

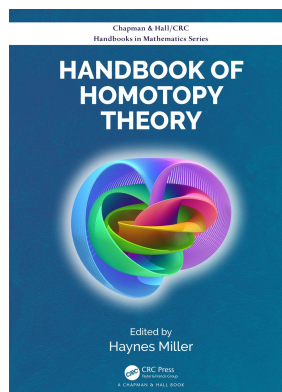
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Publisher: *CRC Press*

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Handbook of Homotopy Theory

Haynes Miller

Assembly maps

Publication details

<https://www.routledgehandbooks.com/doi/10.1201/9781351251624-20>

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Published online on: 18 Dec 2019

How to cite :- Wolfgang Lück. 18 Dec 2019, *Assembly maps from:* Handbook of Homotopy Theory
CRC Press

Accessed on: 04 Jun 2023

<https://www.routledgehandbooks.com/doi/10.1201/9781351251624-20>

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20

Assembly maps

Wolfgang Lück

20.1 Introduction

20.1.1 The homotopy theoretic description of assembly maps

The quickest and probably for a homotopy theorist most convenient approach to assembly maps is via homotopy colimits as explained in Subsection 20.7.3. Let \mathcal{F} be a family of subgroups of G , i.e., a collection of subgroups closed under conjugation and passing to subgroups. Let $\text{Or}(G)$ be the orbit category and $\text{Or}_{\mathcal{F}}(G)$ be the full subcategory consisting of objects G/H satisfying $H \in \mathcal{F}$. Consider a covariant functor $\mathbf{E}^G: \text{Or}(G) \rightarrow \text{Spectra}$ to the category of spectra. We get from the inclusion $\text{Or}_{\mathcal{F}}(G) \rightarrow \text{Or}(G)$ and the fact that G/G is a terminal object in $\text{Or}(G)$ a map

$$\text{hocolim}_{\text{Or}_{\mathcal{F}}(G)} \mathbf{E}^G|_{\text{Or}_{\mathcal{F}}(G)} \rightarrow \text{hocolim}_{\text{Or}(G)} \mathbf{E}^G = \mathbf{E}^G(G/G). \quad (20.1.1)$$

It is called *assembly map* since we are trying to assemble the values of \mathbf{E}^G on homogeneous spaces G/H for $H \in \mathcal{F}$ to get $\mathbf{E}(G/G)$.

On homotopy groups this assembly map can also be described as the map

$$H_n^G(\text{pr}; \mathbf{E}^G): H_n^G(E_{\mathcal{F}}(G); \mathbf{E}^G) \rightarrow H_n^G(G/G; \mathbf{E}^G) = \pi_n(\mathbf{E}^G(G/G)) \quad (20.1.2)$$

induced by the projection $\text{pr}: E_{\mathcal{F}}(G) \rightarrow G/G$ of the classifying G -space $E_{\mathcal{F}}(G)$ for the family \mathcal{F} , see Section 20.6, to G/G , where $H_n^G(-; \mathbf{E}^G)$ is the G -homology theory in the sense of Definition 20.3.1 associated to \mathbf{E}^G , see Lemma 20.3.4.

In all interesting situations one can take a global point of view. Namely, one starts with a covariant functor respecting equivalences $\mathbf{E}: \text{Groupoids} \rightarrow \text{Spectra}$ and defines for a group G the functor \mathbf{E}^G to be the composite of \mathbf{E} with the functor $\text{Or}(G) \rightarrow \text{Groupoids}$ given by the transport groupoid of a G -set, see Subsection 20.7.4.

20.1.2 Isomorphism Conjectures

The Meta Isomorphism Conjecture for G , \mathcal{F} and \mathbf{E}^G , see Section 20.7, says that the assembly map of (20.1.1) is a weak homotopy equivalence, or, equivalently, that the map (20.1.2) is bijective for all $n \in \mathbb{Z}$.

2010 *Mathematics Subject Classification*. primary: 18F25, secondary: 46L80, 55P91, 57N99.

Key words and phrases. assembly maps, equivariant homology and homotopy theory, Farrell-Jones Conjecture, Baum-Connes Conjecture.

If we take for \mathbf{E} an appropriate functor modelling the algebraic K -theory or the algebraic L -theory with decoration $\langle -\infty \rangle$ of the group ring RG and for \mathcal{F} the family of virtually cyclic subgroups, we obtain the Farrell-Jones Conjecture 20.8.1. It assembles $K_n(RG)$ and $L_n^{\langle -\infty \rangle}(RG)$ in terms of $K_n(RH)$ and $L_n^{\langle -\infty \rangle}(RH)$, where H runs through the virtually cyclic subgroups of G .

If we take for \mathbf{E} an appropriate functor modelling the topological K -theory of the reduced group C^* -algebra $C_r^*(G)$ and for \mathcal{F} the family of finite subgroups, we obtain the Baum-Connes Conjecture 20.9.1. It assembles $K_n^{\text{top}}(C_r^*(G))$ in terms of $K_n^{\text{top}}(C_r^*(H))$, where H runs through the finite subgroups of G .

The Farrell-Jones Conjecture 20.8.1 and the Baum-Connes Conjecture 20.9.1 are very powerful conjectures and are the main motivation for the study of assembly maps. A survey of a lot of striking applications such as the ones to the conjectures of Bass, Borel, Gromov-Lawson-Rosenberg, Kadison, Kaplansky, and Novikov is given in Subsections 20.8.2 and 20.9.4. The Farrell-Jones Conjecture 20.8.1 and the Baum-Connes Conjecture 20.9.1 are known to be true for a surprisingly large class of groups, as explained in Subsections 20.8.7 and 20.9.5. All of this is an impressive example how homotopy theoretic methods can be used for problems in other fields such as algebra, geometry, manifold theory and operator algebras.

20.1.3 Other interpretations of assembly maps

The homotopy theoretic approach is the best for structural purposes. The applications and the proofs of the Farrell-Jones Conjecture 20.8.1 and the Baum-Connes Conjecture 20.9.1 require sophisticated analytic, topological and geometric interpretations of the homotopy theoretic assembly maps, for instance in terms of surgery theory, see Subsection 20.8.3, as forget control maps, see Subsection 20.8.4, and in terms of index theory, see Subsection 20.9.3. This presents an intriguing interaction between homotopy theory, geometry and operator theory.

20.1.4 The universal property of the assembly map

In Section 20.5 we characterize the assembly map in the sense that it is the universal approximation from the left by an excisive functor of a given homotopy invariant functor $G\text{-CW}^2 \rightarrow \text{Spectra}$. This is the key ingredient in the difficult identification of the various assembly maps mentioned in Subsection 20.1.3 above. It reflects the fact that in all of the Isomorphism Conjectures the hard and interesting object is the target and the source is given by the G -homology of the classifying G -spaces for a specific family of subgroups. The source is more accessible than the target since one can apply standard methods from algebraic topology such as spectral sequences and equivariant Chern characters.

20.1.5 Relative assembly maps

Relative assembly maps are studied in Section 20.12. They address the problem to make the families appearing in the various Isomorphism Conjectures as small as possible.

20.1.6 Further aspects of assembly maps

The homotopy theoretic approach to assembly allows to relate assembly maps for various theories, such as the algebraic K -theory and A -theory via linearization, see Subsection 20.8.6, algebraic K -theory of groups rings and topological K -theory of reduced group C^* -algebras, see Subsection 20.9.6, and algebraic K -theory of groups rings and the topological cyclic homology of the spherical group ring via cyclotomic traces, see Subsection 20.10.3. How assembly maps can be used for computations is illustrated in Section 20.13 which is based on the global point of view described in Section 20.11. Finally we formulate the challenge of extending equivariant homotopy theory for finite groups to infinite groups in Section 20.14.

The idea of the geometric assembly map is due to Quinn [101, 102] and its algebraic counterpart was introduced by Ranicki [104]. See also Loday [68].

20.1.7 Conventions

Throughout this paper G denotes a (discrete) group. Ring means associative ring with unit. All spectra are non-connective.

Acknowledgements

This paper is dedicated to Andrew Ranicki. It is financially supported by the ERC Advanced Grant “KL2MG-interactions” (no. 662400) of the author granted by the European Research Council, and by the Cluster of Excellence “Hausdorff Center for Mathematics” at Bonn.

20.2 Some basic categories

In this section we recall some well-known basic categories.

20.2.1 G -CW-complexes

Definition 20.2.1 (G -CW-complex). A G -CW-complex X is a G -space together with a G -invariant filtration

$$\emptyset = X_{-1} \subseteq X_0 \subseteq X_1 \subseteq \dots \subseteq X_n \subseteq \dots \subseteq \bigcup_{n \geq 0} X_n = X$$

such that X carries the colimit topology with respect to this filtration (i.e., a set $C \subseteq X$ is closed if and only if $C \cap X_n$ is closed in X_n for all $n \geq 0$) and X_n is obtained from X_{n-1}

for each $n \geq 0$ by attaching equivariant n -dimensional cells, i.e., there exists a G -pushout

$$\begin{array}{ccc} \coprod_{i \in I_n} G/H_i \times S^{n-1} & \xrightarrow{\coprod_{i \in I_n} q_i^n} & X_{n-1} \\ \downarrow & & \downarrow \\ \coprod_{i \in I_n} G/H_i \times D^n & \xrightarrow{\coprod_{i \in I_n} Q_i^n} & X_n. \end{array}$$

A map $f: X \rightarrow Y$ between G -CW-complexes is called *cellular* if $f(X_n) \subseteq Y_n$ holds for all $n \geq 0$. We denote by G -CW the category of G -CW-complexes with cellular G -maps as morphisms and by G -CW² the corresponding category of G -CW-pairs. For basic information about G -CW-complexes we refer for instance to [69, Chapter 1 and 2].

20.2.2 The orbit category

The orbit category $\text{Or}(G)$ has as objects homogeneous spaces G/H and as morphisms G -maps. It can be viewed as the category of 0-dimensional G -CW-complexes, whose G -quotient space is connected. In particular we can think of $\text{Or}(G)$ as a full subcategory of G -CW.

20.2.3 Spectra

In this paper we can work with the most elementary category Spectra of spectra. A *spectrum* $\mathbf{E} = \{(E(n), \sigma(n)) \mid n \in \mathbb{Z}\}$ is a sequence of pointed spaces $\{E(n) \mid n \in \mathbb{Z}\}$ together with pointed maps called *structure maps* $\sigma(n): E(n) \wedge S^1 \rightarrow E(n+1)$. A *map of spectra* $\mathbf{f}: \mathbf{E} \rightarrow \mathbf{E}'$ is a sequence of maps $f(n): E(n) \rightarrow E'(n)$ that are compatible with the structure maps $\sigma(n)$, i.e., we have $f(n+1) \circ \sigma(n) = \sigma'(n) \circ (f(n) \wedge \text{id}_{S^1})$ for all $n \in \mathbb{Z}$. Maps of spectra are sometimes called functions in the literature, they should not be confused with the notion of a map of spectra in the stable category, see [1, III.2.].

The homotopy groups of a spectrum are defined by

$$\pi_i(\mathbf{E}) := \text{colim}_{k \rightarrow \infty} \pi_{i+k}(E(k)), \quad (20.2.1)$$

where the i th structure map of the system $\pi_{i+k}(E(k))$ is given by the composite

$$\pi_{i+k}(E(k)) \xrightarrow{S} \pi_{i+k+1}(E(k) \wedge S^1) \xrightarrow{\sigma^{(k)*}} \pi_{i+k+1}(E(k+1))$$

of the suspension homomorphism S and the homomorphism induced by the structure map. A *weak equivalence* of spectra is a map $\mathbf{f}: \mathbf{E} \rightarrow \mathbf{F}$ of spectra inducing an isomorphism on all homotopy groups.

20.3 G -homology theories and $\text{Or}(G)$ -spectra

Let Λ be a commutative ring. Next we recall the obvious generalization of the notion of a (generalized) homology theory to a G -homology theory.

Definition 20.3.1 (*G*-homology theory). A *G*-homology theory \mathcal{H}_*^G with values in Λ -modules is a collection of covariant functors \mathcal{H}_n^G from the category $G\text{-CW}^2$ of *G*-CW-pairs to the category of Λ -modules indexed by $n \in \mathbb{Z}$ together with natural transformations

$$\partial_n^G(X, A): \mathcal{H}_n^G(X, A) \rightarrow \mathcal{H}_{n-1}^G(A) := \mathcal{H}_{n-1}^G(A, \emptyset)$$

for $n \in \mathbb{Z}$ such that the following axioms are satisfied:

- *G*-homotopy invariance

If f_0 and f_1 are *G*-homotopic *G*-maps of *G*-CW-pairs $(X, A) \rightarrow (Y, B)$, then $\mathcal{H}_n^G(f_0) = \mathcal{H}_n^G(f_1)$ for $n \in \mathbb{Z}$;

- Long exact sequence of a pair

Given a pair (X, A) of *G*-CW-complexes, there is a long exact sequence

$$\dots \xrightarrow{\mathcal{H}_{n+1}^G(j)} \mathcal{H}_{n+1}^G(X, A) \xrightarrow{\partial_{n+1}^G} \mathcal{H}_n^G(A) \xrightarrow{\mathcal{H}_n^G(i)} \mathcal{H}_n^G(X) \xrightarrow{\mathcal{H}_n^G(j)} \mathcal{H}_n^G(X, A) \xrightarrow{\partial_n^G} \dots,$$

where $i: A \rightarrow X$ and $j: X \rightarrow (X, A)$ are the inclusions;

- Excision

Let (X, A) be a *G*-CW-pair and let $f: A \rightarrow B$ be a cellular *G*-map of *G*-CW-complexes. Equip $(X \cup_f B, B)$ with the induced structure of a *G*-CW-pair. Then the canonical map $(F, f): (X, A) \rightarrow (X \cup_f B, B)$ induces an isomorphism

$$\mathcal{H}_n^G(F, f): \mathcal{H}_n^G(X, A) \xrightarrow{\cong} \mathcal{H}_n^G(X \cup_f B, B);$$

- Disjoint union axiom

Let $\{X_i \mid i \in I\}$ be a family of *G*-CW-complexes. Denote by $j_i: X_i \rightarrow \coprod_{i \in I} X_i$ the canonical inclusion. Then the map

$$\bigoplus_{i \in I} \mathcal{H}_n^G(j_i): \bigoplus_{i \in I} \mathcal{H}_n^G(X_i) \xrightarrow{\cong} \mathcal{H}_n^G\left(\coprod_{i \in I} X_i\right)$$

is bijective.

If \mathbf{E} is a spectrum, then one gets a (non-equivariant) homology theory $H_*(-; \mathbf{E})$ by defining

$$H_n(X, A; \mathbf{E}) = \pi_n((X_+ \cup_{A_+} \text{cone}(A_+)) \wedge \mathbf{E})$$

for a *CW*-pair (X, A) and $n \in \mathbb{Z}$, where X_+ is obtained from X by adding a disjoint base point and cone denotes the (reduced) mapping cone. Its main property is $H_n(\{\bullet\}; \mathbf{E}) = \pi_n(\mathbf{E})$. This extends to *G*-homology theories as follows. Since the building blocks of *G*-spaces are homogeneous spaces, we will have to consider a covariant $\text{Or}(G)$ -spectrum, i.e., a covariant functor $\mathbf{E}^G: \text{Or}(G) \rightarrow \text{Spectra}$, instead of a spectrum.

Definition 20.3.2 (Excisive). We call a covariant functor

$$\mathbf{E}: G\text{-CW}^2 \rightarrow \text{Spectra}$$

homotopy invariant if it sends G -homotopy equivalences to weak homotopy equivalences of spectra.

The functor \mathbf{E} is *excisive* if it has the following four properties:

- It is homotopy invariant;
- The spectrum $\mathbf{E}(\emptyset)$ is weakly contractible;
- It respects homotopy pushouts up to weak homotopy equivalence, i.e., if the G -CW-complex X is the union of G -CW-subcomplexes X_1 and X_2 with intersection X_0 , then the canonical map from the homotopy pushout of $\mathbf{E}(X_2) \leftarrow \mathbf{E}(X_0) \rightarrow \mathbf{E}(X_1)$ to $\mathbf{E}(X)$ is a weak homotopy equivalence of spectra;
- It respects disjoint unions up to weak homotopy, i.e., the natural map $\bigvee_{i \in I} \mathbf{E}(X_i) \rightarrow \mathbf{E}(\bigsqcup_{i \in I} X_i)$ is a weak homotopy equivalence for all index sets I .

One easily checks

Lemma 20.3.3. *Suppose that the covariant functor $\mathbf{E}: G\text{-CW}^2 \rightarrow \text{Spectra}$ is excisive. Then we obtain a G -homology theory with values in \mathbb{Z} -modules by assigning to G -CW-pair (X, A) and $n \in \mathbb{Z}$ the abelian group $\pi_n(\mathbf{E}(X, A))$.*

A G -space X defines a contravariant $\text{Or}(G)$ -space $O^G(X)$ by sending G/H to $\text{map}_G(G/H, X) = X^H$. Given a contravariant pointed $\text{Or}(G)$ -space Y and a covariant pointed $\text{Or}(G)$ -space Z , there is the pointed space $Y \wedge_{\text{Or}(G)} Z$. Its construction is explained for instance in [26, Section 1]. This construction is natural in Y and Z . Its main property is that one obtains for every pointed space X an adjunction homeomorphism

$$\text{map}(Y \wedge_{\text{Or}(G)} Z, X) \xrightarrow{\cong} \text{mor}_{\text{Or}(G)}(Y, \text{map}(Z, X))$$

where the source is the pointed mapping space and the target is the topological space of natural transformations from Y to the contravariant pointed $\text{Or}(G)$ -space $\text{map}(Z, X)$ sending G/H to the pointed mapping space $\text{map}(Z(G/H), X)$. If \mathbf{E}^G is a covariant $\text{Or}(G)$ -spectrum, then one obtains a spectrum $Y \wedge_{\text{Or}(G)} \mathbf{E}^G$. Hence we can extend a covariant functor $\mathbf{E}^G: \text{Or}(G) \rightarrow \text{Spectra}$ to a covariant functor

$$(\mathbf{E}^G)_{\%}: G\text{-CW}^2 \rightarrow \text{Spectra}, \quad (X, A) \mapsto O^G(X_+ \cup_{A_+} \text{cone}(A_+)) \wedge_{\text{Or}(G)} \mathbf{E}^G. \quad (20.3.1)$$

The easy proofs of the following two results are left to the reader.

Lemma 20.3.4. *If \mathbf{E}^G is a covariant $\text{Or}(G)$ -spectrum, then $(\mathbf{E}^G)_{\%}$ is excisive and we obtain a G -homology theory $H_*^G(-; \mathbf{E}^G)$ by*

$$H_n^G(X, A; \mathbf{E}^G) = \pi_n((\mathbf{E}^G)_{\%}(X, A)) = \pi_n(O^G(X_+ \cup_{A_+} \text{cone}(A_+)) \wedge_{\text{Or}(G)} \mathbf{E}^G)$$

satisfying $H_n^G(G/H; \mathbf{E}^G) = \pi_n(\mathbf{E}^G(G/H))$ for $n \in \mathbb{Z}$ and $H \subseteq G$.

Lemma 20.3.5. *Let $\mathbf{t}: \mathbf{E} \rightarrow \mathbf{F}$ be a natural transformation of covariant functors $G\text{-CW}^2 \rightarrow \text{Spectra}$. Suppose that \mathbf{E} and \mathbf{F} are excisive and $\mathbf{t}(G/H)$ is a weak homotopy equivalence for any homogeneous G -space G/H .*

Then $t(X, A): \mathbf{E}(X, A) \rightarrow \mathbf{F}(X, A)$ is a weak homotopy equivalence for every G -CW-pair (X, A) .

20.4 Approximation by an excisive functor

The following result follows from [26, Theorem 6.3]. Its non-equivariant version is due to Weiss-Williams [127].

Theorem 20.4.1 (Approximation by an excisive functor). *Let $\mathbf{E}: G\text{-CW}^2 \rightarrow \text{Spectra}$ be a covariant functor that is homotopy invariant. Let $\mathbf{E}|: \text{Or}(G) \rightarrow \text{Spectra}$ be its composite with the obvious inclusion $\text{Or}(G) \rightarrow G\text{-CW}^2$.*

Then there exists a covariant functor

$$\mathbf{E}^\%: G\text{-CW}^2 \rightarrow \text{Spectra}$$

and natural transformations

$$\begin{aligned} \mathbf{A}_\mathbf{E}: \mathbf{E}^\% &\rightarrow \mathbf{E}; \\ \mathbf{B}_\mathbf{E}: \mathbf{E}^\% &\rightarrow \mathbf{E}|_\%, \end{aligned}$$

satisfying:

1. *The functor $\mathbf{E}^\%$ is excisive;*
2. *The map $\mathbf{A}_\mathbf{E}(G/H): \mathbf{E}^\%(G/H) \rightarrow \mathbf{E}(G/H)$ is a weak homotopy equivalence for every homogeneous space G/H ;*
3. *The map $\mathbf{B}_\mathbf{E}(X, A): \mathbf{E}^\%(X, A) \rightarrow \mathbf{E}|_\%(X, A)$ is a weak homotopy equivalence for every G -CW-pair (X, A) ;*
4. *The functor \mathbf{E} is excisive if and only if $\mathbf{A}_\mathbf{E}(X, A)$ is a weak homotopy equivalence for every G -CW-pair (X, A) ;*
5. *The transformations $\mathbf{A}_\mathbf{E}$ and $\mathbf{B}_\mathbf{E}$ are functorial in \mathbf{E} .*

Although one does not need to understand the explicit construction of $\mathbf{E}^\%$, $\mathbf{A}_\mathbf{E}$ and $\mathbf{B}_\mathbf{E}$ and the proof of Theorem 20.4.1 for the applications of Theorem 20.4.1 and for the remainder of this paper, we make some comments about it for the interested reader.

As an illustration we firstly present a naive suggestion in the non-equivariant case, which turns out to require too restrictive assumptions on \mathbf{E} , and therefore will not be the final solution, but conveys a first idea. Namely, we can define a map of pointed sets $X_+ \wedge \mathbf{E}(\{\bullet\}) \rightarrow \mathbf{E}(X)$ for a CW-complex X by sending an element in the target represented by (x, e) for $x \in X$ and $e \in \mathbf{E}(\{\bullet\})$ to $\mathbf{E}(c_x: \{\bullet\} \rightarrow X)(e)$, where $c_x: \{\bullet\} \rightarrow X$ is the constant map with value x . The problem is that the only reasonable way of ensuring the continuity of this map is to require that \mathbf{E} itself is continuous, i.e., the map $\text{map}(X, Y) \rightarrow \text{map}(E(X)_n, E(Y)_n)$ sending f to $E(f)_n$ has to be continuous for all $n \in \mathbb{Z}$. But this assumption is not satisfied for the functors \mathbf{E} which are of interest for us and will be considered below.

The solution is to take homotopy invariance into account and to work simplicially. Let us consider the special case, where G is trivial and X is a simplicial complex. For any simplex σ of X we have the inclusion $i[\sigma]: \sigma \rightarrow X$ and can therefore define maps

$$\begin{aligned} A_{\mathbf{E}}[\sigma]_n: \sigma_+ \wedge E(\sigma)_n &\xrightarrow{\text{pr}_+ \wedge \text{id}_{E(\sigma)_n}} \{\bullet\}_+ \wedge E(\sigma)_n = E(\sigma)_n \xrightarrow{E(i[\sigma])_n} E(X)_n; \\ B_{\mathbf{E}}[\sigma]_n: \sigma_+ \wedge E(\sigma)_n &\xrightarrow{\text{id}_{\sigma_+} \wedge E(\text{pr})_n} \sigma_+ \wedge E(\{\bullet\})_n. \end{aligned}$$

where pr denotes the projection onto $\{\bullet\}$. Now define a space $E^{\%}(X)_n$ by gluing the spaces $\sigma_+ \wedge E(\sigma)_n$ for σ running over the simplices of X together according to the simplicial structure, more precisely, for an inclusion $j: \tau \rightarrow \sigma$ of simplices we identify a point in $\tau_+ \wedge E(\tau)_n$ with its image in $\sigma_+ \wedge E(\sigma)_n$ under the obvious map $j_+ \wedge E(j)_n$. One easily checks that the various maps $A_{\mathbf{E}}[\sigma]_n$ and $B_{\mathbf{E}}[\sigma]_n$ fit together to maps of pointed spaces

$$\begin{aligned} A_{\mathbf{E}}(X)_n: E^{\%}(X)_n &\rightarrow E(X)_n; \\ B_{\mathbf{E}}(X)_n: E^{\%}(X)_n &\rightarrow E|_{\%}(X)_n := X_+ \wedge E(\{\bullet\})_n, \end{aligned}$$

and thus to maps of spectra

$$\begin{aligned} \mathbf{A}_{\mathbf{E}}(X): \mathbf{E}^{\%}(X) &\rightarrow \mathbf{E}(X)_n; \\ \mathbf{B}_{\mathbf{E}}(X): \mathbf{E}^{\%}(X) &\rightarrow \mathbf{E}|_{\%}(X) := X_+ \wedge \mathbf{E}(\{\bullet\}). \end{aligned}$$

Notice that each map $\mathbf{E}(\text{pr}): \mathbf{E}(\sigma) \rightarrow \mathbf{E}(\{\bullet\})$ is by assumption a weak homotopy equivalence. This implies that $\mathbf{B}_{\mathbf{E}}(X): \mathbf{E}^{\%}(X) \rightarrow \mathbf{E}|_{\%}(X)$ is a weak homotopy equivalence. Since the functor $\mathbf{E}|_{\%}$ is excisive, the functor $\mathbf{E}^{\%}$ is excisive. If $X = \{\bullet\}$, the map $\mathbf{A}_{\mathbf{E}}(\{\bullet\}): \mathbf{E}^{\%}(\{\bullet\}) \rightarrow \mathbf{E}(\{\bullet\})$ is an isomorphism and in particular a weak homotopy equivalence.

Now we see, where the name assembly map comes from. In the case of a simplicial complex X we want to assemble $\mathbf{E}(X)$ by its values $\mathbf{E}(\sigma)$ for the various simplices of X , which leads to the definition of $\mathbf{E}^{\%}(X)$. Intuitively it is clear that $\mathbf{E}(X)$ carries the same information as $\mathbf{E}|_{\%}(X)$ if and only if \mathbf{E} is excisive since the condition excisive allows to compute the values of \mathbf{E} on X by its values on the simplices taking into account how the simplices are glued together to yield X .

Finally one wants a definition that is independent of the simplicial structure and actually applies to more general spaces X than simplicial complexes. Therefore one uses simplicial sets and in particular the singular simplicial set $S.X$. Recall that $S.X$ is the functor from the category of finite ordered sets Δ to the category of sets \mathbf{Sets} sending the finite ordered set $[p]$ to the set $\text{map}(\Delta_p, X)$ for Δ_p the standard p -simplex. For the equivariant version one has to bring the orbit category into play. So one considers for a G -space X the functor

$$\text{Or}(G) \times \Delta \rightarrow \mathbf{Sets}, \quad (G/H, [p]) \mapsto \text{map}_G(G/H \times \Delta_p, X).$$

In some sense one uses free resolution of contravariant functors $\text{Or}(G) \times \Delta \rightarrow \mathbf{Spaces}$ to get the right construction of $\mathbf{E}^{\%}$ and of the desired transformations $\mathbf{A}_{\mathbf{E}}$ and $\mathbf{B}_{\mathbf{E}}$ so that the claims appearing in Theorem 20.4.1 can be proved. Details can be found in [26].

20.5 The universal property

Next we explain why Theorem 20.4.1 characterizes the assembly map in the sense that $\mathbf{A}_{\mathbf{E}}: \mathbf{E}^{\%} \rightarrow \mathbf{E}$ is the universal approximation from the left by an excisive functor of a homotopy invariant functor $\mathbf{E}: G\text{-CW}^2 \rightarrow \text{Spectra}$. Namely, let $\mathbf{T}: \mathbf{F} \rightarrow \mathbf{E}$ be a transformation of covariant functors $G\text{-CW}^2 \rightarrow \text{Spectra}$ such that \mathbf{F} is excisive. Then for any $G\text{-}\mathcal{F}\text{-CW}$ -pair (X, A) the following diagram commutes

$$\begin{array}{ccc}
 \mathbf{F}^{\%}(X) & \xrightarrow[\simeq]{\mathbf{A}_{\mathbf{F}}(X)} & \mathbf{F}(X) \\
 \mathbf{T}^{\%}(X) \downarrow & & \downarrow \mathbf{T}(X) \\
 \mathbf{E}^{\%}(X) & \xrightarrow{\mathbf{A}_{\mathbf{E}}(X)} & \mathbf{E}(X)
 \end{array}$$

and $\mathbf{A}_{\mathbf{F}}(X)$ is a weak homotopy equivalence by Theorem 20.4.1 (4). Hence $\mathbf{T}(X)$ factorizes over $\mathbf{A}_{\mathbf{E}}(X)$ up to natural weak homotopy equivalence.

Suppose additionally that $\mathbf{T}(G/H)$ is a weak homotopy equivalence for every subgroup $H \subseteq G$. Then both $\mathbf{T}^{\%}(X)$ and $\mathbf{A}_{\mathbf{F}}(X)$ are weak homotopy equivalences by Lemma 20.3.5 and Theorem 20.4.1 (4), and hence $\mathbf{T}(X)$ can be identified with $\mathbf{A}_{\mathbf{E}}(X)$ up to natural weak homotopy equivalence.

Recall that there is a natural weak equivalence $\mathbf{B}_{\mathbf{E}}(X): \mathbf{E}^{\%}(X) \xrightarrow{\simeq} \mathbf{E}|_{\%}(X)$, so that one may replace in the considerations above $\mathbf{E}^{\%}(X)$ by $\mathbf{E}|_{\%}(X)$, which depends on the values of \mathbf{E} on homogeneous spaces only. This universal property will be the key ingredient for the identification of various versions of assembly maps.

20.6 Classifying spaces for families of subgroups

We recall the notion classifying space for a family which was introduced by tom Dieck [115].

Definition 20.6.1 (Family of subgroups). A family \mathcal{F} of subgroups of a group G is a set of subgroups of G that is closed under conjugation with elements of G and under passing to subgroups.

Our main examples of families are the trivial family \mathcal{TR} consisting of the trivial subgroup, the family \mathcal{ALL} of all subgroups, and the families \mathcal{FCY} , \mathcal{CY} , \mathcal{FLN} , and \mathcal{VCY} of finite cyclic subgroups, of cyclic subgroups, of finite subgroups, and of virtually cyclic subgroups.

Definition 20.6.2 (Classifying G -space for a family of subgroups). Let \mathcal{F} be a family of subgroups of G . A model $E_{\mathcal{F}}(G)$ for the classifying spaces for the family \mathcal{F} of subgroups of G is a $G\text{-CW}$ -complex $E_{\mathcal{F}}(G)$ that has the following properties:

1. All isotropy groups of $E_{\mathcal{F}}(G)$ belong to \mathcal{F} ;
2. For any $G\text{-CW}$ -complex Y , whose isotropy groups belong to \mathcal{F} , there is up to G -homotopy precisely one G -map $Y \rightarrow E_{\mathcal{F}}(G)$.

We abbreviate $\underline{EG} := E_{\mathcal{FIN}}(G)$ and call it the *universal G -space for proper G -actions*. We also write $\underline{EG} := E_{\mathcal{VCY}}(G)$.

Equivalently, $E_{\mathcal{F}}(G)$ is a terminal object in the G -homotopy category of G -CW-complexes, whose isotropy groups belong to \mathcal{F} . In particular two models for $E_{\mathcal{F}}(G)$ are G -homotopy equivalent and for two families $\mathcal{F}_0 \subseteq \mathcal{F}_1$ there is up to G -homotopy precisely one G -map $E_{\mathcal{F}_0}(G) \rightarrow E_{\mathcal{F}_1}(G)$. There are functorial constructions for $E_{\mathcal{F}}(G)$ generalizing the bar construction, see [26, Section 3 and Section 7].

Theorem 20.6.3 (Homotopy characterization of $E_{\mathcal{F}}(G)$). *A G -CW-complex X is a model for $E_{\mathcal{F}}(G)$ if and only if for every subgroup $H \subseteq G$ its H -fixed point set X^H is weakly contractible if $H \in \mathcal{F}$, and is empty if $H \notin \mathcal{F}$.*

A model for $E_{\mathcal{ALL}}(G)$ is G/G . A model for $E_{\mathcal{TR}}(G)$ is the same as a model for EG , i.e., the universal covering of BG , or, equivalently, the total space of the universal G -principal bundle. There are many interesting geometric models for classifying spaces $\underline{EG} = E_{\mathcal{FIN}}(G)$, e.g., the Rips complex for a hyperbolic group, the Teichmüller space for a mapping class group, and so on. The question whether there are finite-dimensional models, models of finite type or finite models has been studied intensively during the last decades. For more information about classifying spaces for families we refer for instance to [77].

20.7 The Meta-Isomorphism Conjecture

In this section we formulate the Meta-Isomorphism Conjecture, from which all other Isomorphism Conjectures such as the one due to Farrell-Jones and Baum-Connes are obtained by specifying the parameters \mathbf{E} and \mathcal{F} .

20.7.1 The Meta-Isomorphism Conjecture for G -homology theories

Let \mathcal{H}_*^G be a G -homology theory with values in Λ -modules for some commutative ring Λ . The projection $\text{pr}: E_{\mathcal{F}}(G) \rightarrow G/G$ induces for all integers $n \in \mathbb{Z}$ a homomorphism of Λ -modules

$$\mathcal{H}_n^G(\text{pr}): \mathcal{H}_n^G(E_{\mathcal{F}}(G)) \rightarrow \mathcal{H}_n^G(G/G) \quad (20.7.1)$$

which is called the *assembly map*.

Conjecture 20.7.1 (Meta-Isomorphism Conjecture for G -homology theories). *The group G satisfies the Meta-Isomorphism Conjecture with respect to the G -homology theory \mathcal{H}_*^G and the family \mathcal{F} of subgroups of G , if the assembly map*

$$\mathcal{H}_n^G(\text{pr}): \mathcal{H}_n^G(E_{\mathcal{F}}(G)) \rightarrow \mathcal{H}_n^G(G/G)$$

of (20.7.1) is bijective for all $n \in \mathbb{Z}$.

If we choose \mathcal{F} to be the family \mathcal{ALL} of all subgroups, then G/G is a model for $E_{\mathcal{ALL}}(G)$ and the Meta-Isomorphism Conjecture 20.7.1 is obviously true. The point is to find an as small as possible family \mathcal{F} . The idea of the Meta-Isomorphism Conjecture 20.7.1 is that

one wants to compute $\mathcal{H}_n^G(G/G)$, which is the unknown and the interesting object, by assembling it from the values $\mathcal{H}_n^G(G/H)$ for $H \in \mathcal{F}$.

20.7.2 The Meta-Isomorphism Conjecture on the level of spectra

Often the construction of the assembly map is done already on the level of spectra or can be lifted to this level. Consider a covariant functor

$$\mathbf{E}^G: \text{Or}(G) \rightarrow \text{Spectra}.$$

Conjecture 20.7.2 (Meta-Isomorphism Conjecture for spectra). *The group G satisfies the Meta-Isomorphism Conjecture with respect to the covariant functor $\mathbf{E}^G: \text{Or}(G) \rightarrow \text{Spectra}$ and the family \mathcal{F} of subgroups of G , if the projection $\text{pr}: E_{\mathcal{F}}(G) \rightarrow G/G$ induces a weak homotopy equivalence*

$$(\mathbf{E}^G)_{\%}(\text{pr}): (\mathbf{E}^G)_{\%}(E_{\mathcal{F}}(G)) \rightarrow (\mathbf{E}^G)_{\%}(G/G) = \mathbf{E}^G(G/G).$$

Notice that $(\mathbf{E}^G)_{\%}(\text{pr}): (\mathbf{E}^G)_{\%}(E_{\mathcal{F}}(G)) \rightarrow (\mathbf{E}^G)_{\%}(G/G) = \mathbf{E}^G(G/G)$ is a weak homotopy equivalence if and only if for every $n \in \mathbb{Z}$ the map

$$H_n^G(\text{pr}; \mathbf{E}^G): H_n^G(E_{\mathcal{F}}(G); \mathbf{E}^G) \rightarrow H_n^G(G/G; \mathbf{E}^G)$$

is a bijection, where $H_*^G(-; \mathbf{E}^G)$ is the G -homology theory associated to \mathbf{E}^G , see Lemma 20.3.4. In other words, Conjecture 20.7.2 is equivalent to Conjecture 20.7.1 if we take for \mathcal{H}_*^G the G -homology theory associated to \mathbf{E}^G .

20.7.3 The assembly map in terms of homotopy colimits

The assembly map appearing in Conjecture 20.7.2 can be interpreted in terms of homotopy colimits as follows. The \mathcal{F} -restricted orbit category $\text{Or}_{\mathcal{F}}(G)$ be the full subcategory of $\text{Or}(G)$ consisting of those objects G/H for which H belongs to \mathcal{F} . Let $\mathbf{E}^G|_{\text{Or}_{\mathcal{F}}(G)}$ be the restriction of \mathbf{E}^G to $\text{Or}_{\mathcal{F}}(G)$. Then we get from the inclusion $\text{Or}_{\mathcal{F}}(G) \rightarrow \text{Or}(G)$ and the fact that G/G is a terminal object in $\text{Or}(G)$ a map

$$\text{hocolim}_{\text{Or}_{\mathcal{F}}(G)} \mathbf{E}^G|_{\text{Or}_{\mathcal{F}}(G)} \rightarrow \text{hocolim}_{\text{Or}(G)} \mathbf{E}^G = \mathbf{E}^G(G/G).$$

This map can be identified with $(\mathbf{E}^G)_{\%}(\text{pr}): (\mathbf{E}^G)_{\%}(E_{\mathcal{F}}(G)) \rightarrow (\mathbf{E}^G)_{\%}(G/G)$, see [26, Section 5.2]. Again this explains the name assembly map: we try to put the values of \mathbf{E}^G on homogeneous spaces G/H for $H \in \mathcal{F}$ together to get its value at G/G .

20.7.4 Spectra over Groupoids

In all interesting cases we will obtain \mathbf{E}^G as follows. Let Groupoids be the category of small groupoids. Consider a covariant functor

$$\mathbf{E}: \text{Groupoids} \rightarrow \text{Spectra}$$

that *respects equivalences*, i.e., it sends equivalences of groupoids to weak equivalences of spectra. Given a G -set S , its *transport groupoid* $\mathcal{T}^G(S)$ has S as set of objects and the set of morphism from s_0 to s_1 is $\{g \in G \mid gs_0 = s_1\}$. Composition comes from the multiplication in G . We get for every group G a functor

$$\mathbf{E}^G : \text{Or}(G) \rightarrow \text{Spectra}$$

by composing \mathbf{E} with the functor $\text{Or}(G) \rightarrow \text{Groupoids}$, $G/H \mapsto \mathcal{T}^G(G/H)$.

Notice that a group G can be viewed as a groupoid with one object and G as the set of automorphisms of this object, and hence we can consider $\mathbf{E}(G)$. We have the obvious identifications $\mathbf{E}(G) = \mathbf{E}^G(G/G) = (\mathbf{E}^G)_{\%}(G/G)$. Moreover, for every subgroup $H \subseteq G$ there is an equivalence of groupoids $H \rightarrow \mathcal{T}^G(G/H)$ sending the unique object of H to the object eH , which induces a weak homotopy equivalence $\mathbf{E}(H) \rightarrow \mathbf{E}^G(G/H)$.

The various prominent Isomorphism Conjectures such as the one due to Farrell-Jones and Baum-Connes are now obtained by specifying $\mathbf{E}: \text{Groupoids} \rightarrow \text{Spectra}$, the group G and the family \mathcal{F} .

20.8 The Farrell-Jones Conjectures

20.8.1 The Farrell-Jones Conjecture for K - and L -theory

Let R be a ring (with involution). There exist covariant functors respecting equivalences

$$\mathbf{K}_R : \text{Groupoids} \rightarrow \text{Spectra}; \tag{20.8.1}$$

$$\mathbf{L}_R^{\langle -\infty \rangle} : \text{Groupoids} \rightarrow \text{Spectra}, \tag{20.8.2}$$

such that for every group G and all $n \in \mathbb{Z}$ we have

$$\begin{aligned} \pi_n(\mathbf{K}_R(G)) &\cong K_n(RG); \\ \pi_n(\mathbf{L}_R^{\langle -\infty \rangle}(G)) &\cong L_n^{\langle -\infty \rangle}(RG). \end{aligned}$$

Here $K_n(RG)$ is the n -th algebraic K -group of the group ring RG and $L_n^{\langle -\infty \rangle}(RG)$ is the n th quadratic L -group with decoration $\langle -\infty \rangle$ of the group ring RG equipped with the involution sending $\sum_{g \in G} r_g g$ to $\sum_{g \in G} \overline{r_g} g^{-1}$.

The details of this construction can be found in [26, Section 2]. If we now take these functors and the family \mathcal{VCY} of virtually cyclic subgroups, we obtain

Conjecture 20.8.1 (Farrell-Jones Conjecture). *A group G satisfies the K -theoretic or L -theoretic Farrell-Jones Conjecture if for every ring (with involution) R the assembly maps induced by the projection $\text{pr}: \underline{EG} \rightarrow G/G$*

$$\begin{aligned} H_n^G(\text{pr}; \mathbf{K}_R) : H_n^G(\underline{EG}; \mathbf{K}_R) &\rightarrow H_n^G(G/G; \mathbf{K}_R) = K_n(RG); \\ H_n^G(\text{pr}; \mathbf{L}_R^{\langle -\infty \rangle}) : H_n^G(\underline{EG}; \mathbf{L}_R^{\langle -\infty \rangle}) &\rightarrow H_n^G(G/G; \mathbf{L}_R^{\langle -\infty \rangle}) = L_n^{\langle -\infty \rangle}(RG), \end{aligned}$$

are bijective for all $n \in \mathbb{Z}$.

It is crucial that we use non-connective K -spectra and that the decoration for the L -theory is $\langle -\infty \rangle$, see [39].

The original version of the Farrell-Jones Conjecture appeared in [38, 1.6 on page 257]. A detailed exposition on the Farrell-Jones Conjecture will be given in [79], see also [82].

20.8.2 Applications of the Farrell-Jones Conjecture

Here are some consequences of the Farrell-Jones Conjecture. For more information about these and other applications we refer for instance to [9, 79, 82].

20.8.2.1 Computations

One can carry out *explicit computations* of K and L -groups of group rings by applying methods from algebraic topology to the left side given by a G -homology theory and by finding small models for the classifying spaces of families using the topology and geometry of groups, see Section 20.13.

20.8.2.2 Vanishing of lower and middle K -groups

If G is a torsionfree group satisfying the K -theoretic Farrell-Jones Conjecture 20.8.1, then $K_n(\mathbb{Z}G)$ for $n \leq -1$, the reduced projective class group $\tilde{K}_0(\mathbb{Z}G)$, and the Whitehead group $\text{Wh}(G)$ vanish.

This has the following consequences. Every homotopy equivalence $f: X \rightarrow Y$ of connected CW -complexes with $\pi_1(Y) \cong G$ is simple. Every h -cobordism over a closed manifold M of dimension ≥ 5 and $G \cong \pi_1(M)$ is trivial. Every finitely generated projective $\mathbb{Z}G$ -module is stably free. Every finitely dominated connected CW -complex X with $\pi_1(X) \cong G$ is homotopy equivalent to a finite CW -complex.

20.8.2.3 Kaplansky's Idempotent Conjecture

If the torsionfree group G satisfies the K -theoretic Farrell-Jones Conjecture 20.8.1, then G satisfies the *Idempotent Conjecture* that for a commutative integral domain R the only idempotents of RG are 0 and 1.

20.8.2.4 Novikov Conjecture

If G satisfies the L -theoretic Farrell-Jones Conjecture 20.8.1, then G satisfies the *Novikov Conjecture* about the homotopy invariance of higher signatures. For more information about the Novikov Conjecture we refer for instance to [42, 43, 61].

20.8.2.5 Borel Conjecture

If G is a torsionfree group satisfying the K -theoretic and the L -theoretic Farrell-Jones Conjecture 20.8.1, then G satisfies the *Borel Conjecture* in dimensions ≥ 5 , i.e., if M and N are closed aspherical manifolds of dimension ≥ 5 with $\pi_1(M) \cong \pi_1(N) \cong G$, then M and N are homeomorphic and every homotopy equivalence from M to N is homotopic to a homeomorphism.

20.8.2.6 Bass Conjecture

If G satisfies the K -theoretic Farrell-Jones Conjecture 20.8.1, then G satisfies the *Bass Conjecture*, see [9, 13].

20.8.2.7 Automorphism groups

The Farrell-Jones Conjecture 20.8.1 yields rational computations of the homotopy groups and homology groups of the automorphisms groups of an aspherical closed manifold in the topological, PL and smooth category, see for instance instance [36], [37, Section 2] and [35, Lecture 5].

For instance, if M is an aspherical orientable closed (smooth) manifold of dimension > 10 with fundamental group G such that G satisfies the Farrell-Jones Conjecture 20.8.1, then we get for $1 \leq i \leq (\dim M - 7)/3$

$$\pi_i(\text{Top}(M)) \otimes_{\mathbb{Z}} \mathbb{Q} = \begin{cases} \text{center}(G) \otimes_{\mathbb{Z}} \mathbb{Q} & \text{if } i = 1; \\ 0 & \text{if } i > 1, \end{cases}$$

and

$$\pi_i(\text{Diff}(M)) \otimes_{\mathbb{Z}} \mathbb{Q} = \begin{cases} \text{center}(G) \otimes_{\mathbb{Z}} \mathbb{Q} & \text{if } i = 1; \\ \bigoplus_{j=1}^{\infty} H_{(i+1)-4j}(M; \mathbb{Q}) & \text{if } i > 1 \text{ and } \dim M \text{ odd;} \\ 0 & \text{if } i > 1 \text{ and } \dim M \text{ even.} \end{cases}$$

For a survey on automorphisms of manifolds we refer to [128].

20.8.2.8 Boundary of hyperbolic groups

In [11] a proof of a conjecture of Gromov is given in dimensions $n \geq 6$ using the Farrell-Jones Conjecture 20.8.1 that a torsionfree hyperbolic group with S^n as boundary is the fundamental group of an aspherical closed topological manifold. This manifold is unique to homeomorphism. The stable Cannon Conjecture is treated in [41].

20.8.2.9 Poincaré duality groups

If G is a Poincaré duality group of dimension $n \geq 6$ and satisfies the Farrell-Jones Conjecture 20.8.1, then it is the fundamental group of an aspherical closed homology ANR-manifold, see [11]. It is unique up to s -cobordism. Whether it can be chosen to be an aspherical closed topological manifold, depends on its Quinn obstruction.

20.8.2.10 Tautological classes and aspherical manifolds

The vanishing of tautological classes for many bundles with fiber an aspherical manifold is proved in [45].

20.8.2.11 Fiberings manifolds

The problem when a map from some closed connected manifold to an aspherical closed manifold approximately fibers, i.e., is homotopic to Manifold Approximate Fibration, is analyzed in [40].

20.8.3 The interpretation of the Farrell-Jones assembly map for L -theory in terms of surgery theory

So far we have given a homotopy theoretic approach to the assembly map. This is the easiest approach and well-suited for structural questions such as comparing the assembly maps of various different theories, as explained below. For concrete applications it is important to give geometric or analytic interpretations. For instance, one key ingredient in the proof that the Borel Conjecture follows from the Farrell-Jones Conjecture is a geometric interpretation of the assembly for the trivial family in terms of surgery theory, notably the surgery exact sequence, which we briefly sketch next.

Definition 20.8.2 (The structure set). Let N be a closed topological manifold of dimension n . We call two simple homotopy equivalences $f_i: M_i \rightarrow N$ from closed topological manifolds M_i of dimension n to N for $i = 0, 1$ equivalent if there exists a homeomorphism $g: M_0 \rightarrow M_1$ such that $f_1 \circ g$ is homotopic to f_0 .

The *structure set* $\mathcal{S}(N)$ of N is the set of equivalence classes of simple homotopy equivalences $M \rightarrow N$ from closed topological manifolds of dimension n to N . This set has a preferred base point, namely, the class of the identity $\text{id}: N \rightarrow N$.

One easily checks that the Borel Conjecture holds for $G = \pi_1(N)$ for a closed aspherical manifold N if and only if $\mathcal{S}(N)$ consists of precisely one element, namely, the class of $\text{id}_N: N \rightarrow N$. The surgery exact sequence, which we will explain next, gives a way of calculating the structure set.

Definition 20.8.3 (Normal map of degree one). A *normal map of degree one* with target the connected closed manifold N of dimension n consists of:

- A connected closed n -dimensional manifold M ;
- A map of degree one $f: M \rightarrow N$;
- A $(k + n)$ -dimensional vector bundle ξ over N ;
- A bundle map $\bar{f}: TM \oplus \mathbb{R}^k \rightarrow \xi$ covering f .

There is an obvious normal bordism relation, and we denote by $\mathcal{N}(N)$ the set of bordism classes of normal maps with target N . One can assign to a normal map $f: M \rightarrow N$ its surgery obstruction $\sigma(f) \in L_n^s(\mathbb{Z}G)$ taking values in the n th quadratic L -group with decoration s , where $G = \pi_1(N)$ and $n = \dim(N)$. If $n \geq 5$, the surgery obstruction vanishes if and only if one can find (by doing surgery) a representative in the normal bordisms class, whose underlying map f is a simple homotopy equivalence. It yields a map $\sigma: \mathcal{N}(N) \rightarrow L_n^s(\mathbb{Z}G)$. There is a map $\eta: \mathcal{S}_n^{\text{top}}(N) \rightarrow \mathcal{N}(N)$ that assigns to the class of a simple homotopy equivalence $f: M \rightarrow N$ with a closed manifold M as source the normal map given by f itself

and the bundle data coming from TM and $\xi = (f^{-1})^*TM$ for some homotopy inverse f^{-1} of f . We denote by $\mathcal{N}(N \times [0, 1], N \times \partial[0, 1])$ the normal bordism classes of normal maps relative boundary. Essentially these are normal maps $(M, \partial M) \rightarrow (N \times [0, 1], N \times \partial[0, 1])$ of degree one which are simple homotopy equivalences on the boundary. There is a surgery obstruction relative boundary which yields a map $\sigma: \mathcal{N}(N \times [0, 1], N \times \partial[0, 1]) \rightarrow L_{n+1}^s(\mathbb{Z}G)$. There is also a map $\partial: L_{n+1}^s(\mathbb{Z}G) \rightarrow \mathcal{S}_n^{\text{top}}(N)$ that sends an element $x \in L_{n+1}^s(\mathbb{Z}G)$ to the class of a simple homotopy equivalence $f: M \rightarrow N$ for which there exists a normal map relative boundary of triads $(F, f_0, \text{id}_N): (W; M, N) \rightarrow (N \times [0, 1]; N \times \{0\}, N \times \{1\})$ whose relative surgery obstruction is x . If $n \geq 5$, then one obtains a long exact sequence of abelian groups, the *surgery exact sequence* due to Browder, Novikov, Sullivan, and Wall

$$\mathcal{N}(N \times [0, 1], N \times \partial[0, 1]) \xrightarrow{\sigma_{n+1}} L_{n+1}^s(\mathbb{Z}G) \xrightarrow{\partial} \mathcal{S}(N) \xrightarrow{\eta} \mathcal{N}(N) \xrightarrow{\sigma_n} L_n^s(\mathbb{Z}G). \quad (20.8.3)$$

If we can show that σ_{n+1} is surjective and σ_n is injective, then the Borel Conjecture holds for $G = \pi_1(N)$, if N is an aspherical closed manifold of dimension $n \geq 5$.

Let \mathbf{L} be the L -theory spectrum. It has the property $\pi_n(\mathbf{L}) = L_n^{(-\infty)}(\mathbb{Z})$. Denote by $\mathbf{L}\langle 1 \rangle$ its 1-connective cover. It comes with a natural map of spectra $\mathbf{L}\langle 1 \rangle \rightarrow \mathbf{L}$, which induces on π_i an isomorphism for $i \geq 1$, and we have $\pi_i(\mathbf{L}\langle 1 \rangle) = 0$ for $i \leq 0$. There are natural identifications coming among other things from the Pontrjagin-Thom construction

$$\begin{aligned} u_n: \mathcal{N}(N) &\xrightarrow{\cong} H_n(N; \mathbf{L}\langle 1 \rangle) = \pi_n(N_+ \wedge \mathbf{L}\langle 1 \rangle); \\ u_{n+1}: \mathcal{N}(N \times [0, 1], N \times \{0, 1\}) &\xrightarrow{\cong} H_{n+1}(N; \mathbf{L}\langle 1 \rangle) = \pi_{n+1}(N_+ \wedge \mathbf{L}\langle 1 \rangle). \end{aligned}$$

An easy spectral sequence argument shows that the canonical map

$$v_n: H_n(N; \mathbf{L}\langle 1 \rangle) \rightarrow H_n(N; \mathbf{L})$$

is injective and the canonical map

$$v_{n+1}: H_{n+1}(N; \mathbf{L}\langle 1 \rangle) \rightarrow H_{n+1}(N; \mathbf{L})$$

is bijective for $n = \dim(N)$.

For the remainder of this subsection we assume additionally that N is aspherical. There is a natural identification for $m = n, n + 1$, see Definition 20.11.1,

$$w_m: H_m^G(EG; \mathbf{L}_{\mathbb{Z}}^{(-\infty)}) \xrightarrow{\cong} H_m(BG; \mathbf{L}) = H_m(N; \mathbf{L}).$$

The K -theoretic Farrell-Jones Conjecture applied to the torsionfree group G implies that $K_n(\mathbb{Z}G)$ for $n \leq -1$, the reduced projective class group $\tilde{K}_0(\mathbb{Z}G)$, and the Whitehead group $\text{Wh}(G)$ vanish. One concludes from the so called *Rothenberg sequences*, see [105, Theorem 17.2 on page 146], that for $m = n, n + 1$ the canonical map

$$r_m: L_m^s(\mathbb{Z}G) \xrightarrow{\cong} L_m^{(-\infty)}(\mathbb{Z}G)$$

is bijective. The up to G -homotopy unique G -map $i: EG = E_{\mathcal{TR}}(G) \rightarrow \underline{EG}$ induces for all

$m \in \mathbb{Z}$ an isomorphism, see Theorem 20.12.2 (5),

$$H_m^G(i; \mathbf{L}_{\mathbb{Z}}^{\langle -\infty \rangle}) : H_m^G(EG; \mathbf{L}_{\mathbb{Z}}^{\langle -\infty \rangle}) \xrightarrow{\cong} H_m^G(\underline{EG}; \mathbf{L}_{\mathbb{Z}}^{\langle -\infty \rangle}).$$

The L -theoretic Farrell-Jones Conjecture predicts the bijectivity of the assembly map

$$H_n^G(\text{pr}, \mathbf{L}_{\mathbb{Z}}^{\langle -\infty \rangle}) : H_n^G(\underline{EG}; \mathbf{L}_{\mathbb{Z}}^{\langle -\infty \rangle}) \xrightarrow{\cong} H_n^G(G/G; \mathbf{L}_{\mathbb{Z}}^{\langle -\infty \rangle}) = L_n^{\langle -\infty \rangle}(\mathbb{Z}G).$$

The following diagram commutes

$$\begin{array}{ccc}
 \mathcal{N}(N) & \xrightarrow{\sigma_n} & L_n^s(\mathbb{Z}G) \\
 \downarrow u_n \cong & & \uparrow \cong r_n \\
 H_n(N; \mathbf{L}\langle 1 \rangle) & & \\
 \downarrow v_n & & \\
 H_n(N; \mathbf{L}) & & \\
 \uparrow \cong w_n & & \\
 H_n^G(EG; \mathbf{L}_{\mathbb{Z}}^{\langle -\infty \rangle}) & \xrightarrow[H_n^G(i, \mathbf{L}_{\mathbb{Z}}^{\langle -\infty \rangle})]{\cong} H_n^G(\underline{EG}; \mathbf{L}_{\mathbb{Z}}^{\langle -\infty \rangle}) \xrightarrow[H_n^G(\text{pr}, \mathbf{L}_{\mathbb{Z}}^{\langle -\infty \rangle})]{\cong} & L_n^{\langle -\infty \rangle}(\mathbb{Z}G).
 \end{array}$$

If we replace everywhere n by $n + 1$ and in the upper left corner $\mathcal{N}(N)$ by $\mathcal{N}(N \times [0, 1], N \times \partial[0, 1])$, we get the analogous commutative diagram. The proof of the commutativity of these diagrams is rather involved and we refer for a proof for instance to [62]. We conclude from these two diagrams that σ_n is injective and σ_{n+1} is bijective since v_n is injective and v_{n+1} is bijective. Recall that this implies the vanishing of the structure set $\mathcal{S}(N)$. Hence the Farrell-Jones Conjecture 20.8.1 implies the Borel Conjecture in dimensions ≥ 5 .

For more information about L -groups and surgery theory and the arguments and facts above we refer for instance to [18, 19, 25, 70, 104, 124].

20.8.4 The interpretation of the Farrell-Jones assembly map in terms of controlled topology

We have defined the assembly map appearing in the Farrell-Jones Conjecture as a map induced by the projection $\underline{EG} \rightarrow G/G$ for a G -homology theory $H_*^G(X; \mathbf{E}^G)$ or for the functor $(\mathbf{E}^G)_\% : G\text{-CW} \rightarrow \text{Spectra}$. We have also given a homotopy theoretic interpretation in terms of homotopy colimits and described its universal property to be the best approximation from the left by an excisive functor. This interpretation is good for structural and computational aspects, but it turns out that it is not helpful for the proof that the assembly maps are weak homotopy equivalences. There is no direct homotopy theoretic construction of an inverse up to weak homotopy equivalence known to the author.

For the actual proofs that the assembly maps are weak homotopy equivalences, the interpretation of the assembly map as a *forget control map* is crucial. This fundamental idea is due to Quinn.

Roughly speaking, one attaches to a metric space certain categories, to these categories spectra and then takes their homotopy groups, where everything depends on a choice of certain control conditions which in some sense measure sizes of cycles. If one requires certain control conditions, one obtains the source of the assembly map. If one requires no control conditions, one obtains the target of the assembly map. The assembly map itself is forgetting the control condition.

One of the basic features of a homology theory is excision. It often comes from the fact that a representing cycle can be arranged to have arbitrarily good control. An example is the technique of subdivision which allows us to make the representing cycles for singular homology arbitrarily controlled, i.e., the diameter of the image of a singular simplex appearing in a singular chain with non-zero coefficient can be arranged to be arbitrarily small. This is the key ingredient in the proof that singular homology satisfies excision. In general one may say that requiring control conditions amounts to implementing homological properties.

With this interpretation it is clear what the main task in the proof of surjectivity of the assembly map is: *achieve control*, i.e., manipulate cycles without changing their homology class so that they become sufficiently controlled. There is a general principle that a proof of surjectivity also gives injectivity. Namely, proving injectivity means that one must construct a cycle whose boundary is a given cycle, i.e., one has to solve a surjectivity problem in a relative situation. The actual implementation of this idea is rather technical. The proof that this forget control version of the assembly map agrees up to weak homotopy equivalence with the homotopy theoretic one appearing in the Farrell-Jones Conjecture 20.8.1 is a direct application of Section 20.5. The same is true also for the version of the assembly map appearing in [38, 1.6 on page 257], as explained in [26, page 239].

To achieve control one can now use geometric methods. The key ingredients are contracting maps and open coverings, transfers, flow spaces, and the geometry of the group G .

For more information about the general strategy of proofs we refer for instance to [2, 78, 79].

20.8.5 The Farrell-Jones Conjecture for Waldhausen's A -theory

Waldhausen has defined for a CW -complex X its algebraic K -theory space $A(X)$ in [120, Chapter 2]. As in the case of algebraic K -theory of rings it will be necessary to consider a non-connective version. Vogell [117] has defined a delooping of $A(X)$ yielding a non-connective spectrum $\mathbf{A}(X)$ for a CW -complex X . This construction actually yields a covariant functor from the category of topological spaces to the category of spectra. We can assign to a groupoid \mathcal{G} its classifying space $B\mathcal{G}$. Thus we obtain a covariant functor

$$\mathbf{A}: \text{Groupoids} \rightarrow \text{Spectra}, \quad \mathcal{G} \mapsto \mathbf{A}(B\mathcal{G}), \quad (20.8.4)$$

denoted by \mathbf{A} again. It respects equivalences, see [120, Proposition 2.1.7] and [117]. If we now take this functor and the family \mathcal{VCY} of virtually cyclic subgroups, we obtain the A -theoretic Farrell-Jones Conjecture

Conjecture 20.8.4 (A -theoretic Farrell-Jones Conjecture). *A group G satisfies the A -theoretic Farrell-Jones Conjecture if the assembly maps induced by the projection $\text{pr}: \underline{EG} \rightarrow$*

G/G

$$H_n^G(\text{pr}; \mathbf{A}): H_n^G(\underline{EG}; \mathbf{A}) \rightarrow H_n^G(G/G; \mathbf{A}) = \pi_n(\mathbf{A}(BG))$$

is bijective for all $n \in \mathbb{Z}$.

The A -theoretic Farrell-Jones Conjecture 20.8.4 is an important ingredient in the computation of the group of self-homeomorphisms of an aspherical closed manifold in the stable range using the machinery of Weiss-Williams [128]. Moreover, it is related to Whitehead spaces, pseudo-isotopy spaces and spaces of h-cobordisms, see for instance [32, 119, 120, 121, 122, 123, 128].

20.8.6 Relating the assembly maps for K -theory and for A -theory

Let X be a connected CW -complex with fundamental group $\pi = \pi_1(X)$. Essentially by passing to the cellular $\mathbb{Z}\pi$ -chain complex of the universal covering one obtains a natural map of (non-connective) spectra, natural in X , called *linearization map*

$$\mathbf{L}(X): \mathbf{A}(X) \rightarrow \mathbf{K}(\mathbb{Z}\pi_1(X)). \tag{20.8.5}$$

The next result follows by combining [118, Section 4] and [119, Proposition 2.2 and Proposition 2.3].

Theorem 20.8.5 (Connectivity of the linearization map). *Let X be a connected CW -complex. Then:*

1. The linearization map $\mathbf{L}(X)$ of (20.8.5) is 2-connected, i.e., the map

$$L_n := \pi_n(\mathbf{L}(X)): A_n(X) \rightarrow K_n(\mathbb{Z}\pi_1(X))$$

is bijective for $n \leq 1$ and surjective for $n = 2$;

2. Rationally the map L_n is bijective for all $n \in \mathbb{Z}$, provided that X is aspherical.

Thus one obtain a transformation $\mathbf{L}: \mathbf{A} \rightarrow \mathbf{K}_{\mathbb{Z}}$ of covariant functors Groupoids \rightarrow Spectra, where $\mathbf{K}_{\mathbb{Z}}$ and \mathbf{A} have been defined in (20.8.1) and (20.8.4). It induces a commutative diagram

$$\begin{array}{ccc} H_n^G(\underline{EG}; \mathbf{A}) & \xrightarrow{H_n^G(\text{pr}; \mathbf{A})} & H_n^G(G/G; \mathbf{A}) = \pi_n(\mathbf{A}(BG)) \\ H_n^G(\text{id}_{\underline{EG}}; \mathbf{L}) \downarrow & & \downarrow \pi_n(\mathbf{L}(BG)) \\ H_n^G(\underline{EG}; \mathbf{K}_{\mathbb{Z}}) & \xrightarrow{H_n^G(\text{pr}; \mathbf{K}_R)} & K_n(\mathbb{Z}G) \end{array} \tag{20.8.6}$$

where the upper horizontal arrow is the assembly map appearing in A -theoretic Farrell-Jones Conjecture 20.8.4, the lower horizontal arrow is the assembly map appearing in the K -theoretic Farrell-Jones Conjecture 20.8.1, and both vertical arrows are bijective for $n \leq 1$, surjective for $n = 2$ and rationally bijective for all $n \in \mathbb{Z}$. In particular the K -theoretic Farrell-Jones Conjecture 20.8.1 for $R = \mathbb{Z}$ and the A -theoretic Farrell-Jones Conjecture 20.8.4 are rationally equivalent.

20.8.7 The status of the Farrell-Jones Conjecture

There is a more general version of the Farrell-Jones Conjecture, the so called *Full Farrell-Jones Conjecture*, where one allows coefficients in additive categories and the passage to finite wreath products, It implies the Farrell-Jones Conjectures 20.8.1. For A -theory there is a so called *fibred* version which implies Conjecture 20.8.4. Let \mathcal{FJ} be the class of groups for which the Full Farrell-Jones Conjecture and the fibred A -theoretic Farrell-Jones Conjecture holds. Notice that then any group in \mathcal{FJ} satisfies in particular Conjectures 20.8.1 and 20.8.4.

Theorem 20.8.6 (The class \mathcal{FJ}).

1. *The following classes of groups belong to \mathcal{FJ} :*
 - (a) *Hyperbolic groups;*
 - (b) *Finite dimensional CAT(0)-groups;*
 - (c) *Virtually solvable groups;*
 - (d) *(Not necessarily cocompact) lattices in second countable locally compact Hausdorff groups with finitely many path components;*
 - (e) *Fundamental groups of (not necessarily compact) connected manifolds (possibly with boundary) of dimension ≤ 3 ;*
 - (f) *The groups $GL_n(\mathbb{Q})$ and $GL_n(F(t))$ for $F(t)$ the function field over a finite field F ;*
 - (g) *S -arithmetic groups;*
 - (h) *mapping class groups;*
2. *The class \mathcal{FJ} has the following inheritance properties:*
 - (a) *Passing to subgroups*
Let $H \subseteq G$ be an inclusion of groups. If G belongs to \mathcal{FJ} , then H belongs to \mathcal{FJ} ;
 - (b) *Passing to finite direct products*
If the groups G_0 and G_1 belong to \mathcal{FJ} , then also $G_0 \times G_1$ belongs to \mathcal{FJ} ;
 - (c) *Group extensions*
Let $1 \rightarrow K \rightarrow G \rightarrow Q \rightarrow 1$ be an extension of groups. Suppose that for any cyclic subgroup $C \subseteq Q$ the group $p^{-1}(C)$ belongs to \mathcal{FJ} and that the group Q belongs to \mathcal{FJ} . Then G belongs to \mathcal{FJ} ;
 - (d) *Directed colimits*
Let $\{G_i \mid i \in I\}$ be a direct system of groups indexed by the directed set I (with arbitrary structure maps). Suppose that for each $i \in I$ the group G_i belongs to \mathcal{FJ} . Then the colimit $\text{colim}_{i \in I} G_i$ belongs to \mathcal{FJ} ;
 - (e) *Passing to finite free products*
*If the groups G_0 and G_1 belong to \mathcal{FJ} , then $G_0 * G_1$ belongs to \mathcal{FJ} ;*
 - (f) *Passing to overgroups of finite index*
Let G be an overgroup of H with finite index $[G : H]$. If H belongs to \mathcal{FJ} , then G belongs to \mathcal{FJ} ;

Proof. See [3, 4, 5, 7, 8, 10, 34, 54, 58, 109, 125, 126]. \square

It is not known whether all amenable groups belong to \mathcal{FJ} .

20.9 The Baum-Connes Conjecture

20.9.1 The Baum-Connes Conjecture

Recall that the *reduced group C^* -algebra* $C_r^*(G)$ is a certain completion of the complex group ring $\mathbb{C}G$. Namely, there is a canonical embedding of $\mathbb{C}G$ into the space $B(l^2(G))$ of bounded operators $L^2(G) \rightarrow L^2(G)$ equipped with the supremum norm given by the right regular representation, and $C_r^*(G)$ is the norm closure of $\mathbb{C}G$ in $B(L^2(G))$. There is a covariant functor respecting equivalences

$$\mathbf{K}^{\text{top}}: \text{Groupoids} \rightarrow \text{Spectra}, \tag{20.9.1}$$

such that for every group G and all $n \in \mathbb{Z}$ we have

$$\pi_n(\mathbf{K}^{\text{top}}(G)) \cong K_n^{\text{top}}(C_r^*(G)),$$

where $K_n^{\text{top}}(C_r^*(G))$ is the topological K -theory of the reduced group C^* -algebra $C_r^*(G)$, see [52]. If we now take this functors and the family \mathcal{FLN} of finite subgroups, we obtain

Conjecture 20.9.1 (Baum-Connes Conjecture). *A group G satisfies the Baum-Connes Conjecture if the assembly maps induced by the projection $\text{pr}: \underline{E}G \rightarrow G/G$*

$$H_n^G(\text{pr}; \mathbf{K}^{\text{top}}): H_n^G(\underline{E}G; \mathbf{K}^{\text{top}}) \rightarrow H_n^G(G/G; \mathbf{K}^{\text{top}}) = K_n^{\text{top}}(C_r^*(G)).$$

is bijective for all $n \in \mathbb{Z}$.

The original version of the Baum-Connes Conjecture is stated in [14, Conjecture 3.15 on page 254]. There is also a version, where the ground field \mathbb{C} is replaced by \mathbb{R} . The complex version of the Baum-Connes Conjecture 20.9.1 implies automatically the real version, see [16, 112].

20.9.2 The Baum-Connes Conjecture with coefficients

There is also a more general version of the Baum-Connes Conjecture 20.9.1, where one allows twisted coefficients. However, there are counterexamples to this more general version, see [48, Section 7]. There is a new formulation of the Baum-Connes Conjecture with coefficients in [15], where these counterexamples do not occur anymore. At the time of writing no counterexample to the Baum-Connes Conjecture 20.9.1 or to the version of [15] is known to the author.

20.9.3 The interpretation of the Baum Connes assembly map in terms of index theory

For applications of the Baum-Connes Conjecture 20.9.1 it is essential that the Baum-Connes assembly maps can be interpreted in terms of indices of equivariant operators with values in C^* -algebras. Namely, one assigns to a Kasparov cycle representing an element in the equivariant KK -group $KK_n^G(C_0(X), \mathbb{C})$ in the sense of Kasparov [55, 56, 57] its C^* -valued index in $K_n(C_r^*(G))$ in the sense of Mishchenko-Fomenko [92], thus defining a map

$$KK_n^G(C_0(X), \mathbb{C}) \rightarrow K_n^{\text{top}}(C_r^*(G)),$$

provided that X is proper and cocompact and $C_0(X)$ is the C^* -algebra (possibly without unit) of continuous function $X \rightarrow \mathbb{C}$ vanishing at infinity. This is the approach appearing in [14].

The other equivalent approach is based on the Kasparov product. Given a proper cocompact G - CW -complex X , one can assign to it an element $[p_X] \in KK_0^G(\mathbb{C}, C_0(X) \rtimes_r G)$, where $C_0(X) \rtimes_r G$ denotes the reduced crossed product C^* -algebra associated to the G - C^* -algebra $C_0(X)$. Now define the Baum-Connes assembly map by the composition of a descent map and a map coming from the Kasparov product

$$KK_n^G(C_0(X), \mathbb{C}) \xrightarrow{j_r^G} KK_n(C_0(X) \rtimes_r G, C_r^*(G)) \xrightarrow{[p_X] \widehat{\otimes}_{C_0(X) \rtimes_r G} -} KK_n(\mathbb{C}, C_r^*(G)) = K_n^{\text{top}}(C_r^*(G)).$$

This extends to arbitrary proper G - CW -complexes X by defining the source by

$$K_n^G(C_0(X), \mathbb{C}) := \text{colim}_{C \subseteq X} K_n^G(C_0(C), \mathbb{C}),$$

where C runs through the finite G - CW -subcomplexes of X directed by inclusion. Hence we can take $X = \underline{E}G$ above without assuming any finiteness conditions on $\underline{E}G$. For some information about these two approaches and their identification, at least for torsionfree G , we refer to [64].

One can identify the original assembly map of [14] with the assembly map appearing in Conjecture 20.9.1 using Section 20.9.1 and the fact that

$$\text{colim}_{C \subseteq X} H_n^G(C; \mathbf{K}^{\text{top}}) \xrightarrow{\cong} H_n^G(X; \mathbf{K}^{\text{top}})$$

is an isomorphism. This is explained in [26, page 247-248], Unfortunately, the proof is based on an unfinished preprint by Carlsson-Pedersen-Roe [20], where the assembly map appearing in [14, Conjecture 3.15 on page 254] is implemented on the spectrum level. Another proof of the identification is given in [44, Corollary 8.4] and [94, Theorem 1.3].

20.9.4 Applications of the Baum-Connes Conjecture

20.9.4.1 Computations

One can carry out *explicit computations* of topological K -groups of group C^* -algebras and related C^* -algebras by applying methods from algebraic topology to the left side given by a G -homology theory and by finding small models for the classifying spaces of families using the topology and geometry of groups. This leads to classification results about certain C^* -algebras, see for instance [28, 33, 66, 67].

20.9.4.2 (Modified) Trace Conjecture

The Baum-Connes Conjecture 20.9.1 implies the *Trace Conjecture for torsionfree groups* that for a torsionfree group G the image of

$$\mathrm{tr}_{C_r^*(G)} : K_0(C_r^*(G)) \rightarrow \mathbb{R}$$

consists of the integers. If one drops the condition torsionfree, there is the so called *Modified Trace Conjecture*, which is implied by Baum-Connes Conjecture 20.9.1, see [73].

20.9.4.3 Kadison Conjecture

The Baum-Connes Conjecture 20.9.1 implies the *Kadison Conjecture* that for a torsion-free group G the only idempotent elements in $C_r^*(G)$ are 0 and 1.

20.9.4.4 Novikov Conjecture

The Baum-Connes Conjecture 20.9.1 implies the *Novikov Conjecture*.

20.9.4.5 The Zero-in-the-spectrum Conjecture

The *Zero-in-the-spectrum Conjecture* says, if \widetilde{M} is the universal covering of an aspherical closed Riemannian manifold M , then zero is in the spectrum of the minimal closure of the p th Laplacian on \widetilde{M} for some $p \in \{0, 1, \dots, \dim M\}$. It is a consequence of the Strong Novikov Conjecture and hence of the Baum-Connes Conjecture 20.9.1, see [72, Chapter 12].

20.9.4.6 The Stable Gromov-Lawson-Rosenberg Conjecture

Let $\Omega_n^{\mathrm{Spin}}(BG)$ be the bordism group of closed Spin-manifolds M of dimension n with a reference map to BG . Given an element $[u : M \rightarrow BG] \in \Omega_n^{\mathrm{Spin}}(BG)$, we can take the $C_r^*(G; \mathbb{R})$ -valued index of the equivariant Dirac operator associated to the G -covering $\overline{M} \rightarrow M$ determined by u . Thus we get a homomorphism

$$\mathrm{ind}_{C_r^*(G; \mathbb{R})} : \Omega_n^{\mathrm{Spin}}(BG) \rightarrow K_n^{\mathrm{top}}(C_r^*(G; \mathbb{R})). \tag{20.9.2}$$

A *Bott manifold* is any simply connected closed Spin-manifold B of dimension 8 whose \widehat{A} -genus $\widehat{A}(B)$ is 1. We fix such a choice; the particular choice does not matter for the sequel. Notice that $\mathrm{ind}_{C_r^*(\{1\}; \mathbb{R})}(B) \in K_8^{\mathrm{top}}(\mathbb{R}) \cong \mathbb{Z}$ is a generator and the product with this element induces the Bott periodicity isomorphisms $K_n^{\mathrm{top}}(C_r^*(G; \mathbb{R})) \xrightarrow{\cong} K_{n+8}^{\mathrm{top}}(C_r^*(G; \mathbb{R}))$. In

particular

$$\operatorname{ind}_{C_r^*(G;\mathbb{R})}(M) = \operatorname{ind}_{C_r^*(G;\mathbb{R})}(M \times B), \quad (20.9.3)$$

if we identify $K_n^{\operatorname{top}}(C_r^*(G;\mathbb{R})) = K_{n+8}^{\operatorname{top}}(C_r^*(G;\mathbb{R}))$ via Bott periodicity.

Conjecture 20.9.2 (Stable Gromov-Lawson-Rosenberg Conjecture). *Let M be a closed connected Spin-manifold of dimension $n \geq 5$. Let $u_M: M \rightarrow B\pi_1(M)$ be the classifying map of its universal covering. Then $M \times B^k$ carries for some integer $k \geq 0$ a Riemannian metric with positive scalar curvature if and only if*

$$\operatorname{ind}_{C_r^*(\pi_1(M);\mathbb{R})}([M, u_M]) = 0 \in K_n^{\operatorname{top}}(C_r^*(\pi_1(M);\mathbb{R})).$$

If M carries a Riemannian metric with positive scalar curvature, then the index of the Dirac operator must vanish by the Bochner-Lichnerowicz formula [106]. The converse statement that the vanishing of the index implies the existence of a Riemannian metric with positive scalar curvature is the hard part of the conjecture. The unstable version of Conjecture 20.9.2, where one does not stabilize with B^k , is not true in general; see [110].

A sketch of the proof of the following result can be found in Stolz [114, Section 3].

Theorem 20.9.3 (The Baum-Connes Conjecture implies the Stable Gromov-Lawson-Rosenberg Conjecture). *If the assembly map for the real version of the Baum-Connes Conjecture 20.9.1 is injective for the group G , then the Stable Gromov-Lawson-Rosenberg Conjecture 20.9.2 is true for all closed Spin-manifolds of dimension ≥ 5 with $\pi_1(M) \cong G$.*

The space of positive scalar curvature metrics is analyzed for instance in [100, 129, 130].

20.9.4.7 Homotopy invariance of L^2 -Rho-invariants and L^2 -signature theorems

Results about the homotopy invariance of L^2 -Rho-invariants and L^2 -signature theorems based on the Baum-Connes Conjecture can be found in [22, 59, 60, 87, 88].

20.9.4.8 Knot theory

Cochran-Orr-Teichner give in [24] new obstructions for a knot to be a slice which are sharper than the Casson-Gordon invariants. They use L^2 -signatures and the Baum-Connes Conjecture 20.9.1. We also refer to the survey article [23] about non-commutative geometry and knot theory.

20.9.5 The status of the Baum-Connes Conjecture

Let \mathcal{BC} be the class of groups for which the Baum-Connes Conjecture with coefficients, which implies the Baum-Connes Conjecture 20.9.1, is true.

Theorem 20.9.4 (Status of the Baum-Connes Conjecture 20.9.1).

1. The following classes of groups belong to \mathcal{BC} .
 - (a) *A-T-menable groups;*
 - (b) *Hyperbolic groups;*
 - (c) *One-relator groups;*
 - (d) *Fundamental groups of compact 3-manifolds (possibly with boundary);*
2. The class \mathcal{BC} has the following inheritance properties:
 - (a) *Passing to subgroups*
 Let $H \subseteq G$ be an inclusion of groups. If G belongs to \mathcal{BC} , then H belongs to \mathcal{BC} ;
 - (b) *Passing to finite direct products*
 If the groups G_0 and G_1 belong to \mathcal{BC} , then also $G_0 \times G_1$ belongs to \mathcal{BC} ;
 - (c) *Group extensions*
 Let $1 \rightarrow K \rightarrow G \rightarrow Q \rightarrow 1$ be an extension of groups. Suppose that for any finite subgroup $F \subseteq Q$ the group $p^{-1}(F)$ belongs to \mathcal{BC} and that the group Q belongs to \mathcal{BC} . Then G belongs to \mathcal{BC} ;
 - (d) *Directed unions*
 Let $\{G_i \mid i \in I\}$ be a direct system of subgroups of G indexed by the directed set I such that $G = \bigcup_{i \in I} G_i$. Suppose that G_i belongs to \mathcal{BC} for every $i \in I$. Then G belongs to \mathcal{BC} ;
 - (e) *Actions on trees*
 Let G be a countable discrete group acting without inversion on a tree T . Then G belongs to \mathcal{BC} if and only if the stabilizers of each of the vertices of T belong to \mathcal{BC} . In particular \mathcal{BC} is closed under amalgamated products and HNN-extensions.

Proof. See [4, 21, 47, 63, 91, 96, 97]. □

It is not known whether finite-dimensional $\text{CAT}(0)$ -groups and $SL_n(\mathbb{Z})$ for $n \geq 3$ belong to \mathcal{BC} .

For more information about the Baum-Connes Conjecture and its applications we refer for instance to [14, 46, 49, 50, 51, 79, 82, 93, 99, 108, 111, 116].

20.9.6 Relating the assembly maps of Farrell-Jones to the one of Baum-Connes

One can construct the following commutative diagram

$$\begin{array}{ccc}
 H_n^G(\underline{EG}; \mathbf{L}_{\mathbb{Z}}^{(-\infty)})[1/2] & \longrightarrow & L_n^{(-\infty)}(\mathbb{Z}G)[1/2] & (20.9.4) \\
 \downarrow l \cong & & \downarrow \text{id} \cong & \\
 H_n^G(\underline{EG}; \mathbf{L}_{\mathbb{Z}}^{(-\infty)})[1/2] & \longrightarrow & L_n^{(-\infty)}(\mathbb{Z}G)[1/2] & \\
 \uparrow i_0 \cong & & \uparrow \cong j_0 & \\
 H_n^G(\underline{EG}; \mathbf{L}_{\mathbb{Z}}^p[1/2]) & \longrightarrow & L_n^p(\mathbb{Z}G)[1/2] & \\
 \downarrow i_1 \cong & & \downarrow \cong j_1 & \\
 H_n^G(\underline{EG}; \mathbf{L}_{\mathbb{Q}}^p[1/2]) & \longrightarrow & L_n^p(\mathbb{Q}G)[1/2] & \\
 \downarrow i_2 \cong & & \downarrow j_2 & \\
 H_n^G(\underline{EG}; \mathbf{L}_{\mathbb{R}}^p[1/2]) & \longrightarrow & L_n^p(\mathbb{R}G)[1/2] & \\
 \downarrow i_3 \cong & & \downarrow j_3 & \\
 H_n^G(\underline{EG}; \mathbf{L}_{C_r^*(?; \mathbb{R})}^p[1/2]) & \longrightarrow & L_n^p(C_r^*(G; \mathbb{R}))[1/2] & \\
 \uparrow i_4 \cong & & \uparrow \cong j_4 & \\
 H_n^G(\underline{EG}; \mathbf{K}_{\mathbb{R}}^{\text{top}}[1/2]) & \longrightarrow & K_n^{\text{top}}(C_r^*(G; \mathbb{R}))[1/2] & \\
 \uparrow \cong l & & \uparrow \text{id} \cong & \\
 H_n^G(\underline{EG}; \mathbf{K}_{\mathbb{R}}^{\text{top}})[1/2] & & K_n^{\text{top}}(C_r^*(G; \mathbb{R}))[1/2] & \\
 \downarrow i_5 & & \downarrow j_5 & \\
 H_n^G(\underline{EG}; \mathbf{K}_{\mathbb{C}}^{\text{top}})[1/2] & \longrightarrow & K_n(C_r^*(G))[1/2] &
 \end{array}$$

where all horizontal maps are assembly maps and the vertical arrows are induced by transformations of functors $\text{Groupoids} \rightarrow \text{Spectra}$. These transformations are induced by change of rings maps except the one from $\mathbf{K}_{\mathbb{R}}^{\text{top}}[1/2]$ to $\mathbf{L}_{C_r^*(?; \mathbb{R})}^p[1/2]$ which is much more complicated and carried out in [65]. This sophisticated and key ingredient was missing in [61, Lemma 22.13 on page 196], where the existence of such a diagram was claimed. The same remark applies also to [107, Theorem 2.7], see [65, Subsection 1.1]. Actually, it does not exist without inverting two on the spectrum level. Since it is a weak equivalence, the maps i_4 and j_4 are bijections.

For any finite group H each of the following maps is known to be a bijection because of [104, Proposition 22.34 on page 252] and $\mathbb{R}H = C_r^*(H; \mathbb{R})$

$$L_n^p(\mathbb{Z}H)[1/2] \xrightarrow{\cong} L_n^p(\mathbb{Q}H)[1/2] \xrightarrow{\cong} L_n^p(\mathbb{R}H)[1/2] \xrightarrow{\cong} L_n^p(C_r^*(H; \mathbb{R})).$$

The natural map $L_n^p(\mathbb{R}G)[1/2] \rightarrow L_n^{(-\infty)}(\mathbb{R}G)[1/2]$ is an isomorphism for any $n \in \mathbb{Z}$, group G and ring with involution R by the Rothenberg sequence, see [105, Theorem 17.2 on page 146]. Hence we conclude from the equivariant Atiyah Hirzebruch spectral sequence

that the vertical arrows $i_1, i_2,$ and i_3 are isomorphisms. The arrow j_1 is bijective by [103, page 376]. The maps l are isomorphisms for general results about localizations.

The lowermost vertical arrows i_5 and j_5 are known to be split injective because the inclusion $C_r^*(G; \mathbb{R}) \rightarrow C_r^*(G; \mathbb{C})$ induces an isomorphism $C_r^*(G; \mathbb{R}) \rightarrow C_r^*(G; \mathbb{C})^{\mathbb{Z}/2}$ for the $\mathbb{Z}/2$ -operation coming from complex conjugation $\mathbb{C} \rightarrow \mathbb{C}$. The following conjecture is already raised as a question in [61, Remark 23.14 on page 197], see also [65, Completion Conjecture in Subsection 5.2].

Conjecture 20.9.5 (Passage for L -theory from $\mathbb{Q}G$ to $\mathbb{R}G$ to $C_r^*(G; \mathbb{R})$). *The maps j_2 and j_3 appearing in diagram (20.9.4) are bijective.*

One easily checks

Lemma 20.9.6. *Let G be a group.*

1. *Suppose that G satisfies the L -theoretic Farrell-Jones Conjecture 20.8.1 with coefficients in the ring R for $R = \mathbb{Q}$ and $R = \mathbb{R}$ and the Baum-Connes Conjecture 20.9.1. Then G satisfies Conjecture 20.9.5;*
2. *Suppose that G satisfies Conjecture 20.9.5. Then G satisfies the L -theoretic Farrell-Jones Conjecture 20.8.1 for the ring \mathbb{Z} after inverting 2, if and only if G satisfies the real version of the Baum-Connes Conjecture 20.9.1 after inverting 2;*
3. *Suppose that the assembly map appearing in the Baum-Connes Conjecture 20.9.1 is (split) injective after inverting 2. Then the assembly map appearing in L -theoretic Farrell-Jones Conjecture 20.8.1 with coefficients in the ring for $R = \mathbb{Z}$ is (split) injective after inverting 2.*

20.10 Topological cyclic homology

Let \mathbf{R} be a (well-pointed connective) symmetric ring spectrum and p be a prime. There are covariant functors respecting equivalences

$$\begin{aligned} \mathbf{THH}_{\mathbf{R}}: \text{Groupoids} &\rightarrow \text{Spectra}; \\ \mathbf{TC}_{\mathbf{R},p}: \text{Groupoids} &\rightarrow \text{Spectra}, \end{aligned}$$

such that for every group G and all $n \in \mathbb{Z}$ we have

$$\begin{aligned} \pi_n(\mathbf{THH}_{\mathbf{R}}(G)) &\cong \pi_n(\mathbf{THH}(\mathbf{R}[G])); \\ \pi_n(\mathbf{TC}_{\mathbf{R},p}(G)) &\cong \pi_n(\mathbf{TC}(\mathbf{R}[G]; p)), \end{aligned}$$

where $\mathbf{THH}(\mathbf{R}[G])$ is the topological Hochschild homology and $\mathbf{THH}(\mathbf{R}[G]; p)$ is the topological cyclic homology of the group ring spectrum $\mathbf{R}[G]$.¹

¹See the chapter in this Handbook by Lars Hesselholt and Thomas Nikolaus for more information about topological Hochschild and cyclic homology.

20.10.1 Topological Hochschild homology

If we now take the functor $\mathbf{THH}_{\mathbf{R}}$ and the family \mathcal{CY} of cyclic subgroups, we obtain from [84, Theorem 1.19] that the Farrell-Jones Conjecture for topological Hochschild homology is true for all groups.

Theorem 20.10.1 (Topological Hochschild homology). *The assembly maps induced by the projection $\text{pr}: E_{\mathcal{CY}}(G) \rightarrow G/G$*

$$H_n^G(\text{pr}; \mathbf{THH}_{\mathbf{R}}): H_n^G(E_{\mathcal{CY}}(G); \mathbf{THH}_{\mathbf{R}}) \rightarrow H_n^G(G/G; \mathbf{THH}_{\mathbf{R}}) = \pi_n(\mathbf{THH}(\mathbf{R}[G]))$$

is bijective for all $n \in \mathbb{Z}$.

20.10.2 Topological cyclic homology

If we take the functor $\mathbf{TC}_{\mathbf{R};p}$ and the family \mathcal{FLN} of cyclic subgroups, we obtain from [83, Theorem 1.5] that the injectivity part of the Farrell-Jones Conjecture for topological cyclic homology is true under certain finiteness assumptions

Theorem 20.10.2 (Split injectivity for topological cyclic homology). *Assume that one for the following conditions hold for the family \mathcal{F} :*

1. *We have $\mathcal{F} = \mathcal{FLN}$ and there is a model for \underline{EG} of finite type;*
2. *We have $\mathcal{F} = \mathcal{VCY}$ and G is hyperbolic or virtually abelian.*

Then the assembly maps induced by the projection $\text{pr}: \underline{EG} \rightarrow G/G$

$$H_n^G(\text{pr}; \mathbf{TC}_{\mathbf{R};p}): H_n^G(\underline{EG}; \mathbf{TC}_{\mathbf{R};p}) \rightarrow H_n^G(G/G; \mathbf{TC}_{\mathbf{R};p}) = \pi_n(\mathbf{THH}(\mathbf{R}[G]; p))$$

is split injective for all $n \in \mathbb{Z}$.

Moreover, we also have, see [83, Theorem 1.2].

Theorem 20.10.3 (Topological cyclic homology and finite groups). *If G is a finite group, then the assembly map for the family \mathcal{CY} of cyclic subgroups*

$$H_n^G(\text{pr}; \mathbf{TC}_{\mathbf{R};p}): H_n^G(E_{\mathcal{CY}}(G); \mathbf{TC}_{\mathbf{R};p}) \rightarrow H_n^G(G/G; \mathbf{TC}_{\mathbf{R};p}) = \pi_n(\mathbf{TC}(\mathbf{R}[G]; p))$$

is bijective for all $n \in \mathbb{Z}$.

Remark 20.10.4 (The Farrell Jones Conjecture for topological cyclic homology is not true in general). There are examples, where the assembly map

$$H_n^G(\text{pr}; \mathbf{TC}_{\mathbf{R};p}): H_n^G(\underline{EG}; \mathbf{TC}_{\mathbf{R};p}) \rightarrow \pi_n(\mathbf{THH}(\mathbf{R}[G]; p))$$

not surjective, see [83, Theorem 1.6]. At least there is a pro-isomorphism for $\mathbf{TC}_{\mathbf{R};p}$ with respect to the family \mathcal{CY} , see [83, Theorem 1.4]. The complications occurring with topological cyclic homology are due to the fact that smash products and homotopy inverse limit do not commute in general, see [85].

20.10.3 Relating the assembly maps of Farrell-Jones to the one for topological cyclic homology via the cyclotomic trace

There is an important transformation from algebraic K -theory to topological cyclic homology, the so called *cyclotomic trace*. It relates the assembly maps for the algebraic K -theory of $\mathbb{Z}G$ to the cyclic topological homology of the spherical group ring of G and is a key ingredient in proving the rational injectivity of K -theoretic assembly maps. The construction of the cyclotomic trace and the proof of the K -theoretic Novikov conjecture is carried out in the celebrated paper by Boekstedt-Hsiang-Madsen [17]. The passage from \mathcal{TR} to \mathcal{FLN} , thus detecting a much larger portion of the algebraic K -theory of $\mathbb{Z}G$ and proving new results about the Whitehead group $\text{Wh}(G)$, is presented in [84].

For more information about topological cyclic homology we refer for instance to [17, 31, 95].

20.11 The global point of view

At various occasions it has turned out that one should take a global point of view, i.e., one should not consider each group separately, but take into account that in general there is a theory which can be applied to every group and the values for the various groups are linked. This appears for instance in the following definition taken from [71, Section 1].

Let $\alpha: H \rightarrow G$ be a group homomorphism. Given an H -space X , define the *induction of X with α* to be the G -space $\text{ind}_\alpha X := G \times_\alpha X$, which is the quotient of $G \times X$ by the right H -action $(g, x) \cdot h := (g\alpha(h), h^{-1}x)$ for $h \in H$ and $(g, x) \in G \times X$.

Definition 20.11.1 (Equivariant homology theory). An *equivariant homology theory with values in Λ -modules* $\mathcal{H}_n^?$ assigns to each group G a G -homology theory \mathcal{H}_*^G with values in Λ -modules (in the sense of Definition 20.3.1) together with the following so called *induction structure*:

Given a group homomorphism $\alpha: H \rightarrow G$ and a H -CW-pair (X, A) , there are for every $n \in \mathbb{Z}$ natural homomorphisms

$$\text{ind}_\alpha: \mathcal{H}_n^H(X, A) \rightarrow \mathcal{H}_n^G(\text{ind}_\alpha(X, A)) \tag{20.11.1}$$

satisfying:

- *Compatibility with the boundary homomorphisms*

$$\partial_n^G \circ \text{ind}_\alpha = \text{ind}_\alpha \circ \partial_n^H;$$

- *Functoriality*

Let $\beta: G \rightarrow K$ be another group homomorphism. Then we have for $n \in \mathbb{Z}$

$$\text{ind}_{\beta \circ \alpha} = \mathcal{H}_n^K(f_1) \circ \text{ind}_\beta \circ \text{ind}_\alpha: \mathcal{H}_n^H(X, A) \rightarrow \mathcal{H}_n^K(\text{ind}_{\beta \circ \alpha}(X, A)),$$

where $f_1: \text{ind}_\beta \text{ind}_\alpha(X, A) \xrightarrow{\cong} \text{ind}_{\beta \circ \alpha}(X, A)$, $(k, g, x) \mapsto (k\beta(g), x)$ is the natural K -homeomorphism;

- *Compatibility with conjugation*

For $n \in \mathbb{Z}$, $g \in G$ and a (proper) G -CW-pair (X, A) the homomorphism $\text{ind}_{c(g): G \rightarrow G}: \mathcal{H}_n^G(X, A) \rightarrow \mathcal{H}_n^G(\text{ind}_{c(g): G \rightarrow G}(X, A))$ agrees with $\mathcal{H}_n^G(f_2)$ for the G -homeomorphism $f_2: (X, A) \rightarrow \text{ind}_{c(g): G \rightarrow G}(X, A)$ that sends x to $(1, g^{-1}x)$ in $G \times_{c(g)}(X, A)$;

- *Bijectivity*

If $\text{Ker}(\alpha)$ acts freely on $X \setminus A$, then $\text{ind}_\alpha: \mathcal{H}_n^H(X, A) \rightarrow \mathcal{H}_n^G(\text{ind}_\alpha(X, A))$ is bijective for all $n \in \mathbb{Z}$.

Because of the following theorem it will pay off that in Subsection 20.7.4 we considered functors defined on Groupoids and not only on $\text{Or}(G)$.

Theorem 20.11.2 (Constructing equivariant homology theories using spectra). *Consider a covariant functor $\mathbf{E}: \text{Groupoids} \rightarrow \text{Spectra}$ respecting equivalences.*

Then there is an equivariant homology theory $H_^?(-; \mathbf{E})$ satisfying*

$$H_n^G(G/H; \mathbf{E}) \cong H_n^H(\{\bullet\}; \mathbf{E}) \cong \pi_n(\mathbf{E}(H))$$

for every subgroup $H \subseteq G$ of every group G and every $n \in \mathbb{Z}$.

Proof. See [82, Proposition 5.6 on page 793]. □

The global point of view has been taken up and pursued by Stefan Schwede on the level of spectra in his book [113], where global equivariant homotopy theory for compact Lie groups is developed. To deal with spectra is much more advanced and sophisticated than with equivariant homology.²

20.12 Relative assembly maps

In the formulations of the Isomorphism Conjectures above such as the one due to Farrell-Jones and Baum-Connes it is important to make the family \mathcal{F} as small as possible. The largest family we encounter is \mathcal{VCY} , but there are special cases, where one can get smaller families. In particular it is desirable to get away with \mathcal{FLN} , since there are often finite models for $\underline{E}G = E_{\mathcal{FLN}}(G)$, whereas conjecturally there is a finite model for $\underline{E}G = E_{\mathcal{VCY}}(G)$ only if G itself is virtually cyclic, see [53, Conjecture 1].

The general problem is to study and hopefully to prove bijectivity of relative assembly map associated to two families $\mathcal{F} \subseteq \mathcal{F}'$, i.e., of the map induced by the up to G -homotopy unique G -map $E_{\mathcal{F}}(G) \rightarrow E_{\mathcal{F}'}(G)$

$$\text{asmb}_{\mathcal{F} \subseteq \mathcal{F}'}: H_n^G(E_{\mathcal{F}}(G)) \rightarrow H_n^G(E_{\mathcal{F}'}(G))$$

for a G -homology theory H_*^G with values in Λ -modules. In studying this the global point of view becomes useful.

²See the chapter by Mike Hill in this Handbook for more information about equivariant stable homotopy theory.

The main technical result is the so called Transitivity Principle, which we explain next. For a family \mathcal{F} of subgroups of G and a subgroup $H \subset G$ we define a family of subgroups of H

$$\mathcal{F}|_H = \{K \cap H \mid K \in \mathcal{F}\}.$$

Theorem 20.12.1 (Transitivity Principle). *Let $\mathcal{H}_*^2(-)$ be an equivariant homology theory with values in Λ -modules. Suppose $\mathcal{F} \subset \mathcal{F}'$ are two families of subgroups of G . If for every $H \in \mathcal{F}'$ and every $n \in \mathbb{Z}$ the assembly map*

$$\text{asmb}_{\mathcal{F}|_H \subseteq \mathcal{A}\mathcal{L}\mathcal{L}}: \mathcal{H}_n^H(E_{\mathcal{F}|_H}(H)) \rightarrow \mathcal{H}_n^H(\{\bullet\})$$

is an isomorphism, then for every $n \in \mathbb{Z}$ the relative assembly map

$$\text{asmb}_{\mathcal{F} \subseteq \mathcal{F}'}: \mathcal{H}_n^G(E_{\mathcal{F}}(G)) \rightarrow \mathcal{H}_n^G(E_{\mathcal{F}'}(G))$$

is an isomorphism.

Proof. See [82, Theorem 65 on page 742]. □

One has the following results about diminishing the family of subgroups. Denote by \mathcal{VCY}_I the family of subgroups of G that are either finite or admit an epimorphism onto \mathbb{Z} with finite kernel. Obviously $\mathcal{FLN} \subseteq \mathcal{VCY}_I \subseteq \mathcal{VCY}$.

Theorem 20.12.2 (Relative assembly maps).

1. *The relative assembly map for K -theory*

$$\text{asmb}_{\mathcal{TR} \subseteq \mathcal{VCY}}: H_n^G(E_{\mathcal{TR}}(G); \mathbf{K}_R) \rightarrow H_n^G(E_{\mathcal{VCY}}(G); \mathbf{K}_R)$$

is bijective for all $n \in \mathbb{Z}$, provided that G is torsionfree and R is regular;

2. *The relative assembly map for K -theory*

$$\text{asmb}_{\mathcal{FLN} \subseteq \mathcal{VCY}}: H_n^G(E_{\mathcal{FLN}}(G); \mathbf{K}_R) \rightarrow H_n^G(E_{\mathcal{VCY}}(G); \mathbf{K}_R)$$

is bijective for all $n \in \mathbb{Z}$, provided that R is a regular ring containing \mathbb{Q} ;

3. *The relative assembly map for K -theory*

$$\text{asmb}_{\mathcal{VCY}_I \subseteq \mathcal{VCY}}: H_n^G(E_{\mathcal{VCY}_I}(G); \mathbf{K}_R) \xrightarrow{\cong} H_n^G(E_{\mathcal{VCY}}(G); \mathbf{K}_R)$$

is bijective for all $n \in \mathbb{Z}$;

4. *The relative assembly map for K -theory*

$$\text{asmb}_{\mathcal{FLN} \subseteq \mathcal{VCY}} \otimes_{\mathbb{Z}} \text{id}_{\mathbb{Q}}: H_n^G(E_{\mathcal{FLN}}(G); \mathbf{K}_R) \otimes_{\mathbb{Z}} \mathbb{Q} \rightarrow H_n^G(E_{\mathcal{VCY}}(G); \mathbf{K}_R) \otimes_{\mathbb{Z}} \mathbb{Q}$$

is bijective for all $n \in \mathbb{Z}$, provided that R is regular;

5. *The relative assembly map for L -theory*

$$\text{asmb}_{\mathcal{TR} \subseteq \mathcal{VCY}}: H_n^G(E_{\mathcal{TR}}(G); \mathbf{L}_R^{(-\infty)}) \rightarrow H_n^G(E_{\mathcal{VCY}}(G); \mathbf{L}_R^{(-\infty)})$$

is an isomorphism for all $n \in \mathbb{Z}$, provided that G is torsionfree;

6. The relative assembly map for L -theory

$$\text{asmb}_{\mathcal{FLN} \subseteq \mathcal{VCY}_I}: H_n^G(E_{\mathcal{FLN}}(G); \mathbf{L}_R^{\langle -\infty \rangle}) \rightarrow H_n^G(E_{\mathcal{VCY}_I}(G); \mathbf{L}_R^{\langle -\infty \rangle})$$

is bijective for all $n \in \mathbb{Z}$;

7. The relative assembly map for L -theory

$$\text{asmb}_{\mathcal{FLN} \subseteq \mathcal{VCY}}[1/2]: H_n^G(E_{\mathcal{FLN}}(G); \mathbf{L}_R^{\langle -\infty \rangle})[1/2] \rightarrow H_n^G(E_{\mathcal{VCY}}(G); \mathbf{L}_R^{\langle -\infty \rangle})[1/2]$$

is bijective for all $n \in \mathbb{Z}$;

8. The relative assembly map for topological K -theory

$$\text{asmb}_{\mathcal{FCY} \subseteq \mathcal{FLN}}: H_n^G(E_{\mathcal{FCY}}(G); \mathbf{K}^{\text{top}}) \rightarrow H_n^G(E_{\mathcal{FLN}}(G); \mathbf{K}^{\text{top}})$$

is bijective for all $n \in \mathbb{Z}$. This is also true for the real version;

9. The relative assembly maps for K -theory and L -theory

$$\begin{aligned} \text{asmb}_{\mathcal{FLN} \subseteq \mathcal{VCY}}: H_n^G(E_{\mathcal{FLN}}(G); \mathbf{K}_R) &\rightarrow H_n^G(E_{\mathcal{VCY}}(G); \mathbf{K}_R); \\ \text{asmb}_{\mathcal{FLN} \subseteq \mathcal{VCY}}: H_n^G(E_{\mathcal{FLN}}(G); \mathbf{L}_R^{\langle -\infty \rangle}) &\rightarrow H_n^G(E_{\mathcal{VCY}}(G); \mathbf{L}_R^{\langle -\infty \rangle}), \end{aligned}$$

are split injective for all $n \in \mathbb{Z}$.

Proof. See [12, Theorem 1.3], [6, Theorem 0.5], [29], [76, Lemma 4.2], [82, Section 2.5], [89, Theorem 0.1 and Theorem 0.3], and [90]. \square

Remark 20.12.3 (Torsionfree groups). A typical application is that for a torsionfree group G and a regular ring the K -theoretic Farrell-Jones Conjecture 20.8.1 implies together with Theorem 20.12.2 (1) that the assembly map

$$H_n(BG; \mathbf{K}(R)) = \pi_n(BG_+ \wedge \mathbf{K}(R)) \rightarrow K_n(RG)$$

is bijective for all $n \in \mathbb{Z}$. Analogously, for a torsionfree group G the L -theoretic Farrell-Jones Conjecture 20.8.1 implies together with Theorem 20.12.2 (5) that the assembly map

$$H_n(BG; \mathbf{L}^{\langle -\infty \rangle}(R)) = \pi_n(BG_+ \wedge \mathbf{L}^{\langle -\infty \rangle}) \rightarrow L_n^{\langle -\infty \rangle}(RG)$$

is bijective for all $n \in \mathbb{Z}$.

20.13 Tools for computations

Most computations of K - and L -groups of group rings are done using the Farrell-Jones Conjecture 20.8.1 and the Baum-Connes Conjecture 20.9.1. The situation in the Farrell-Jones Conjecture 20.8.1 is more complicated than in the Baum-Connes setting, since the

family \mathcal{VCY} is much harder to handle than the family \mathcal{FLN} . One can consider $H_n^G(\underline{EG}; \mathbf{K}_R)$ and $H_n^G(\underline{EG}, \underline{EG}; \mathbf{K}_R)$ separately because of Theorem 20.12.2 (9), where one considers \underline{EG} as a G -CW-subcomplex of \underline{EG} . The term $H_n^G(\underline{EG}, \underline{EG}; \mathbf{K}_R)$ involves Nil-terms and UNil-terms, which are hard to determine. For $H_n^G(\underline{EG}; \mathbf{K}_R)$, $H_n^G(\underline{EG}; \mathbf{L}_R^{\langle -\infty \rangle})$ and $H_n^G(\underline{EG}; \mathbf{K}^{\text{top}})$ one can use the equivariant Atiyah-Hirzebruch spectral sequence or the p -chain spectral sequence, see Davis-Lueck [27]. Rationally these groups can often be computed explicitly using equivariant Chern characters, see [71, Section 1]. Notice that these can only be constructed since we take on the global point of view as explained in Section 20.11. Often an important input is that one obtains from the geometry of the underlying group nice models for \underline{EG} and can construct \underline{EG} from \underline{EG} by attaching a tractable family of equivariant cells.

Here are two example, where these ideas lead to a explicit computation, whose outcome is as simple as one can hope.

Theorem 20.13.1 (Farrell-Jones Conjecture for torsionfree hyperbolic groups for K -theory). *Let G be a torsionfree hyperbolic group.*

1. We obtain for all $n \in \mathbb{Z}$ an isomorphism

$$H_n(BG; \mathbf{K}(R)) \oplus \bigoplus_C (NK_n(R) \oplus NK_n(R)) \xrightarrow{\cong} K_n(RG),$$

where C runs through a complete system of representatives of the conjugacy classes of maximal infinite cyclic subgroups. If R is regular, we have $NK_n(R) = 0$ for all $n \in \mathbb{Z}$;

2. The abelian groups $K_n(\mathbb{Z}G)$ for $n \leq -1$, $\tilde{K}_0(\mathbb{Z}G)$, and $\text{Wh}(G)$ vanish;
3. We get for every ring R with involution and $n \in \mathbb{Z}$ an isomorphism

$$H_n(BG; \mathbf{L}^{\langle -\infty \rangle}(R)) \xrightarrow{\cong} L_n^{\langle -\infty \rangle}(RG).$$

For every $j \in \mathbb{Z}, j \leq 2$, and $n \in \mathbb{Z}$, the natural map

$$L_n^{\langle j \rangle}(\mathbb{Z}G) \xrightarrow{\cong} L_n^{\langle -\infty \rangle}(\mathbb{Z}G)$$

is bijective;

4. We get for every $n \in \mathbb{Z}$ an isomorphism

$$K_n^{\text{top}}(BG) \xrightarrow{\cong} K_n^{\text{top}}(C_r^*(G)).$$

Proof. See [86, Theorem 1.2]. □

Theorem 20.13.2. *Suppose that G satisfies the Baum-Connes Conjecture 20.9.1 and K -theoretic Farrell-Jones Conjecture 20.8.1 with coefficients in the ring \mathbb{C} . Let $\text{con}(G)_f$ be the set of conjugacy classes (g) of elements $g \in G$ of finite order. We denote for $g \in G$ by $C_G\langle g \rangle$ the centralizer in G of the cyclic subgroup generated by g .*

Then we get the following commutative square, whose horizontal maps are isomorphisms and complexifications of assembly maps, and whose vertical maps are induced by the obvious change of theory homomorphisms

$$\begin{array}{ccc} \bigoplus_{p+q=n} \bigoplus_{(g) \in \text{con}(G)_f} H_p(C_G \langle g \rangle; \mathbb{C}) \otimes_{\mathbb{Z}} K_q(\mathbb{C}) & \xrightarrow{\cong} & K_n(\mathbb{C}G) \otimes_{\mathbb{Z}} \mathbb{C} \\ \downarrow & & \downarrow \\ \bigoplus_{p+q=n} \bigoplus_{(g) \in \text{con}(G)_f} H_p(C_G \langle g \rangle; \mathbb{C}) \otimes_{\mathbb{Z}} K_q^{\text{top}}(\mathbb{C}) & \xrightarrow{\cong} & K_n^{\text{top}}(C_r^*(G)) \otimes_{\mathbb{Z}} \mathbb{C}. \end{array}$$

Proof. See [71, Theorem 0.5]. □

20.14 The challenge of extending equivariant homotopy theory to infinite groups

We have seen that it is important to study equivariant homology and homotopy also for groups that are not necessarily finite. In particular equivariant KK-theory has developed into a whole industry, which, however, does not really take the point of view of spectra instead of K -groups and cycles into account. So we encounter

Problem 20.14.1. Extend equivariant homotopy theory for finite groups to infinite groups, at least in the case of proper G -actions.

A few first steps are already in the literature. We have already explained the notion of an equivariant homology and the existence of equivariant Chern characters, see [71], where the global point of view enters. There is also a cohomological version, see [75]. Topological K -theory has systematically been studied in [80, 81], and an attempt of defining Burnside rings and equivariant cohomotopy for proper G -spaces is presented in [74]. These do include multiplicative structures.

An important ingredient in equivariant homotopy theory for finite groups is to stabilize with unit spheres in finite-dimensional orthogonal representations. However, there are infinite groups such that any finite-dimensional representation is trivial and therefore one has to stabilize with equivariant vector bundles, see [74, Remark 6.17]. Or one may have to pass even to Hilbert bundles and equivariant Fredholm operator between these, see [98] and also [81].

There are various interesting pairings on the group level in the literature, such as Kasparov products, the action of Swan groups on algebraic K -theory and so on. They all should be implemented on the spectrum level. So a systematical study of higher structures for equivariant spectra over infinite groups has to be carried out and one has to find the right equivariant homotopy category. This applies also to multiplicative structures and smash products. First steps will be presented in [30] using orthogonal spectra. This seems to work well for topological K -theory, but is probably not adequate for algebraic K -theory. This remark also holds for global equivariant homotopy theory.

A general description of Mackey structures and induction theorems in the sense of Dress is described in [6]. There are more sophisticated Mackey structure and transfers in the

equivariant homotopy of finite groups, but it is not at all clear whether and how they extend to infinite groups.

Topological K -theory and the Baum-Connes Conjecture make sense and are studied also for topological groups, e.g., reductive p -adic groups and Lie groups. It is conceivable that also the Farrell-Jones Conjecture has an analogue for Hecke algebras of totally disconnected groups, see [82, Conjecture 119 on page 773]. So one can ask Problem 20.14.1 also for (not necessarily compact) topological groups instead of infinite (discrete) groups.

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