** ([12], [91]) *The variety $\mathrm{Spec}_{\mathcal{F}}^h$ admits a decomposition as a disjoint union*

$$\mathrm{Spec}_{\mathcal{F}}^h \cong \amalg_{E\in\mathcal{E}(\mathcal{F})} \mathrm{Spec}_{\mathcal{F},E}^+$$

*where $\mathcal{E}(\mathcal{F})$ is a set if representatives of $\mathcal{F}$-conjugacy classes of elementary abelian $p$-subgroups of $S$.*

Since every saturated fusion system $\mathcal{F}$ has an associated classifying space $B\mathcal{F}$, we can ask whether this construction is functorial. Recall the Definition 9.3.20 at the end of Section 9.3 where we introduced the concept of a morphism between fusion systems. It is not clear that a morphism between fusion systems induces a map between classifying spaces.

This question has been approached by Castellana-Libman in [42]. The strategy applied is a classical one used to study maps between $p$-completed classifying spaces by several authors (see [82], [79], [80], [81]) Given a topological space, restriction to $p$-subgroups through $BS \to B\mathcal{F}$ gives a map

$$[B\mathcal{F}, X] \longrightarrow \lim_{P\in\mathcal{O}(\mathcal{F})} [BP, X].$$

In [125], Wojtkowiak described the obstructions classes for the surjectivity and injectivity for this restriction map, which lie in

$$\lim_{P\in\mathcal{O}(\mathcal{F}^{cr})}{}^{i+\varepsilon}\,\pi_i(\mathrm{Map}(BP,X)_{f\circ Bi_P})$$

for $\varepsilon = 0, 1$ and $i \geq 1$ where $Bi_P\colon BP \to BS$ is induced by the group inclusion.

If $X = B\mathcal{F}'$, where $\mathcal{F}'$ is a saturated fusion system on a finite $p$-group $S'$, and $Bf\colon BS \to BS'$ is induced by a morphism of fusion systems, obstructions do not need to vanish. Castellana-Libman [42] showed that the obstruction groups are torsion, which is the key fact for the following partial result.

**Theorem 9.4.18.** *Let $(\rho, \Theta)\colon \mathcal{F} \to \mathcal{F}'$ be a morphism of saturated fusion systems where $\rho\colon S \to S'$ is a group homomorphism. Then there exists a natural number $m \geq 0$ such that the morphism $\Delta \circ \rho\colon S \to S' \to S' \wr \Sigma_m$ is fusion preserving and extends to a map $B\tilde{\rho}\colon B\mathcal{F} \to B(\mathcal{F}' \wr \Sigma_m)$.*

The next corollary can be interpreted as a proof of the existence of a regular representation for $\mathcal{F}$. It is obtained applying the previous Theorem 9.4.18 to the regular representation of the Sylow $p$-subgroup.

**Corollary 9.4.19.** *Let $\mathcal{F}$ be a saturated fusion system on a finite $p$-group $S$. There is a map $B\rho\colon B\mathcal{F} \to (B\Sigma_m)^\wedge_p$ such that its restriction to $S$ is a sum of several copies of the regular representation.*

**Remark 9.4.20.** Any finite group admits a faithful unitary representation $G \hookrightarrow U(n)$ for some $n$. Combining the permutation matrix representation $\Sigma_n \hookrightarrow U(n)$ with the map obtained in Corollary 9.4.19, we obtain a faithful complex representation $B\rho\colon B\mathcal{F} \to BU(n)^\wedge_p$ in the homotopical sense, i.e. the homotopy fiber $F$ of $B\rho$ has finite mod $p$ cohomology; $H^*(F)$ is a finite $\mathbb{F}_p$-vector space. In particular, using Venkov's proof for showing that the mod $p$ cohomology of a finite group is finitely generated, one obtains an alternative proof for the fact the mod $p$ cohomology of a saturated fusion system is finitely generated.

**Remark 9.4.21.** Strategies involving the subgroup decomposition of a classifying space naturally lead to an obstruction theory which in many situations involves computing higher limits of functors over the orbit category. For finite groups, those limits have been studied in detail in many situations ([58], [80], [75]). In the context of fusion systems it is still not known whether the previous results on the vanishing of higher limits over the orbit category hold. An approach to this question has been attempted by Díaz-Park in [54] for the case of Mackey functors obtaining partial results.

We finish this part with an example of a stable elements theorem in a different context. Let $R(G)$ be the Grothendieck ring of complex representations of $G$. The classifying space functor $\mathrm{Rep}(G, U(n)) \to [BG, BU(n)^\wedge_p]$ induces a bijection if $G$ is a finite $p$-group by work of Dwyer-Zabrodsky [64]. Then, looking at the stable virtual representations in this context, $\lim_{P\in\mathcal{F}} R(P)$ is closely related to $[B\mathcal{F}, BU(n)^\wedge_p]$ via restriction maps.

Given a topological space $X$ one can consider a topological analogue of the representation ring at a prime $p$ by defining $\mathbb{K}(X)$ to be the Grothendieck group completion of the monoid

$\coprod[X, BU(n)^\wedge_p]$ where the sum is induced by $U(n)\times U(m) \hookrightarrow U(n+m)$. If $X = B\mathcal{F}$, $\mathbb{K}(B\mathcal{F})$ is an homotopical analogue for the complex representation ring. We would like to compare it to the stable representations, that is, elements in $R(S)$ that satisfy the analogous relations as in Definition 9.1.1. There is a restriction morphism

$$\mathrm{Res}\colon \mathbb{K}(B\mathcal{F}) \longrightarrow \lim_{P\in\mathcal{O}(\mathcal{F})} R(P)$$

defined using the inclusion of the Sylow $p$-subgroup $\theta\colon BS \to B\mathcal{F}$. Jackowsky-Oliver [82] proved that Res is an isomorphism when $\mathcal{F}$ is the the fusion system of a finite group (when $B\mathcal{F} \simeq BG^\wedge_p$ for a finite group $G$). Cantarero-Castellana-Morales [34] showed that the same statement holds for a general saturated fusion system on a finite $p$-group.

**Theorem 9.4.22.** *If $\mathcal{F}$ is a saturated fusion system on a finite $p$-group $S$ then*

$$\mathbb{K}(B\mathcal{F}) \cong \lim_{P\in\mathcal{O}(\mathcal{F}^c)} R(P).$$

### 9.4.2 The stable homotopy theory

Stable homotopy theory of saturated fusion systems deserves a special section since the developments in this area have been ahead of the unstable theory in solving relevant problems in the theory, like the existence of a classifying space or the functoriality of the classifying space construction.

If $S \in \mathrm{Syl}_p(G)$, the group $S$ acts on the set $G$ by left and right multiplication giving $G$ the structure of an $(S,S)$-biset. Note that given $(s_1, s_2) \in S \times S$ and $g \in G$, the condition $s_1 g = g s_2$ is equivalent to $s_1 = c_g(s_2)$, that is, we can see conjugacy relations encoded in the $(S,S)$-biset $G$.

In an analogy with the idea of axiomatizing the properties of the fusion system $\mathcal{F}_S(G)$ in the abstract concept of a saturated fusion system, Lincklemann and Webb introduced the notion of a characteristic biset for a saturated fusion system in order to isolate the main features of $G$.

Let $A(G)$ be the Burnside ring of finite $G$-sets, i.e. the Grothendieck group completion of the monoid of isomorphism classes of finite left $G$-sets, where the multiplicative group structure is induced by Cartesian product. Let $A(G,H)$ be the Grothendieck group completion of the monoid of isomorphism classes of finite left free $(G,H)$-bisets.

**Example 9.4.23.** Given two finite groups $G$ and $H$, we consider a pair $(K,\varphi)$ where $K \leq G$ and $\varphi\colon K \to H$ is a monomorphism. Define the subgroup $\Delta(K,\varphi) = \{(\varphi(k), k) \mid k \in K\} \leq H \times G$. We denote by $H \times_{(K,\varphi)} G$ the $(G,H)$-biset defined by

$$(H \times G)/\Delta(K,\varphi).$$

Equivalently, $(H\times G)/\sim$ where $(x, gy) \sim (x\varphi(g), y)$ for $x \in H$, $y \in G$ and $g \in K$. Any left-free transitive $(G,H)$-biset is of this form. We write $[K,\varphi]$ for the isomorphism class of $H \times_{(K,\varphi)} G$ in $A(G,H)$.

Given a finite left-free $(G, H)$-biset $X$, we define a stable map $\alpha(X)\colon \Sigma^\infty_+ BG \to \Sigma^\infty_+ BH$. On transitive bisets of the form $X = H \times_{(K,\varphi)} G$, $\alpha(X) = \Sigma_+ B\varphi \circ \tau^H_G$ where $\tau^H_G\colon \Sigma^\infty_+ BG \to \Sigma^\infty_+ BK$ is the transfer map associated to the finite covering $BK \to BG$ (see [46]). Since every left-free biset is a coproduct of transitive ones, this construction extends to a homomorphism

$$\alpha\colon A(G, H) \longrightarrow \{\Sigma^\infty_+ BG, \Sigma^\infty_+ BH\}.$$

The Segal conjecture is about $\alpha$ being the completion with respect to the augmentation ideal $I(G)$ of the Burnside ring $A(G)$ (which acts on $A(G, H)$ by Cartesian product). When $G$ is a finite $p$-group, completion can be related to $p$-adic completion (see [113, Section 9.1]).

Let $\varepsilon\colon A(G, H) \to \mathbb{Z}$ be the augmentation morphism defined on basis elements by $\varepsilon(X) = |X|/|H|$.

**Theorem 9.4.24.** ([40], [89]) *Let $G$, $H$ be finite groups. Then*

$$\alpha\colon A(G, H)^\wedge_{I(G)} \xrightarrow{\cong} \{\Sigma^\infty_+ BG, \Sigma^\infty_+ BH\},$$

*where $I(G)$ is the augmentation ideal $I(G) \subset A(G)$. If $G$ and $H$ are both finite $p$-groups, $A'(G, H)^\wedge_{I(G)} \cong A'(G, H)^\wedge_p \cong \mathbb{Z}^\wedge_p \otimes A'(G, H)$ where $A'(G, H)$ is the kernel of the augmentation morphism $\varepsilon$.*

**Example 9.4.25.** Regard $G$ as an $(S, S)$-biset. Then $\alpha([G]) \in \{\Sigma^\infty_+ BS, \Sigma^\infty_+ BS\}$ induces a morphism $H^*(\alpha([G]))\colon H^*(S) \to H^*(S)$ that corresponds to $\mathrm{Res}^G_S \circ \mathrm{Tr}^G_S$. Then, the image of $H^*(\alpha([G]))$ consists of the stable elements (Theorem 9.1.3).

Let $\varphi\colon P \hookrightarrow S$ be a morphism. If $X \in A(S, S)$, ${}_{(P,\varphi)}X \in A(S, P)$ denotes the $(S, P)$-biset obtained by restricting the action via $\varphi$, $X_{(P,\varphi)} \in A(P, S)$ is defined analogously.

**Definition 9.4.26.** Let $\mathcal{F}$ be a fusion system on a finite $p$-group $S$.

1. Let $X$ be an $(S, S)$-biset. $X$ is right (resp. left ) $\mathcal{F}$-*stable* if for every $P \leq S$ and $\varphi \in \mathrm{Mor}_{\mathcal{F}}(P, S)$, $X_{(P,\mathrm{id})} \cong X_{(P,\varphi)}$ (resp. ${}_{(P,\mathrm{id})}X \cong {}_{(P,\varphi)}X$). We will just say that it is $\mathcal{F}$-*stable* if it is both right and left $\mathcal{F}$-stable.
2. If $P, Q \leq S$, $A_{\mathcal{F}}(P, Q)$ is the subgroup of $A(P, Q)$ generated by $[K, \varphi]$ where $\varphi \in \mathrm{Mor}_{\mathcal{F}}(P, Q)$.

Linckelmann and Webb suggested the following properties for a biset mimicking the properties of $G$.

**Definition 9.4.27.** Let $\mathcal{F}$ be a fusion system on a finite $p$-group $S$. An $(S, S)$-biset $\Omega$ is a *characteristic biset* for $\mathcal{F}$ if:

1. $\Omega$ is free as a left and right $S$-set.
2. $\Omega \in A_{\mathcal{F}}(S, S)$.
3. $\Omega$ is $\mathcal{F}$-stable.
4. $|\Omega|/|S|$ is prime to $p$.

The existence of characteristic bisets for saturated fusion systems was first proven by Broto-Levi-Oliver [26].

**Theorem 9.4.28** (Proposition 5.5 in [26])**.** *Let $\mathcal{F}$ be a saturated fusion system on a finite $p$-group $S$. Then, there exists a characteristic biset $\Omega$ for $\mathcal{F}$. Moreover, $H^*(\alpha(\Omega)) \colon H^*(S) \to H^*(S)$ is an idempotent $H^*(\mathcal{F})$-linear morphism with image $H^*(\mathcal{F})$.*

A saturated fusion system does not need to have a unique characteristic biset. For example, if $H \leq G$ controls fusion in $G$, then both $G$ and $H$ are characteristic bisets for $\mathcal{F} = \mathcal{F}_S(G) = \mathcal{F}_S(H)$.

Gelvin-Reeh [66] proved that every saturated fusion system $\mathcal{F}$ has a unique minimal characteristic biset $\Lambda_{\mathcal{F}}$ in the following sense: for any characteristic biset $X$ we have $\Lambda_{\mathcal{F}} \subset X$. Special attention is devoted to constrained fusion systems. In that case, Theorem 9.3.16 shows that there is a group $G$ with $O_{p'}(G) = 1$ and $C_G(O_p(G)) \leq O_p(G)$ such that $\mathcal{F} \cong \mathcal{F}_S(G)$, and is unique satisfying this property. Then the Gelvin-Reeh show that $G$, as a $(S,S)$-biset, is the minimal characteristic biset for $\mathcal{F}$.

Ragnarsson-Stancu [113] introduce an analogous notion for virtual $(S,S)$-bisets: $\Omega \in A(S,S)_{(p)}$ is a *characteristic element* if $\Omega \in A_{\mathcal{F}}(S,S)$ is $\mathcal{F}$-stable and $\varepsilon(\Omega)$ is prime to $p$, where $\varepsilon \colon A(S,S) \to \mathbb{Z}$ is the augmentation morphism defined on basis elements by $\varepsilon(X) = |X|/|S|$. By inverting $|X|/|S|$, one can always assume that $\varepsilon(X) = 1$. It is clear that a characteristic biset for $\mathcal{F}$ provides also a characteristic element, but the main result obtained by Ragnarsson in [112] is the existence of unique *idempotent characteristic element*.

**Theorem 9.4.29.** [112] *A saturated fusion system $\mathcal{F}$ on a finite $p$-group $S$ has a unique idempotent characteristic element $\omega_{\mathcal{F}} \in A(S,S)_{(p)}$.*

By [113, Proposition 9.2], the $I(S)$-adic completion $A(S,S)^\wedge_{I(S)}$ injects into the $p$-adic completion $A(S,S)^\wedge_p$ with image given by the submodule of elements with augmentation in $\mathbb{Z}$. Then, since $A(S,S)_{(p)}$ is a submodule of $A(S,S)^\wedge_p$, a characteristic element gives an idempotent stable map $\tilde{\omega}_{\mathcal{F}} \colon \Sigma^\infty_+ BS \to \Sigma^\infty_+ BS$ by Theorem 9.4.24.

**Remark 9.4.30.** In [116] and [117], Reeh defines the Burnside ring for a saturated fusion system on a finite $p$-group $S$ by considering $\mathcal{F}$-stable sets. By constructing a tranfer map from the Burnside ring of $S$, he gives a new construction for $\omega_{\mathcal{F}}$. The Burnside ring and the representation ring of $\mathcal{F}$ are studied in [118] and [67]. Previously, Díaz-Libman described the cohomotopy of the classifying space for saturated fusion systems (see [53], [52]).

Let $\tilde{A}(S,S)$ be the quotient module obtained from $A(S,S)$ by dividing out all the basis elements $[K,\varphi]$ where $\varphi$ is the trivial homomorphisms. Then the Segal conjecture ([40], [89]) identifies stable maps between classifying spaces with no extra base point

$$\tilde{A}(S,S)^\wedge_p \cong \{\Sigma^\infty BS, \Sigma^\infty BS\}.$$

We consider $\tilde{\omega}_{\mathcal{F}} \in \{\Sigma^\infty BS, \Sigma^\infty BS\}$.

**Definition 9.4.31.** Let $\mathcal{F}$ be a saturated fusion system on a finite $p$-group $S$. The *classifying spectrum of* $\mathcal{F}$, $\mathbb{B}\mathcal{F}$, is the stable summand of $\Sigma^\infty BS$ obtained via the telescope construction

$$\operatorname{hocolim}(\Sigma^\infty BS \xrightarrow{\tilde{\omega}_{\mathcal{F}}} \Sigma^\infty BS \xrightarrow{\tilde{\omega}_{\mathcal{F}}} \cdots).$$

Then, there is a stable map $\mathrm{Tr}_{\mathcal{F}}\colon \mathbb{B}\mathcal{F}_+ \to \Sigma^\infty_+ BS$ (the transfer map) and $\sigma_{\mathcal{F}}\colon \Sigma^\infty_+ BS \to \mathbb{B}\mathcal{F}_+$ with $\sigma_{\mathcal{F}} \circ \mathrm{Tr}_{\mathcal{F}} \simeq \mathrm{id}_{\mathbb{B}\mathcal{F}_+}$ and $\mathrm{Tr}_{\mathcal{F}} \circ \sigma_{\mathcal{F}} \simeq \tilde{\omega}_{\mathcal{F}}$.

**Remark 9.4.32.** If $\mathcal{L}$ is a linking system for a saturated fusion system $\mathcal{F}$ on a finite $p$-group $S$, then $\Sigma^\infty|\mathcal{L}|^\wedge_p \simeq \mathbb{B}\mathcal{F}$ (see [112]).

Ragnarsson [112, Theorem 7.9] shows that the classifying spectrum construction is functorial with respect to morphisms of fusion systems. Let $\mathcal{F}$ and $\mathcal{F}'$ be two saturated fusion systems on finite $p$-groups $S$ and $S'$ respectively. Recall from Defintion 9.3.20 that a morphism of fusion systems is a pair $(\alpha, \Psi)$ where $\alpha\colon S \to S'$ is a morphism and $\Psi\colon \mathcal{F} \to \mathcal{F}'$ is a functor. Then there is a map of spectra $B\Psi\colon \mathbb{B}\mathcal{F}_+ \to \mathbb{B}\mathcal{F}'_+$ such that

$$\begin{array}{ccc} \Sigma^\infty_+ BS & \xrightarrow{B\alpha} & \Sigma^\infty_+ BS' \\ \downarrow{\scriptstyle \sigma_{\mathcal{F}}} & & \downarrow{\scriptstyle \sigma_{\mathcal{F}'}} \\ \mathbb{B}\mathcal{F}_+ & \xrightarrow[B\Psi]{} & \mathbb{B}\mathcal{F}'_+. \end{array}$$

Moreover, in [112, Theorem 7.2], the author describes a basis for stable maps between classifying spectrum of saturated fusion systems in terms of basis elements in the double Burnside ring for the corresponding Sylow $p$-subgroups.

The classical transfer morphism $\mathrm{Tr}_H^G\colon H^*(H) \to H^*(G)$ for given $H \leq G$ finite groups is a morphism $H^*(G)$-modules. This property is called Frobenius reciprocity.

**Theorem 9.4.33.** [112] *For any saturated fusion system on a finite $p$-group $S$, there is a natural transfer* $\mathrm{Tr}_{\mathcal{F}}\colon \mathbb{B}\mathcal{F}_+ \to BS_+$. *Moreover, the following diagram commutes up to homotopy*

$$\begin{array}{ccc} \mathbb{B}\mathcal{F}_+ & \xrightarrow{\Delta} & \mathbb{B}\mathcal{F}_+ \wedge \mathbb{B}\mathcal{F}_+ \\ \downarrow{\scriptstyle \mathrm{Tr}_{\mathcal{F}}} & & \downarrow{\scriptstyle 1\wedge \mathrm{Tr}_{\mathcal{F}}} \\ BS_+ & \xrightarrow[(\sigma_{\mathcal{F}}\wedge 1)\circ\Delta]{} & \mathbb{B}\mathcal{F}_+ \wedge BS_+. \end{array}$$

**Remark 9.4.34.** [113] The properties of the transfer map $\mathrm{Tr}_{\mathcal{F}}$ and the unique idempotent characteristic element $\tilde{\omega}_{\mathcal{F}}$ allow one to obtain a stable element theorem for any cohomology theory. That is, if $E$ is a spectrum then $E^*(\sigma_{\mathcal{F}})$ is an split injection with image the $\mathcal{F}$-stable elements in $E^*(\Sigma^\infty_+ BS)$. If $E$ is a ring spectrum then $E^*(\mathrm{Tr}_{\mathcal{F}})$ is map of $E^*(B\mathcal{F}_+)$-modules.

The difference with respect to previous results for finite groups, where stable elements formula for generalized cohomology theories with $p$-local coefficient were obtained in [82], is that the authors identified

$$E^*(BG) \longrightarrow \lim_{P \in \mathcal{O}_S(G)} E^*(BP)$$

with the edge homomorphism of the Bousfield spectral sequence for a colimit [21] and then proved that the higher limits vanish.

In Section 9.3, we pointed out that one of the main issues when constructing saturated fusion systems is to check the saturation axioms. In this context, Ragnarsson-Stancu decribed a construction of a fusion system $\mathcal{F}_\Omega$ associated to a symmetric idempotent element

$\Omega \in A(S,S)_{(p)}$. They show that saturation axioms for $\mathcal{F}_\Omega$ are encoded in the Frobenius reciprocity formula. We say that an element in $A(S,S)$ is symmetric if it is isomorphic to the one obtained by transposing the two actions.

**Theorem 9.4.35.** [113] *For a finite $p$-group $S$, there is a bijective correspondence between saturated fusion systems on $S$ and symmetric idempotents in $A(S,S)_{(p)}$ of augmentation* 1 *that satisfy Frobenius reciprocity.*

**Remark 9.4.36.** In [113], the authors state a conjecture by Miller which attempts a purely homotopy theoretic description for the homotopy theory of saturated fusion systems. Let $X$ be a connected, $p$-complete space with finite fundamental group and $S$ a finite $p$-group. Assume there is a map $f\colon BS \to X$ such that $H^*(BS)$ is a finitely generated $H^*(X)$-module and that admits a stable transfer retract $t\colon \Sigma^\infty_+ X \to \Sigma^\infty_+ BS$ satisfying the Frobenius reciprocity relation. The conjecture says that $X$ is then the classifying space of a saturated fusion system on $S$.

## 9.5 Algebraic models for finite loop spaces

A finite loop space is a pair of $(X, BX)$ where $X \simeq \Omega BX$ is a pointed path-connected space with the homotopy type of finite $CW$-complex and $BX$ is the classifying space. The motivating example is $(G, BG)$ where $G$ is a compact Lie group.

An extension to general compact Lie groups of the group cohomology of finite groups is provided by the cohomology of the classifying space. Many statements from sections 9.1 and 9.2 hold in general for compact Lie groups. For example, Quillen's results on the stratification of the mod $p$ cohomology were originally proven for compact Lie groups. The stable elements theorem for any generalized cohomology theory with $\mathbb{Z}_{(p)}$-coefficients in the work of Jackowski-Oliver [82, Corollary 3.7] holds for compact Lie groups. Homology decompositions were described by Jackowski-McClure-Oliver in [78], [79], [80].

In order to study the homotopical properties of compact Lie groups, Rector considered their classifying spaces; but he proved that there are uncountably many distinct loop space structures on $S^3$ [115]. But Dwyer, Miller and Wilkerson showed that this problem goes away after $p$-completion, since they proved that there is a unique loop space structure on $(S^3)^\wedge_p$ [60]. This result lead to the proposal that the right category to isolate the homotopical properties of compact Lie groups is the category of $p$-complete loop spaces. Finally, Dwyer and Wilkerson [62] introduced the notion of a $p$-compact group.

**Definition 9.5.1.** A *$p$-compact group* is a loop space $(X, BX)$ where $X$ is an $\mathbb{F}_p$-finite space (that is, $H^*(X)$ is a finite $\mathbb{F}_p$-vector space), and $BX$ is a $p$-complete space.

Examples of p-compact groups are provided by compact Lie groups such that $\pi_0(G)$ is a $p$-group. However, there are exotic examples of $p$-compact groups that are not the $p$-completion of any compact Lie group (see [61], [1]).

Most of the geometric structure of a connected compact Lie group can be translated into this homotopy theoretic setting, see [62]. We focus on the subgroup structure. We begin with the analogue of tori and their normalizers. A *p-compact toral group* is a $p$-compact group that is a finite extension of a $p$-compact torus (the $p$-completion of a torus) by a finite $p$-group.

Let $\mathbb{Z}/p^\infty \cong \mathbb{Z}\left[\frac{1}{p}\right]/\mathbb{Z}$ denote the union of the cyclic $p$-groups $\mathbb{Z}/p^n$ under the standard inclusions.

**Definition 9.5.2.** A *discrete p-toral group* is a group $P$ with a normal subgroup $P_0 \triangleleft P$ such that $P_0$ is isomorphic to a finite product of copies of $\mathbb{Z}/p^\infty$, and $P/P_0$ is a finite $p$-group. The subgroup $P_0$ will be called the identity component of $P$.

The two definitions are related since, by [62, Prop. 6.9], any $p$-compact toral group is the $\mathbb{F}_p$-completion of a discrete $p$-toral group $S$.

Dwyer-Wilkerson proved that $p$-compact groups admit a maximal torus $T_X$, an associated Weyl group $W_X$, and a maximal $p$-compact toral subgroup $N_p$ (see [62]) that fits into a fibration sequence

$$BT_X \to BN_p(T_X) \to B(W_X)_p$$

where $(W_X)_p \in \mathrm{Syl}_p(W_X)$. Then $N_p(T_X)$ is a $p$-compact toral group being an extension of a $p$-complete torus by a finite $p$-group, and it is the $\mathbb{F}_p$-completion of a discrete $p$-toral group $S$.

The main achievement in the theory of $p$-compact groups is the classification of connected $p$-compact groups by Andersen-Grodal-Møller-Viruel ($p$ odd) and Andersen-Grodal ($p = 2$) in [5] and [4],where a bijective correspondence between connected $p$-compact groups and reflection data over the $p$-adic integers encoded in $(W_X, T_X)$ was established.

Following the theory of saturated fusion systems on finite $p$-groups, Broto-Levi-Oliver introduced the notion of a saturated fusion system on a discrete $p$-toral group.

**Definition 9.5.3.** A *fusion system* $\mathcal{F}$ on a discrete $p$-toral group $S$ is a subcategory of the category of groups whose objects are the subgroups of $S$, and whose morphism sets $\mathrm{Hom}_{\mathcal{F}}(P, Q)$ satisfy the following conditions:

(a) $\mathrm{Hom}_S(P, Q) \subseteq \mathrm{Hom}_{\mathcal{F}}(P, Q) \subseteq \mathrm{Inj}(P, Q)$ for all $P, Q \leq S$.

(b) Every morphism in $\mathcal{F}$ factors as an isomorphism in $\mathcal{F}$ followed by an inclusion.

The saturation axioms in Definition 9.3.2 include now a third axiom modeling continuity.

(III) If $P_1 \leq P_2 \leq P_3 \leq \cdots$ is an increasing sequence of subgroups of $S$, with $P_\infty = \bigcup_{n=1}^{\infty} P_n$, and if $\phi \in \mathrm{Hom}(P_\infty, S)$ is any homomorphism such that $\phi_{|P_n} \in \mathrm{Hom}_{\mathcal{F}}(P_n, S)$ for all $n$, then $\phi \in \mathrm{Hom}_{\mathcal{F}}(P_\infty, S)$.

The inspiring example for this definition is the fusion system of a compact Lie group $G$. If $T$ is a maximal torus of $G$, let $W_p$ be the $p$-Sylow subgroup of the Weyl group $W_G(T) = N_G(T)/T$. Then, the preimage of $W_p$ in $N_G(T)$ defines a subgroup $N_p$ that is an extension of $W_p$ by $T$. The proof of Proposition 9.3 (b) in [26] shows that any compact Lie group $G$ has a maximal discrete $p$-toral subgroup $S$ which can be found as a discrete subgroup of $N_p$ and that it is unique up to $G$-conjugacy. The fusion system $\mathcal{F}_S(G)$ over $S$ defined by setting $\mathrm{Hom}_{\mathcal{F}_S(G)}(P, Q) = \mathrm{Hom}_G(P, Q)$ for all $P, Q \leq S$ is saturated.

There is also an axiomatic version of the linking system in this context in order to recover the classifying space. To do so, Broto, Levi, and Oliver in [28] axiomatized a new category $\mathcal{L}$, the centric linking system, containing the information needed to construct the classifying space. All this information together leads to the definition of a $p$-local compact group.

**Definition 9.5.4** (Broto–Levi–Oliver)**.** A $p$-*local compact group* is a triple $\mathcal{G} = (\mathcal{S}, \mathcal{F}, \mathcal{L})$ where $\mathcal{F}$ is a saturated fusion system over a discrete $p$-toral group $S$ and $\mathcal{L}$ is a centric linking system associated to $\mathcal{F}$. The classifying space $B\mathcal{G}$ is defined as $|\mathcal{L}|_p^\wedge$, the $p$-completion of the nerve of the associated centric linking system.

Recently, following ideas of the proof in the finite case [103], Levi–Libman [86] proved that there is a unique $p$-complete classifying space associated to a saturated fusion system over a discrete $p$-toral group. Hence, we will often denote the classifying space of a $p$-local compact group simply as $B\mathcal{F}$.

Examples of $p$-local compact groups described in [28] are compact Lie groups, $p$-compact groups, and torsion linear groups. More progress on constructing new examples can be found in the work of González-Lozano-Ruiz [73].

A relevant result which unifies some of the previous examples from the point of view of homotopy theory is the following one: $p$-local compact groups also model $p$-completions of classifying spaces of finite loop spaces [29].

**Theorem 9.5.5.** *Let $X$ be a connected space. For each prime $p$ such that $H^*(\Omega X)$ is finite, the space $X_p^\wedge$ has the homotopy type of the classifying space of a $p$-local compact group. In particular, as long as $H^*(\Omega X;\mathbb{Z})$ is finite (for example if $(\Omega X, X)$ is a finite loop space), the space $X_p^\wedge$ has the homotopy type of the classifying space of a $p$-local compact group.*

There are notions of centralizer and normalizer saturated fusion systems. In particular the descriptions of mapping spaces and homotopy classes of maps from $BQ$ where $Q$ is a discrete $p$-toral group in the spirit of Theorem 9.4.15 (see also [72]), and the existence of homology decompositions (see [28]) also hold.

A version of Alperin's fusion theorem is also available in this context.

**Definition 9.5.6.** Let $\mathcal{F}$ be a saturated fusion system on a discrete $p$-toral group $S$. A subgroup $P \leq S$ is called $\mathcal{F}$-*centric* if $P$ and all of its $\mathcal{F}$-conjugates contain their $S$-centralizers. A subgroup $Q \leq S$ is called $\mathcal{F}$-*radical* if $\mathrm{Out}_{\mathcal{F}}(Q)$ contains no nontrivial normal $p$-subgroup.

**Remark 9.5.7.** By Proposition 2.3 in [28], $\mathrm{Out}_{\mathcal{F}}(Q)$ is finite for all $Q \leq S$, so Definition 9.5.6 makes sense.

We will denote by $\mathcal{F}^c$ the full subcategory generated by $\mathcal{F}$ whose objects are the $\mathcal{F}$-centric subgroups of $S$ and by $\mathcal{F}^{cr}$ the full subcategory whose objects are the $\mathcal{F}$-centric $\mathcal{F}$-radical subgroups of $S$.

**Theorem 9.5.8** (Alperin's fusion theorem, [28])**.** *Let $\mathcal{F}$ be a saturated fusion system on a discrete $p$-toral group $S$. Then $\mathcal{F}$ is $\mathcal{F}^{cr}$-generated.*

The *orbit category* $\mathcal{O}(\mathcal{F})$ is the category whose objects are the subgroups of $S$ and whose morphisms are given by

$$\mathrm{Hom}_{\mathcal{O}(\mathcal{F})}(P,Q) = \mathrm{Rep}_{\mathcal{F}}(P,Q) = \mathrm{Hom}_{\mathcal{F}}(P,Q)/\,\mathrm{Inn}(Q).$$

For any full subcategory $\mathcal{F}_0$ of $\mathcal{F}$ we also consider $\mathcal{O}(\mathcal{F}_0)$, the full subcategory of $\mathcal{O}(\mathcal{F})$ whose objects are those of $\mathcal{F}_0$. Let $B\colon \mathcal{O}(\mathcal{F}_0) \to \mathrm{HoTop}$ be the classifying space functor regarded as a functor to the homotopy category of topological spaces.

**Proposition 9.5.9.** [28] *Fix a saturated fusion system $\mathcal{F}$ on a discrete $p$-toral group $S$, and let $\mathcal{F}' \subseteq \mathcal{F}^c$ be any full subcategory. Given a linking system $\mathcal{L}'$ associated to $\mathcal{F}'$, the left homotopy Kan extension $\widetilde{B}\colon \mathcal{O}(\mathcal{F}') \to \mathrm{Top}$ of the constant functor $\mathcal{L}' \to \mathrm{Top}$ along the projection $\widetilde{\pi}\colon \mathcal{L}' \to \mathcal{O}(\mathcal{F}')$ is a lift of $B$ to* Top, *and there is a homotopy equivalence:*

$$|\mathcal{L}'| \simeq \operatorname*{hocolim}_{\mathcal{O}(\mathcal{F}')} \widetilde{B}.$$

But there are some questions related to the key words in Section 9.1 which we haven't addressed: is $H^*(B\mathcal{F})$ finitely generated? is there a transfer map $\mathrm{Tr}\colon H^*(BS) \to H^*(B\mathcal{F})$? does the stable elements formula hold? is it true that $\mathrm{Spec}^h(H^*(B\mathcal{F}))$ is stratified by using elementary abelian $p$-subgroups?

The next result is a key statement. Note that the classifying space of a discrete $p$-toral group is a homotopy direct colimit of classifying spaces of finite $p$-groups. In [72, Theorem 1], González proves any $p$-local compact group can be approximated by a sequence of $p$-local finite groups in such a way that the homotopy colimit over the associated sequence of classifying spaces is equivalent to the classifying space after $p$-completion.

**Theorem 9.5.10.** *If $\mathcal{F}$ is the saturated fusion system on $S$ of a $p$-local compact group, then there is a commutative diagram*

$$\begin{array}{ccccccc}
\cdots & \longrightarrow & BS_i & \longrightarrow & BS_{i+1} & \longrightarrow & \cdots \\
 & & \downarrow & & \downarrow & & \\
\cdots & \longrightarrow & B\mathcal{F}_i & \longrightarrow & B\mathcal{F}_{i+1} & \longrightarrow & \cdots
\end{array}$$

*of classifying spaces of $p$-local finite groups $\mathcal{F}_i$ over $S_i$, indexed on $i \in \mathbb{N}$ whose colimits recovers the map $BS \to B\mathcal{F}$ after $p$-completion.*

In [72], many corollaries of the previous theorem are stated. Combining Theorem 9.5.10 with the stable elements theorem for the finite case, González proves the stable elements theorem in this context.

What about the remaining questions? They have positive answers from work of Barthel-Castellana-Heard-Valenzuela [12]. The $F$-isomorphim theorem with respect to elementary abelian $p$-subgroups and the strong Quillen's stratification hold ([12, Theorem 5.1, Theorem 5.6]), by using work of Rector on spaces with Noetherian cohomology [114]. Finally, a transfer map is constructed at the level of geometric cochains, the function spectrum $F(\Sigma^\infty X, H\mathbb{F}_p)$.

**Proposition 9.5.11.** *Let $\mathcal{F}$ be a saturated fusion system on a $p$-discrete toral group and $\theta\colon BS \to B\mathcal{F}$ the canonical map. Then, $\theta^*\colon H^*(B\mathcal{F}) \to H^*(BS)$ is split as a map of $H^*(B\mathcal{F})$-modules.*

**Corollary 9.5.12.** *Let $\mathcal{F}$ be a saturated fusion system on a $p$-discrete toral group. Then $H^*(B\mathcal{F})$ is finitely generated.*

The homotopy theory of maps between classifying spaces is less developed, as well as the stable homotopy theory. The transfer in mod $p$ cohomology is induced from a stable map between geometric cochains but there is no analogue of a stable transfer between classifying spaces satisfying Frobenius reciprocity, for example.

Concerning homotopy classes of maps between classifying spaces, there is no analogue of Theorem 9.4.18 in this more general context but one cannot expect the obstruction groups to be always torsion. A first attempt is in the work of Cantarero-Castellana [33] where they approach the problem of constructing maps to unitary groups $U(n)$. Faithful unitary representations correspond to homotopy monomorphisms from the classifying space $B\mathcal{F}$ into $BU(n)^\wedge_p$ ( the homotopy fiber is mod $p$ finite). The first nontrivial problem is to find fusion preserving faithful complex representations.

**Theorem 9.5.13.** *Let $\mathcal{F}$ be a saturated fusion system on a discrete p-toral group $S$. Then there exists a faithful complex representation $\rho \in \mathrm{Hom}(S, U(n))$ such that $[\rho] \in \lim_{\mathcal{O}(\mathcal{F})} R(P)$.*

The second problem is to formulate the obstruction theory. In that case, only partial results were obtained which apply under certain hypothesis on the depth of the orbit category which assure vanishing of the obstruction groups.

**Theorem 9.5.14.** *Let $(X, BX)$ be a finite loop space or a p-compact group. Then there exists faithful unitary representation $BX \to BU(n)^\wedge_p$ for some natural number $n \geq 1$.*

The stable homotopy theory described in 9.4.2 has yet to be developed in the context of p-local compact groups or saturated fusion systems on discrete $p$-toral groups.

## 9.6 Modular representation theory

Let $k$ be a field and $G$ a finite group. When attempting to classify $kG$-modules, there are very different situations. If the characteristic of the field $\mathrm{ch}(k)$ does not divide the order of $G$ then Maschke's theorem says that every $kG$-module is a direct sum of irreducible $kG$-modules. Moreover, there is a finite number of such, classified by their character describing completely the picture. In contrast, when $\mathrm{ch}(k)$ does divide the order of $G$, it is no longer true that $kG$-modules split in such a way (e.g. the regular representation $kG$ is not a direct sum of irreducible representations), and even non-isomorphic modules can have the same character. Only in special cases indecomposable can be classified, e.g. when $S \in \mathrm{Syl}_p(G)$ is cyclic or when $p = 2$ and $S = D_{2^n}, Q_{2^n}, SD_{2^n}$. When $\mathrm{ch}(k)$ divides $|G|$, the study of the category of $kG$-modules is called modular representation theory.

At this point one can adopt several strategies to pursue in modular representation theory: one can restrict attention to a particular class of modules that are more accessible or attempt a classification by means of a different organizational principle. We will give examples of both strategies: the classification of endotrivial modules and the stratification of the stable module category. In both cases, the fusion category will play a role to go from local to a global description.

### 9.6.1 Endotrivial modules

Let $\mathrm{Mod}_{kG}$ be the category of $kG$-modules where $k$ is a field such that $\mathrm{ch}(k)$ divides $|G|$. The stable module category $\mathrm{StMod}_{kG}$ is the category whose objects are $kG$-modules but given $M, N \in \mathrm{Mod}_{kG}$, $\mathrm{Hom}_{\mathrm{StMod}_{kG}}(M,N) = \mathrm{Hom}_{kG}(M,N)/P$ where $P$ is the linear span of morphism that factor through a projective.

**Definition 9.6.1.** An object $M \in \mathrm{StMod}_{kG}$ of finite dimension is called *endotrivial* if its endomorphism ring $\mathrm{End}_{kG}(M) \cong M \otimes M^*$ is equivalent in $\mathrm{StMod}_{kG}$ to the unit $k$, the one-dimensional trivial representation; equivalently if $M$ belongs to the Picard group of $\mathrm{StMod}_{kG}$.

Let $T_k(G)$ be the set of equivalence classes of endotrivial modules. The abelian group structure on $T_k(G)$ is induced by the tensor product of $kG$-modules. An example of an endotrivial module is given by the trivial one dimensional representation $k$ and $\Sigma^i k$ with $i \in \mathbb{Z}$.

A general structure theorem for $p$-groups $S$ due to Puig [106] proves that $T_k(S)$ is a finitely generated abelian group and Alperin [3] determined its rank. If $S$ is abelian, Dade [51] proved that $T_k(G)$ is a cyclic infinite group generated by $\Sigma^i k$. But what it is relevant is that if $S$ is a finite $p$-group, $T_k(S)$ was completely determined by work of Carlson-Thévenaz in [37], and [38].

Given a finite group $G$, there is a restriction map

$$\mathrm{Res}\colon T_k(G) \to T_k(S),$$

and the image of the restriction map was also determined by Carlson, Mazza, Nakano, Thévenaz ([92], [36], [35]). We define $T_k(G,S)$ to be the kernel of the morphism Res. Most of the efforts in this line of research have been then concentrated in computing $T_k(G,S)$. Carlson-Thévenaz [39] conjectured a description of it in terms of the fusion data, meaning local subgroups of $G$. In this direction, Balmer [11] established a connection to the equivariant topology of the Brown complex of $p$-subgroups of $G$.

Grodal [76] describes $T_k(G,S)$ in terms of the orbit category of $p$-subgroups in $G$. Let $\mathcal{O}_p^*(G)$ be the orbit category of non-trivial $p$-subgroups of $G$.

**Theorem 9.6.2.** [76] *Let $G$ be a finite group and $k$ a field of characteristic $p$ that divides the order of $G$. There is an isomorphism of abelian groups*

$$\Phi\colon T_k(G,S) \to H^1(|\mathcal{O}_p^*(G)|; k^\times).$$

The morphism $\Phi$ and its inverse are explicitly described in [76]. The morphism $\Phi$ is induced by considering, for each orbit $G/P$, the zeroth Tate cohomology $\hat{H}^0(P;M)$ which turns out to be a 1-dimensional $k$-module. Consequences and different descriptions in tems of $p$-local data are stated as a corollary of the previous result; in particular, an affirmative solution to the Carlson-Thévenaz conjecture. To emphasize the description in terms of $p$-local data, Grodal describes $H_1(|\mathcal{O}_p^*(G)|; k^\times)$ as an explicit quotient of $N_G(S)$. The power of Theorem 9.6.2 is that it is suitable for explicit computations of these groups.

### 9.6.2 Stratification and duality

The stable module category is a tensor triangulated category. Given a tensor triangulated category, an important problem is the classification of tensor triangulated localizing subcategories, i.e. tensor triangulated subcategories that are closed under all filtered colimits.[1] Neeman's result [99] on the classification of localizing subcategories of the derived category $D(R)$ of a commutative Noetherian ring $R$ in terms of subsets of the spectrum $\mathrm{Spec}(R)$ is such an example.

Benson-Iyengar-Krause [19] proved that if $G$ is a finite group and $k$ a field of characteristic $p$ dividing $|G|$, the set of localizing subcategories of the stable module category $\mathrm{StMod}_{kG}$ can be parametrized in terms of subsets of the spectrum $\mathrm{Spec}^h(H^*(G))$ of homogeneous prime ideals. The idea is to describe minimal localizing subcategories in terms of certain local cohomology functors for each prime ideal. In [18], the same authors developed a general theory for triangulated categories $T$ with an action of a graded commutative ring $R$, by constructing local cohomology functors and defining support functions $\mathrm{supp}_R$. When $R$ is Noetherian and $T$ is stratified there is a one-to-one correspondence between localizing subcategories of $T$ and subsets of $\mathrm{supp}_R(T) \subseteq \mathrm{Spec}^h(R)$.

The proof of the stratification of $\mathrm{StMod}_{kG}$ is based on a descent type argument which uses Quillen's strong stratification of the cohomology $H^*(G)$ and Chouinard's theorem to reduce the problem to elementary abelian $p$-groups.

**Theorem 9.6.3** (Chouinard's Theorem, [45])**.** *Let $R$ a ring, and $M$ an $RG$-module. Then, $M$ is projective iff the restriction of $M$ is projective as an $RE$-module for every elementary abelian $p$-subgroup $E \leq G$*

How can these statements on $kG$-modules be translated into a homotopy theoretic setting? Let $R$ be a commutative ring spectrum and $\mathrm{Mod}_R$ the category of module spectra over $R$. We write $C^*(X)$ for the ring spectrum of $H\mathbb{F}_p$-valued cochains on $X$, i.e. the function spectrum $F(\Sigma^\infty_+ X, H\mathbb{F}_p)$. Then, $C^*(X)$ is an augmented commutative $H\mathbb{F}_p$-algebra with multiplication given by the composite

$$\Sigma^\infty_+ X \xrightarrow{\Delta} \Sigma^\infty_+ X \wedge \Sigma^\infty_+ X \xrightarrow{x\wedge y} H\mathbb{F}_p \wedge H\mathbb{F}_p \xrightarrow{m} H\mathbb{F}_p$$

where the first map is induced by the diagonal of $X$ and the last map $m$ is the multiplication on the ring spectrum $H\mathbb{F}_p$. There is an isomorphism $\pi_* C^*(X) \cong H^{-*}(X)$.

Benson-Iyengar-Krause showed that stratification results also hold for several categories, one of which is $K(\mathrm{Inj}\, G)$, the homotopy category of complexes of injective modules. The authors define an equivalence of categories $\mathrm{StMod}_{kG} \overset{\simeq}{\to} K_{ac}(\mathrm{Inj}\, G) \subset K(\mathrm{Inj}\, G)$ where $K_{ac}(\mathrm{Inj}\, G)$ is the subcategory of acyclic complexes. And, precisely this category can be interpreted in terms of cochains on the classifying space. Benson-Krause [20] proved that there is a functor $K(\mathrm{Inj}\, G) \to \mathrm{Mod}_{C^*(BG)}$ that is an equivalence if $G$ is a finite $p$-group. From the homotopy theory point of view, one is led to investigate the category $\mathrm{Mod}_{C^*(BG)}$.

The case when $G$ compact connected Lie group was first described in the work of Benson-Greenlees [15]. Barthel-Castellana-Heard-Valenzuela [12] proved that the module category

[1] See Paul Balmer's chapter in this Handbook for a survey of classification results in the theory of tensor triangulated categories.

$\mathrm{Mod}_{C^*(B\mathcal{F})}$ is also stratified when $\mathcal{F}$ is a saturated fusion system over a finite $p$-group $S$, or $\mathcal{F} \cong \mathcal{F}_S(G)$ where $G$ is a compact Lie group or a $p$-compact group.

**Theorem 9.6.4.** ([15], [12]) *Let $\mathcal{F}$ be a saturated fusion on a finite $p$-group, or the saturated fusion system of Lie group or a $p$-compact group. The category $\mathrm{Mod}_{C^*(B\mathcal{F})}$ is stratified by the canonical action of $H^*(B\mathcal{F})$. In particular, there is a bijection*

$$\left\{ \begin{matrix} \textit{Localizing subcategories} \\ \textit{of } \mathrm{Mod}_{C^*(B\mathcal{F})} \end{matrix} \right\} \underset{\longleftarrow}{\xrightarrow{\text{supp}}} \left\{ \textit{Subsets of } \mathrm{Spec}^h(H^*(B\mathcal{F})) \right\}. \tag{9.6.1}$$

The proof of Theorem 9.6.4 follows a descent strategy where the role of elementary abelian $p$-subgroups is essential. Quillen's strong stratification Theorem 9.4.17 (see [12]) for $H^*(B\mathcal{F})$ and Chouinard's theorem are the key ingredients in the descent argument.

In general, if $f\colon R \to R'$ be a morphism of commutative ring spectra, then $R'$ is also an $R$-module via $f$. Forgetting along $f$ induces a restriction functor $\mathrm{Res}_f\colon \mathrm{Mod}_{R'} \to \mathrm{Mod}_R$ which admits both a left adjoint $\mathrm{Ind}_f$ and a right adjoint $\mathrm{CoInd}_f$, given by induction (or extension of scalars) along $f$,

$$\mathrm{Ind}_f = R' \otimes_R (-)\colon \mathrm{Mod}_R \longrightarrow \mathrm{Mod}_{R'},$$

and coinduction, defined as

$$\mathrm{CoInd}_f = \mathrm{Hom}_R(R', -)\colon \mathrm{Mod}_R \longrightarrow \mathrm{Mod}_{R'}.$$

Restriction detects equivalences, but that is not necessarily true for the induction and coinduction functors. A functor $F\colon \mathcal{C} \to \mathcal{D}$ is conservative if it reflects equivalences, that is, $f \in \mathrm{Mor}(\mathcal{C})$ is an equivalence if and only if $F(f) \in \mathrm{Mor}(\mathcal{D})$ is so.

**Definition 9.6.5.** A map $f\colon R \to R'$ of commutative ring spectra is called conservative (resp. coconservative) if the associated induction functor $\mathrm{Ind}_f\colon \mathrm{Mod}_R \to \mathrm{Mod}_{R'}$ (resp. coinduction $\mathrm{CoInd}_f$) is conservative. If a map is both conservative and coconservative, it will be called biconservative.

Consider

$$f := \prod \mathrm{Res}_{\mathcal{F},E}\colon C^*(B\mathcal{F}) \longrightarrow \prod_{E \in \mathcal{F}^e} C^*(BE)$$

where $C^*(B\mathcal{F}) \to C^*(BE)$ is induced by the inclusion $E \leq S$. Then Chouinard's theorem is about $f$ being conservative. Benson-Greenlees [15] proved that $f$ is biconservative when $G$ is a compact Lie group. In fact, because of the existence of a transfer map $C^*(BS) \to C^*(B\mathcal{F})$ of $C^*(B\mathcal{F})$-modules (see [12], Proposition 9.5.11), in general one only needs to check that that $f$ is biconservative when $B\mathcal{F} = BS$.

#### 9.6.2.1 Duality

We finish with an example which comes from duality properties of the cohomology rings of finite groups. Benson–Carlson duality for cohomology rings of finite groups [14] shows that if the mod $p$ cohomology ring of a finite group is Cohen–Macaulay, then it is Gorenstein.

One example of an explicit computation of the mod $p$ cohomology of the classifying space of a saturated fusion system is in [74], where the mod 2–cohomology rings of the exotic 2–local finite groups $\mathrm{Sol}(q)$ constructed in Levi–Oliver [87] are described. One can check from the explicit description that these rings are Gorenstein [34, Example 6.10], suggesting Benson–Carlson duality might hold for saturated fusion systems.

Dwyer, Greenlees and Iyengar [59] expressed Benson–Carlson duality, as well as other duality properties, in the framework of ring spectra, showing that it is a particular instance of a more general situation described at the level of cochains.

From this point of view, Benson–Carlson duality is a consequence of the fact that the augmentation map $C^*(BG) \to H\mathbb{F}_p$ is Gorenstein in the sense of [59, Definition 8.1] and the existence of a local cohomology spectral sequence. Under some technical assumptions, if $k$ is a field of characteristic $p$, we say that a map of ring spectra $R \to Hk$ is Gorenstein of shift $a$ if there is an equivalence $\mathrm{Hom}_R(Hk, R) \simeq \Sigma^a Hk$ of $Hk$-modules. In classical commutative algebra, we recall that a commutative Noetherian local ring $R$ is Gorenstein if and only if $\mathrm{Ext}^*_R(k, R)$ is a one dimensional $k$-vector space.

The proof that $C^*(BG; \mathbb{F}_p) \to H\mathbb{F}_p$ is Gorenstein follows the argument in [59, Example 10.3], where the main ingredient is the existence of a faithful unitary representation $G \hookrightarrow SU(n)$ such that $H^*(SU(n)/G)$ is an Poincaré duality $\mathbb{F}_p$-algebra.

The analogue for saturated fusion systems of the existence faithful unitary representation with Poincaré duality on the homogeneous space is the following result.

**Theorem 9.6.6.** [34] *Given a saturated fusion system $\mathcal{F}$ on a finite p-group, there exists a map $B\mathcal{F} \to BSU(n)^\wedge_p$ whose homotopy fiber has finite mod p cohomology satisfying Poincaré duality.*

The Gorenstein property then follows from a formal argument ([59, Example 10.3]).

**Theorem 9.6.7.** [34] *Let $\mathcal{F}$ be a saturated fusion system on a finite p-group. Then, the augmentation $C^*(B\mathcal{F}) \to H\mathbb{F}_p$ is Gorenstein. Moreover, if $H^*(B\mathcal{F})$ is Cohen-MacCaulay, then it is also Gorenstein.*

This is another example on how fusion systems provide a framework where cohomological and homotopical properties of classifying spaces of finite groups and Lie groups can be reinterpreted and viewed in a much broader context.

## Bibliography

[1] J. Aguadé. Constructing modular classifying spaces. *Israel J. Math.*, 66(1–3):23–40, 1989.

[2] J. L. Alperin. Sylow intersections and fusion. *J. Algebra*, 6:222–241, 1967.

[3] J. L. Alperin. A construction of endo-permutation modules. *J. Group Theory*, 4(1):3–10, 2001.

[4] K. K. S. Andersen, J. Grodal, J. M. Møller, and A. Viruel. The classification of $p$-compact groups for $p$ odd. *Ann. of Math. (2)*, 167(1):95–210, 2008.

[5] Kasper K. S. Andersen and Jesper Grodal. The classification of 2-compact groups. *J. Amer. Math. Soc.*, 22(2):387–436, 2009.

[6] Kasper K. S. Andersen, Bob Oliver, and Joana Ventura. Reduced, tame and exotic fusion systems. *Proc. Lond. Math. Soc. (3)*, 105(1):87–152, 2012.

[7] Michael Aschbacher. *Finite group theory*, volume 10 of *Cambridge Stud. Adv. Math.* Cambridge University Press, 1986.

[8] Michael Aschbacher. Normal subsystems of fusion systems. *Proc. Lond. Math. Soc. (3)*, 97(1):239–271, 2008.

[9] Michael Aschbacher and Andrew Chermak. A group-theoretic approach to a family of 2-local finite groups constructed by Levi and Oliver. *Ann. of Math. (2)*, 171(2):881–978, 2010.

[10] Michael Aschbacher, Radha Kessar, and Bob Oliver. *Fusion systems in algebra and topology*, volume 391 of *London Math. Soc. Lecture Note Ser.*. Cambridge University Press, 2011.

[11] Paul Balmer. Endotrivial representations of finite groups and equivariant line bundles on the Brown complex. *Geom. Topol.*, 22(7):4145–4161, 2018.

[12] Tobias Barthel, Natalia Castellana, Drew Heard, and Gabriel Valenzuela. Stratification and duality for homotopical groups. arXiv:1711.03491, 2017.

[13] David J. Benson. Cohomology of sporadic groups, finite loop spaces, and the Dickson invariants. In *Geometry and cohomology in group theory (Durham, 1994)*, volume 252 of *London Math. Soc. Lecture Note Ser.*, pages 10–23. Cambridge University Press, 1998.

[14] D. J. Benson and Jon F. Carlson. Projective resolutions and Poincaré duality complexes. *Trans. Amer. Math. Soc.*, 342(2):447–488, 1994.

[15] David Benson and John Greenlees. Stratifying the derived category of cochains on $BG$ for $G$ a compact Lie group. *J. Pure Appl. Algebra*, 218(4):642–650, 2014.

[16] David J. Benson, Jesper Grodal, and Ellen Henke. Group cohomology and control of $p$-fusion. *Invent. Math.*, 197(3):491–507, 2014.

[17] Dave Benson, Srikanth B. Iyengar, and Henning Krause. Localising subcategories for cochains on the classifying space of a finite group. *C. R. Math. Acad. Sci. Paris*, 349(17-18):953–956, 2011.

[18] Dave Benson, Srikanth B. Iyengar, and Henning Krause. Stratifying triangulated categories. *J. Topol.*, 4(3):641–666, 2011.

[19] David J. Benson, Srikanth B. Iyengar, and Henning Krause. Stratifying modular representations of finite groups. *Ann. of Math. (2)*, 174(3):1643–1684, 2011.

[20] David J. Benson and Henning Krause. Complexes of injective $kG$-modules. *Algebra Number Theory*, 2(1):1–30, 2008.

[21] A. K. Bousfield. On the homology spectral sequence of a cosimplicial space. *Amer. J. Math.*, 109(2):361–394, 1987.

[22] A. K. Bousfield and D. M. Kan. *Homotopy limits, completions and localizations.* volume 304 of *Lecture Notes in Math*, Springer-Verlag, 1972.

[23] Carles Broto, Natàlia Castellana, Jesper Grodal, Ran Levi, and Bob Oliver. Subgroup families controlling $p$-local finite groups. *Proc. London Math. Soc. (3)*, 91(2):325–354, 2005.

[24] C. Broto, N. Castellana, J. Grodal, R. Levi, and B. Oliver. Extensions of $p$-local finite groups. *Trans. Amer. Math. Soc.*, 359(8):3791–3858, 2007.

[25] Carles Broto, Ran Levi, and Bob Oliver. Homotopy equivalences of $p$-completed classifying spaces of finite groups. *Invent. Math.*, 151(3):611–664, 2003.

[26] Carles Broto, Ran Levi, and Bob Oliver. The homotopy theory of fusion systems. *J. Amer. Math. Soc.*, 16(4):779–856, 2003.

[27] Carles Broto, Ran Levi, and Bob Oliver. A geometric construction of saturated fusion systems. In *An alpine anthology of homotopy theory*, volume 399 of *Contemp. Math.*, pages 11–40. Amer. Math. Soc., 2006.

[28] Carles Broto, Ran Levi, and Bob Oliver. Discrete models for the $p$-local homotopy theory of compact Lie groups and $p$-compact groups. *Geom. Topol.*, 11:315–427, 2007.

[29] Carles Broto, Ran Levi, and Bob Oliver. An algebraic model for finite loop spaces. *Algebr. Geom. Topol.*, 14(5):2915–2981, 2014.

[30] Carles Broto, Jesper M. Møller, and Bob Oliver. Equivalences between fusion systems of finite groups of Lie type. *J. Amer. Math. Soc.*, 25(1):1–20, 2012.

[31] C. Broto and S. Zarati. Nil-localization of unstable algebras over the Steenrod algebra. *Math. Z.*, 199(4):525–537, 1988.

[32] W. Burnside. *Theory of groups of finite order.* Dover Publications, Inc., 1955. 2d ed.

[33] José Cantarero and Natàlia Castellana. Unitary embeddings of finite loop spaces. *Forum Math.*, 29(2):287–311, 2017.

[34] Jose Cantarero, Natalia Castellana, and Lola Morales. Vector bundles over classifying spaces of $p$-local finite groups and Benson-Carlson duality. arXiv:1701.08309, 2017.

[35] Jon F. Carlson, Nadia Mazza, and Daniel K. Nakano. Endotrivial modules for finite groups of Lie type $A$ in nondefining characteristic. *Math. Z.*, 282(1-2):1–24, 2016.

[36] Jon F. Carlson, Nadia Mazza, and Jacques Thévenaz. Endotrivial modules over groups with quaternion or semi-dihedral Sylow 2-subgroup. *J. Eur. Math. Soc. (JEMS)*, 15(1):157–177, 2013.

[37] Jon F. Carlson and Jacques Thévenaz. The classification of endo-trivial modules. *Invent. Math.*, 158(2):389–411, 2004.

[38] Jon F. Carlson and Jacques Thévenaz. The classification of torsion endo-trivial modules. *Ann. of Math. (2)*, 162(2):823–883, 2005.

[39] Jon F. Carlson and Jacques Thévenaz. The torsion group of endotrivial modules. *Algebra Number Theory*, 9(3):749–765, 2015.

[40] Gunnar Carlsson. Equivariant stable homotopy and Segal's Burnside ring conjecture. *Ann. of Math. (2)*, 120(2):189–224, 1984.

[41] Henri Cartan and Samuel Eilenberg. *Homological algebra*. Princeton University Press, 1956.

[42] Natàlia Castellana and Assaf Libman. Wreath products and representations of $p$-local finite groups. *Adv. Math.*, 221(4):1302–1344, 2009.

[43] Andrew Chermak. Fusion systems and localities. *Acta Math.*, 211(1):47–139, 2013.

[44] Andy Chermak and Alex González. Discrete localities. I. arXiv:1702.02595, 2017.

[45] Leo G. Chouinard. Projectivity and relative projectivity over group rings. *J. Pure Appl. Algebra*, 7(3):287–302, 1976.

[46] Mónica Clapp. Duality and transfer for parametrized spectra. *Arch. Math. (Basel)*, 37(5):462–472, 1981.

[47] Murray Clelland and Christopher Parker. Two families of exotic fusion systems. *J. Algebra*, 323(2):287–304, 2010.

[48] David A. Craven. Normal subsystems of fusion systems. *J. Lond. Math. Soc. (2)*, 84(1):137–158, 2011.

[49] David A. Craven. *The theory of fusion systems: An algebraic approach*, volume 131 of *Cambridge Stud. Adv. Math.*. Cambridge University Press, 2011.

[50] Everett C. Dade. Endo-permutation modules over $p$-groups. I. *Ann. of Math. (2)*, 107(3):459–494, 1978.

[51] Everett Dade. Endo-permutation modules over $p$-groups. II. *Ann. of Math. (2)*, 108(2):317–346, 1978.

[52] Antonio Díaz and Assaf Libman. The Burnside ring of fusion systems. *Adv. Math.*, 222(6):1943–1963, 2009.

[53] Antonio Díaz and Assaf Libman. Segal's conjecture and the Burnside rings of fusion systems. *J. Lond. Math. Soc. (2)*, 80(3):665–679, 2009.

[54] Antonio Díaz and Sejong Park. Mackey functors and sharpness for fusion systems. *Homology Homotopy Appl.*, 17(1):147–164, 2015.

[55] Antonio Díaz, Albert Ruiz, and Antonio Viruel. All $p$-local finite groups of rank two for odd prime $p$. *Trans. Amer. Math. Soc.*, 359(4):1725–1764, 2007.

[56] W. G. Dwyer. The centralizer decomposition of $BG$. In *Algebraic topology: new trends in localization and periodicity (Sant Feliu de Guíxols, 1994)*, volume 136 of *Progr. Math.*, pages 167–184. Birkhäuser, 1996.

[57] W. G. Dwyer. Homology decompositions for classifying spaces of finite groups. *Topology*, 36(4):783–804, 1997.

[58] W. G. Dwyer. Sharp homology decompositions for classifying spaces of finite groups. In *Group representations: cohomology, group actions and topology (Seattle, WA, 1996)*, volume 63 of *Proc. Sympos. Pure Math.*, pages 197–220. Amer. Math. Soc., 1998.

[59] W. G. Dwyer, J. P. C. Greenlees, and S. Iyengar. Duality in algebra and topology. *Adv. Math.*, 200(2):357–402, 2006.

[60] William G. Dwyer, Haynes R. Miller, and Clarence W. Wilkerson. The homotopic uniqueness of $BS^3$. In *Algebraic topology, Barcelona, 1986*, volume 1298 of *Lecture Notes in Math.*, pages 90–105. Springer, 1987.

[61] W. G. Dwyer and C. W. Wilkerson. A new finite loop space at the prime two. *J. Amer. Math. Soc.* 6(1):37–64, 2993.

[62] W. G. Dwyer and C. W. Wilkerson. Homotopy fixed-point methods for Lie groups and finite loop spaces. *Ann. of Math. (2)*, 139(2):395–442, 1994.

[63] W. G. Dwyer and C. W. Wilkerson. The fundamental group of a $p$-compact group. *Bull. Lond. Math. Soc.*, 41(3):385–395, 2009.

[64] W. Dwyer and A. Zabrodsky. Maps between classifying spaces. In *Algebraic topology, Barcelona, 1986*, volume 1298 of *Lecture Notes in Math.*, pages 106–119. Springer, 1987.

[65] Leonard Evens. The cohomology ring of a finite group. *Trans. Amer. Math. Soc.*, 101:224–239, 1961.

[66] Matthew Gelvin and Sune Precht Reeh. Minimal characteristic bisets for fusion systems. *J. Algebra*, 427:345–374, 2015.

[67] Matthew Gelvin, Sune Precht Reeh, and Ergün Yalçin. On the basis of the Burnside ring of a fusion system. *J. Algebra*, 423:767–797, 2015.

[68] George Glauberman and Justin Lynd. Control of fixed points and existence and uniqueness of centric linking systems. *Invent. Math.*, 206(2):441–484, 2016.

[69] David M. Goldschmidt. A conjugation family for finite groups. *J. Algebra*, 16:138–142, 1970.

[70] E. Golod. The cohomology ring of a finite $p$-group. *Dokl. Akad. Nauk SSSR*, 125:703–706, 1959.

[71] Alex González. Extensions of partial groups and localities. arXiv:1507.04392, 2015.

[72] Alex Gonzalez. Finite approximations of $p$-local compact groups. *Geom. Topol.*, 20(5):2923–2995, 2016.

[73] Alex Gonzalez, Toni Lozano, and Albert Ruiz. Some new examples of simple $p$-local compact groups. arXiv:1512.00284, 2015.

[74] Jelena Grbić. The cohomology of exotic 2-local finite groups. *Manuscripta Math.*, 120(3):307–318, 2006.

[75] Jesper Grodal. Higher limits via subgroup complexes. *Ann. of Math. (2)*, 155(2):405–457, 2002.

[76] Jesper Grodal. Endotrivial modules for finite groups via homotopy theory. arXiv:1608.00499, 2016.

[77] Ellen Henke. Subcentric linking systems. *Trans. Amer. Math. Soc.*, 371(5):3325–3373, 2019.

[78] Stefan Jackowski and James McClure. Homotopy decomposition of classifying spaces via elementary abelian subgroups. *Topology*, 31(1):113–132, 1992.

[79] Stefan Jackowski, James McClure, and Bob Oliver. Homotopy classification of self-maps of $BG$ via $G$-actions. I. *Ann. of Math. (2)*, 135(1):183–226, 1992.

[80] Stefan Jackowski, James McClure, and Bob Oliver. Homotopy classification of self-maps of $BG$ via $G$-actions. II. *Ann. of Math. (2)*, 135(2):227–270, 1992.

[81] Stefan Jackowski, James McClure, and Bob Oliver. Self-homotopy equivalences of classifying spaces of compact connected Lie groups. *Fund. Math.*, 147(2):99–126, 1995.

[82] Stefan Jackowski and Bob Oliver. Vector bundles over classifying spaces of compact Lie groups. *Acta Math.*, 176(1):109–143, 1996.

[83] Radha Kessar and Radu Stancu. A reduction theorem for fusion systems of blocks. *J. Algebra*, 319(2):806–823, 2008.

[84] Jean Lannes. Sur les espaces fonctionnels dont la source est le classifiant d'un $p$-groupe abélien élémentaire. *Inst. Hautes Études Sci. Publ. Math.*, (75):135–244, 1992. With an appendix by Michel Zisman.

[85] Ian J. Leary and Radu Stancu. Realising fusion systems. *Algebra Number Theory*, 1(1):17–34, 2007.

[86] Ran Levi and Assaf Libman. Existence and uniqueness of classifying spaces for fusion systems over discrete $p$-toral groups. *J. Lond. Math. Soc. (2)*, 91(1):47–70, 2015.

[87] Ran Levi and Bob Oliver. Construction of 2-local finite groups of a type studied by Solomon and Benson. *Geom. Topol.*, 6:917–990, 2002.

[88] Ran Levi and Kári Ragnarsson. $p$-local finite group cohomology. *Homology Homotopy Appl.*, 13(1):223–257, 2011.

[89] L. G. Lewis, J. P. May, and J. E. McClure. Classifying $G$-spaces and the Segal conjecture. In *Current trends in algebraic topology, Part 2 (London, Ont., 1981)*, volume 2 of *CMS Conf. Proc.*, pages 165–179. Amer. Math. Soc., 1982.

[90] Markus Linckelmann. Simple fusion systems and the Solomon 2-local groups. *J. Algebra*, 296(2):385–401, 2006.

[91] Markus Linckelmann. Quillen's stratification for fusion systems. *Comm. Algebra*, 45(12):5227–5229, 2017.

[92] Nadia Mazza and Jacques Thévenaz. Endotrivial modules in the cyclic case. *Arch. Math. (Basel)*, 89(6):497–503, 2007.

[93] Haynes Miller. The Sullivan conjecture on maps from classifying spaces. *Ann. of Math. (2)*, 120(1):39–87, 1984.

[94] Guido Mislin. On group homomorphisms inducing mod-$p$ cohomology isomorphisms. *Comment. Math. Helv.*, 65(3):454–461, 1990.

[95] Rémi Molinier. Cohomology with twisted coefficients of the classifying space of a fusion system. *Topology Appl.*, 212:1–18, 2016.

[96] Rémi Molinier. Cohomology of linking systems with twisted coefficients by a $p$-solvable action. *Homology Homotopy Appl.*, 19(2):61–82, 2017.

[97] Rémi Molinier. Alperin's fusion theorem for localities. *Comm. Algebra*, 46(6):2615–2619, 2018.

[98] Rémi Molinier. Control of fixed points over discrete $p$-toral groups, and existence and uniqueness of linking systems. *J. Algebra*, 499:43–73, 2018.

[99] Amnon Neeman. The chromatic tower for $D(R)$. *Topology*, 31(3):519–532, 1992. With an appendix by Marcel Bökstedt.

[100] Bob Oliver. Equivalences of classifying spaces completed at odd primes. *Math. Proc. Cambridge Philos. Soc.*, 137(2):321–347, 2004.

[101] Bob Oliver. Equivalences of classifying spaces completed at the prime two. *Mem. Amer. Math. Soc.*, 180(848),2006.

[102] Bob Oliver. Extensions of linking systems and fusion systems. *Trans. Amer. Math. Soc.*, 362(10):5483–5500, 2010.

[103] Bob Oliver. Existence and uniqueness of linking systems: Chermak's proof via obstruction theory. *Acta Math.*, 211(1):141–175, 2013.

[104] Bob Oliver and Joana Ventura. Extensions of linking systems with $p$-group kernel. *Math. Ann.*, 338(4):983–1043, 2007.

[105] Sejong Park. Realizing fusion systems inside finite groups. *Proc. Amer. Math. Soc.*, 144(8):3291–3294, 2016.

[106] Lluís Puig. Affirmative answer to a question of Feit. *J. Algebra*, 131(2):513–526, 1990.

[107] Lluis Puig. Frobenius categories. *J. Algebra*, 303(1):309–357, 2006.

[108] Luis Puig. Structure locale dans les groupes finis. *Bull. Soc. Math. France Suppl. Mém.*, (47):132, 1976.

[109] Daniel Quillen. A cohomological criterion for $p$-nilpotence. *J. Pure Appl. Algebra*, 1(4):361–372, 1971.

[110] Daniel Quillen. The spectrum of an equivariant cohomology ring. I, II. *Ann. of Math. (2)*, 94:549–572; ibid. (2) 94 (1971), 573–602, 1971.

[111] D. Quillen and B. B. Venkov. Cohomology of finite groups and elementary abelian subgroups. *Topology*, 11:317–318, 1972.

[112] Kári Ragnarsson. Classifying spectra of saturated fusion systems. *Algebr. Geom. Topol.*, 6:195–252, 2006.

[113] Kári Ragnarsson and Radu Stancu. Saturated fusion systems as idempotents in the double Burnside ring. *Geom. Topol.*, 17(2):839–904, 2013.

[114] D. L. Rector. Noetherian cohomology rings and finite loop spaces with torsion. *J. Pure Appl. Algebra*, 32(2):191–217, 1984.

[115] David L. Rector. Loop structures on the homotopy type of $S^3$. In *Symposium on Algebraic Topology (Battelle Seattle Res. Center, Seattle, Wash., 1971)*, volume 249 of *Lecture Notes in Math.*, pages 99–105, Springer, 1971.

[116] Sune Precht Reeh. The abelian monoid of fusion-stable finite sets is free. *Algebra Number Theory*, 9(10):2303–2324, 2015.

[117] Sune Precht Reeh. Transfer and characteristic idempotents for saturated fusion systems. *Adv. Math.*, 289:161–211, 2016.

[118] Sune Precht Reeh and Ergün Yalçın. Representation rings for fusion systems and dimension functions. *Math. Z.*, 288(1-2):509–530, 2018.

[119] K. Roberts and S. Shpectorov. On the definition of saturated fusion systems. *J. Group Theory*, 12(5):679–687, 2009.

[120] Geoffrey R. Robinson. Amalgams, blocks, weights, fusion systems and finite simple groups. *J. Algebra*, 314(2):912–923, 2007.

[121] Albert Ruiz. Exotic normal fusion subsystems of general linear groups. *J. Lond. Math. Soc. (2)*, 76(1):181–196, 2007.

[122] Albert Ruiz and Antonio Viruel. The classification of $p$-local finite groups over the extraspecial group of order $p^3$ and exponent $p$. *Math. Z.*, 248(1):45–65, 2004.

[123] Ronald Solomon. Finite groups with Sylow 2-subgroups of type 3. *J. Algebra*, 28:182–198, 1974.

[124] B. B. Venkov. Cohomology algebras for some classifying spaces. *Dokl. Akad. Nauk SSSR*, 127:943–944, 1959.

[125] Zdzisław Wojtkowiak. On maps from $\operatorname{ho}\varinjlim F$ to $\mathbf{Z}$. In *Algebraic topology, Barcelona, 1986*, volume 1298 of *Lecture Notes in Math.*, pages 227–236. Springer, 1987.

Departament de Matematiques, Universitat Autònoma de Barcelona, Bellaterra, Spain

*E-mail address*: natalia@mat.uab.cat