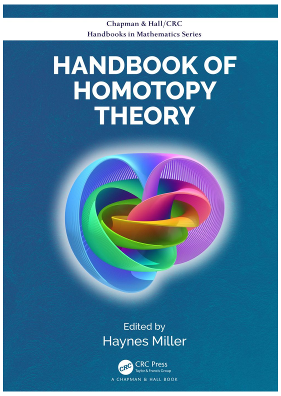

Handbook of Homotopy Theory

Haynes Miller

E -spectra and Dyer-Lashof operations

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E_n-spectra and Dyer-Lashof operations

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19.1 Introduction

Cohomology operations are absolutely essential in making cohomology an effective tool for studying spaces. In particular, the mod-p cohomology groups of a space X are enhanced with a binary cup product, a Bockstein derivation, and Steenrod's reduced power operations; these satisfy relations such as graded-commutativity, the Cartan formula, the Adem relations, and the instability relations [90]. The combined structure of these cohomology operations is very effective in homotopy theory because of three critical properties.

These operations are natural. We can exclude the possibility of certain maps between spaces because they would not respect these operations.

These operations are constrained. We can exclude the existence of certain spaces because the cup product and power operations would be incompatible with the relations that must hold.

These operations are complete. Because cohomology is *representable*, we can determine all possible natural operations which take an n-tuple of cohomology elements and produce a new one. All operations are built, via composition, from these basic operations. All relations between these operations are similarly built from these basic relations.

In particular, this last property makes the theory reversible: there are mechanisms which take cohomology as input and converge to essentially complete information about homotopy theory in many useful cases, with the principal examples being the stable and unstable Adams spectral sequences. The stable Adams spectral sequence begins with the Ext-groups $\mathrm{Ext}(H^*(Y), H^*(X))$ in the category of modules with Steenrod operations and converges to the stable classes of maps from X to a p-completion of Y [1]. The unstable Adams spectral sequence is similar, but it begins with nonabelian Ext-groups that are calculated in the category of graded-commutative rings with Steenrod operations [19, 18].

Our goal is to discuss multiplicative homotopy theory: spaces, categories, or spectra with extra multiplicative structure. In this situation, we will see that the *Dyer–Lashof operations* play the role that the Steenrod operations did in ordinary homotopy theory.

In ordinary algebra, commutativity is an extremely useful *property* possessed by certain monoids and algebras. This is no longer the case in multiplicative homotopy theory

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or category theory. In category theory, commutativity becomes *structure*: to give symmetry to a monoidal category C we must make a choice of a natural twist isomorphism $\tau\colon A \otimes B \to B \otimes A$. Moreover, there are more degrees of symmetry possible than in algebra because we can ask for weaker or stronger identities on τ. By asking for basic identities to hold we obtain the notion of a braided monoidal category, and by asking for very strong identities to hold we obtain the notion of a symmetric monoidal category. In homotopy theory and higher category theory we rarely have the luxury of imposing identities, and these become replaced by extra structure. One consequence is that there are many degrees of commutativity, parametrized by operads.

The most classical such structures arose geometrically in the study of iterated loop spaces. For a pointed space X, the n-fold loop space $\Omega^n X$ has algebraic operations parametrized by certain configuration spaces $\mathcal{E}_n(k)$, which assemble into an E_n*-operad*; moreover, there is a converse theorem due to Boardman–Vogtand May that provides a recognition principle for what structure on Y is needed to express it as an iterated loop space. As n grows, these spaces possess more and more commutativity, reflected algebraically in extra Dyer–Lashof operations on the homology $H_* Y$ that are analogous to the Steenrod operations.

In recent years there is an expanding library of examples of ring spectra that only admit, or only naturally admit, these intermediate levels of structure between associativity and commutativity. Our goal in this chapter is to give an outline of the modern theory of highly structured ring spectra, particularly E_n ring spectra, and to give a toolkit for their study. One of the things that we would like to emphasize is how to usefully work in this setting, and so we will discuss useful tools that are imparted by E_n ring structures, such as operations on them that unify the study of Steenrod and Dyer–Lashof operations. We will also introduce the next stage of structure in the form of secondary operations. Throughout, we will make use of these operations to show that structured ring spectra are heavily constrained, and that many examples do not admit this structure; we will in particular discuss our proof in [47] that the 2-primary Brown–Peterson spectrum does not admit the structure of an E_∞ ring spectrum, answering an old question of May [60]. At the close we will discuss some ongoing directions of study.

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19.3 Operads and algebras

Throughout this section, we will let C be a fixed symmetric monoidal topological category. For us, this means that C is enriched in the category S of spaces, that there is a functor $\otimes\colon \mathsf{C}\times\mathsf{C}\to\mathsf{C}$ of enriched categories, and that the underlying functor of ordinary categories is extended to a symmetric monoidal structure. We will write $\mathrm{Map}_{\mathsf{C}}(X,Y)$ for the mapping space between two objects, and $\mathrm{Hom}_{\mathsf{C}}(X,Y)$ for the underlying set. Associated to C there is the (ordinary) homotopy category $h\mathsf{C}$, with morphisms $[X,Y]=\pi_0\mathrm{Map}_{\mathsf{C}}(X,Y)$.

19.3.1 Operads

Associated to any object $X\in\mathsf{C}$ there is an *endomorphism operad* $\mathrm{End}_{\mathsf{C}}(X)$. The k'th term is

$$\mathrm{Map}_{\mathsf{C}}(X^{\otimes k},X),$$

with an operad structure given by composition of functors. For any operad $\mathcal{O}$, this allows us to discuss $\mathcal{O}$-algebra structures on the objects of C, maps of $\mathcal{O}$-algebras, and further structure.

If $\mathcal{O}$ is the associative operad $\mathcal{A}ssoc$, then $\mathcal{O}$-algebras are monoid objects in the symmetric monoidal structure on C. If $\mathcal{O}$ is the commutative operad $\mathcal{C}omm$, then $\mathcal{O}$-algebras are strictly commutative monoids in C. However, these operads are highly rigid and do not take any space-level structure into account. Mapping spaces allow us to encode many different levels of structure.

Example 19.3.1. There is a sequence of operads $\mathcal{A}_1\to\mathcal{A}_2\to\mathcal{A}_3\to\cdots$ built out of the Stasheff associahedra [89]. An $\mathcal{A}_2$-algebra has a unital binary multiplication; an $\mathcal{A}_3$-algebra has a chosen homotopy expressing associativity, and has Massey products; an $\mathcal{A}_4$-algebra has a homotopy expressing a juggling formula for Massey products; and so on. Moreover, each operad is simply built from the previous: extension from an $\mathcal{A}_{n-1}$-structure to an $\mathcal{A}_n$-structure roughly asks to extend a certain map $S^{n-3}\times X^n\to X$ to a map $D^{n-2}\times X^n\to X$ expressing an n-fold coherence law for the multiplication [3]. This gives $\mathcal{A}_n$ a *perturbative* property: if $X\to Y$ is a homotopy equivalence, then $\mathcal{A}_n$-algebra structures on one space can be transferred to the other.

Example 19.3.2. The colimit of the $\mathcal{A}_n$-operads is called $\mathcal{A}_\infty$, and it is equivalent to the associative operad. It satisfies a *rectification* property: In a well-behaved category like the category S of spaces or the category Sp of spectra, any $\mathcal{A}_\infty$-algebra is equivalent in the homotopy category of $\mathcal{A}_\infty$-algebras to an associative object.

Example 19.3.3. There is a sequence of operads $\mathcal{E}_1\to\mathcal{E}_2\to\mathcal{E}_3\to\cdots$, where the space $\mathcal{E}_n(k)$ is homotopy equivalent to the configuration space of unordered k-tuples of points in $\mathbb{R}^n$. These have various models, such as the *little cubes* or *little discs* operads. The $\mathcal{E}_1$-operad is equivalent to the associative operad, and the $\mathcal{E}_\infty$-operad is equivalent to the commutative operad. We refer to an algebra over any operad equivalent to $\mathcal{E}_n$ as an *E_n-algebra.* These play an important role in the *recognition principle* [62, 16]: given an E_n-algebra X we can construct an n-fold classifying space $B^n X$; and if the binary multiplication makes $\pi_0(X)$ into a group then $X \simeq \Omega^n B^n X$. The relation between E_n-algebra structures and iteration of the functor Ω is closely related to an additivity result of Dunn [25], who showed that E_{n+1}-algebras are equivalent to E_1-algebras in the category of E_n-algebras.

Example 19.3.4. Associated to a topological monoid M, there is an operad $\mathcal{O}_M$ whose only nonempty space is $\mathcal{O}_M(1) = M$. An algebra over this operad is precisely an object with M-action. This operad is usually not perturbative. However, M can be resolved by a *cellular* topological monoid $\widetilde{M} \to M$ such that $\mathcal{O}_{\widetilde{M}}$-algebras are perturbative and can be rectified to $\mathcal{O}_M$-algebras. This construction is a recasting of Cooke's obstruction theory for lifting homotopy actions of a group G to honest actions [24]; stronger versions of this were developed by Dwyer–Kan and Badzioch [26, 4].

Example 19.3.5. There is a free-forgetful adjunction between operads and symmetric sequences. Given any sequence of spaces Z_n with Σ_n-actions, we can construct an operad $\mathrm{Free}(Z)$ such that a $\mathrm{Free}(Z)$-algebra structure is the same as a collection of Σ_n-equivariant maps $Z_n \to \mathrm{Map}_{\mathsf{C}}(A^{\otimes n}, A)$.

If, further, Z_1 is equipped with a chosen point e, we can construct an operad $\mathrm{Free}(Z, e)$ such that a $\mathrm{Free}(Z, e)$-algebra structure is the same as a $\mathrm{Free}(Z)$-algebra structure such that e acts as the identity: $\mathrm{Free}(Z, e)$ is a pushout of a diagram $\mathrm{Free}(Z) \leftarrow \mathrm{Free}(\{e\}) \to \mathrm{Free}(\emptyset)$ of operads.

Example 19.3.6. In the previous example, let Z_2 be S^1 with the antipodal action of Σ_2 and let all other Z_n be empty, freely generating an operad $\mathcal{Q}_1$ that we call the *cup-1 operad*. A $\mathcal{Q}_1$-algebra is an object A with a Σ_2-equivariant map $S^1 \to \mathrm{Map}_{\mathsf{C}}(A^{\otimes 2}, A)$. The Σ_2-equivariant cell decomposition of S^1 allows us to describe $\mathcal{Q}_1$-algebras as objects with a binary multiplication m and a chosen homotopy from the multiplication m to the multiplication in the opposite order $m \circ \sigma$. In particular, any homotopy-commutative multiplication lifts to a $\mathcal{Q}_1$-algebra structure.

In the category Sp of spectra, one of the main applications of E_n-algebras is that they have well-behaved categories of modules, whose homotopy categories are triangulated categories.

Theorem 19.3.7 (Mandell [57])**.** *An E_1-algebra R in Sp has a category of left modules LMod_R. An E_2-algebra structure on R makes the homotopy categories of left modules and right modules equivalent, and gives the homotopy category of left modules a monoidal structure $\otimes_R$. An E_3-algebra structure on R extends this monoidal structure to a braided monoidal structure. An E_4-algebra structure on R makes this braided monoidal structure into a symmetric monoidal structure.*

Theorem 19.3.8. *An E_1-algebra R in Sp has a monoidal category of bimodules. An E_∞-algebra R in Sp has a symmetric monoidal category of left modules.*

19.3.2 Monads

If C is not just enriched, but is *tensored* over spaces, an $\mathcal{O}$-algebra structure on X is expressible in terms internal to C. An $\mathcal{O}$-algebra structure is equivalent to having *action* *maps*

$$\gamma_k\colon\ \mathcal{O}(k) \otimes X^{\otimes k} \to X$$

that are invariant under the action of Σ_k and respect composition in the operad $\mathcal{O}$. If C has colimits, we can define *extended power constructions*

$$\mathrm{Sym}^k_{\mathcal{O}}(X) = \left(\mathcal{O}(k) \otimes_{\Sigma_k} X^{\otimes k}\right),$$

and an associated free $\mathcal{O}$-algebra functor

$$\mathrm{Free}_{\mathcal{O}}(X) = \coprod_{k \geq 0} \mathrm{Sym}^k_{\mathcal{O}}(X).$$

An $\mathcal{O}$-algebra structure on X is then determined by a single map $\mathrm{Free}_{\mathcal{O}}(X) \to X$. To say more, we need C to be compatible with enriched colimits in the sense of [42, §3].

Definition 19.3.9. A symmetric monoidal category C is *compatible with enriched colimits* if the monoidal structure on C preserves enriched colimits in each variable separately.

Compatibility with enriched colimits is necessary to give composite action maps

$$\mathcal{O}(k) \otimes \left(\bigotimes_{i=1}^{k} \mathcal{O}(n_i) \otimes X^{\otimes n_i}\right) \to X^{\otimes \Sigma n_i}$$

and make them assemble into a monad structure $\mathrm{Free}_{\mathcal{O}} \circ \mathrm{Free}_{\mathcal{O}} \to \mathrm{Free}_{\mathcal{O}}$ on this free functor. In this case, $\mathcal{O}$-algebras are equivalent to $\mathrm{Free}_{\mathcal{O}}$-algebras, and $\mathrm{Sym}^k_{\mathcal{O}}$ and $\mathrm{Free}_{\mathcal{O}}$ are enriched functors.

When these functors are enriched functors, they also give rise to a monad on the homotopy category $h\mathsf{C}$. We refer to algebras over it as *homotopy $\mathcal{O}$-algebras*. This is strictly stronger than being an $\mathcal{O}$-algebra in the homotopy category; the latter asks for compatible maps $\pi_0\mathcal{O}(n) \to [A^{\otimes n}, A]$, whereas the former asks for compatible elements in $[\mathcal{O}(n) \otimes_{\Sigma_n} A^{\otimes n}, A]$ that use $\mathcal{O}$ before passing to homotopy. In the case of the E_n-operads, such a structure in the homotopy category is what is classically known as an *$\mathcal{H}_n$-algebra* [20].

This type of structure can be slightly rigidified using pushouts of free algebras. For any operad $\mathcal{O}$ with identity $e \in \mathcal{O}(1)$, we can construct a homotopy coequalizer diagram

$$\mathrm{Free}(\mathrm{Free}(\mathcal{O}, e), e) \rightrightarrows \mathrm{Free}(\mathcal{O}, e) \to \mathcal{O}^h$$

in the category of operads. An object A has an $\mathcal{O}^h$-algebra structure if and only if there are Σ_k-equivariant maps $\mathcal{O}(k) \to \mathrm{Map}_{\mathsf{C}}(A^{\otimes k}, A)$ so that the associativity diagram homotopy commutes and so that e acts by the identity. In particular, A has an $\mathcal{O}^h$-algebra structure if and only if it has a homotopy $\mathcal{O}$-algebra structure; the $\mathcal{O}^h$-structure has a *chosen* homotopy for the associativity of composition. For example, there is an operad parametrizing objects with a unital binary multiplication, a chosen associativity homotopy, and a chosen commutativity homotopy.

19.3.3 Connective algebras

In the category of spectra, the Eilenberg–Mac Lane spectra HA are characterized by a useful mapping property. We refer to a spectrum as *connective* if it is (-1)-connected. For any connective spectrum X, the natural map

$$\mathrm{Map}_{\mathsf{Sp}}(X, HA) \to \mathrm{Hom}_{\mathsf{Ab}}(\pi_0 X, A)$$

is a weak equivalence.

This has a number of strong consequences. For example, we get an equivalence of endomorphism operads $\mathrm{End}_{\mathsf{Sp}}(HA) \to \mathrm{End}_{\mathsf{Ab}}(A)$, obtained by taking π_0:

$$\mathrm{End}_{\mathsf{Sp}}(HA)_k = \mathrm{Map}(HA^{\otimes k}, HA) \simeq \mathrm{End}_{\mathsf{Ab}}(A)_k = \mathrm{Hom}(A^{\otimes k}, A).$$

Thus, an action of an operad $\mathcal{O}$ on HA is equivalent to an action of $\pi_0\mathcal{O}$ on A, and this equivalence is natural. This technique also generalizes, using the equivalences

$$\mathrm{Hom}(H\pi_0 R^{\otimes n}, H\pi_0 R) \xrightarrow{\sim} \mathrm{Map}(R^{\otimes n}, H\pi_0 R).$$

Proposition 19.3.10. *Suppose R is a connective spectrum and $\mathcal{O}$ is an operad acting on R. Then the map $R \to H\pi_0(R)$ can be given, in a functorial way, the structure of a map of $\mathcal{O}$-algebras.*

Example 19.3.11. If A is given the structure of a commutative ring, HA inherits an essentially unique structure of an E_∞-algebra. If R is a connective and homotopy commutative ring spectrum, then it can be equipped with an action of the cup-1 operad $\mathcal{Q}_1$ from 19.3.6. Any ring homomorphism $\pi_0 R \to A$ lifts to a map of $\mathcal{Q}_1$-algebras $R \to HA$.

19.3.4 Example algebras

Example 19.3.12. There exist models for the category of spectra so that the function spectrum

$$F(\Sigma^\infty_+ X, A) = A^X$$

is a lax monoidal functor $\mathsf{S}^{op} \times \mathsf{Sp} \to \mathsf{Sp}$, with the homotopy groups of A^X being the unreduced A-cohomology groups of X. The diagonal Δ makes any space X into a commutative monoid in S^{op}. If A is an $\mathcal{O}$-algebra in Sp, then A^X then becomes an $\mathcal{O}$-algebra.

Example 19.3.13. For any spectrum E, composition of functions naturally gives the *endomorphism algebra* spectrum $\mathrm{End}(E) = F(E, E)$ the structure of an $\mathcal{A}_\infty$-algebra, and E is a left module over $\mathrm{End}(E)$. The homotopy groups of $\mathrm{End}(E)$ are sometimes called the *E-Steenrod algebra* and they parametrize operations on E-cohomology.

Example 19.3.14. The suspension spectrum functor

$$X \mapsto \Sigma^\infty_+ X = \mathbb{S}[X]$$

is strong symmetric monoidal. As a result, it takes $\mathcal{O}$-algebras to $\mathcal{O}$-algebras. For example, any topological group G has an associated *spherical group algebra* $\mathbb{S}[G]$.

Example 19.3.15. For any pointed space X, the n-fold loop space $\Omega^n X$ is an E_n-algebra in spaces, and $\mathbb{S}[\Omega^n X]$ is an E_n-algebra. For any spectrum Y the space $\Omega^\infty Y$ is an E_∞-algebra in spaces, and $\mathbb{S}[\Omega^\infty Y]$ is an E_∞-algebra.

Example 19.3.16. The Thom spectra MO and MU have E_∞ ring structures [63]. At any prime p, MU decomposes into a sum of shifts of the Brown–Peterson spectrum BP, which has the structure of an E_4-ring spectrum [10].

Example 19.3.17. The smash product being symmetric monoidal implies that it is also a strong symmetric monoidal functor $\mathsf{Sp} \times \mathsf{Sp} \to \mathsf{Sp}$. If A and B are $\mathcal{O}$-algebras then so is $A \otimes B$.

Example 19.3.18. For a map $Q \to R$ of E_∞ ring spectra, there is an adjunction

$$\mathrm{Mod}_Q \rightleftarrows \mathrm{Mod}_R$$

between the extension of scalars functor $M \mapsto R \otimes_Q M$ and the forgetful functor. The left adjoint is strong symmetric monoidal and the right adjoint is lax symmetric monoidal, and hence both functors preserve $\mathcal{O}$-algebras.

This allows us to narrow our focus. For example, if E has an E_∞-algebra structure and we are interested in understanding operations on the E-homology of $\mathcal{O}$-algebras, we can restrict our attention to those operations on the homotopy groups of $\mathcal{O}$-algebras in Mod_E rather than considering all possible operations on the E-homology.

19.3.5 Multicategories

A *multicategory* (or colored operad) encodes the structure of a category where functions have multiple input objects. They serve as a useful way to encode many multilinear structures in stable homotopy theory: multiplications, module structures, graded rings, and coherent structures on categories. In this section we will give a quick introduction to them, and will return in §19.7.

Definition 19.3.19. A *multicategory* $\mathcal{M}$ consists of the following data:

1. a collection $Ob(\mathcal{M})$ of objects;
2. a set $\mathrm{Mul}_{\mathcal{M}}(\mathbf{x}_1, \ldots, \mathbf{x}_d; \mathbf{y})$ of *multimorphisms* for any objects $\mathbf{x}_i$ and $\mathbf{y}$ of $\mathcal{M}$, or more generally a set $\mathrm{Mul}_{\mathcal{M}}(\{\mathbf{x}_s\}_{s\in S}; \mathbf{y})$ for any finite set S and objects $\mathbf{x}_s$, $\mathbf{y}$;
3. composition operations
$$\circ\colon\ \mathrm{Mul}_{\mathcal{M}}(\{\mathbf{y}_t\}_{t\in T}; \mathbf{z}) \times \prod_{t\in T} \mathrm{Mul}_{\mathcal{M}}(\{\mathbf{x}_s\}_{s\in f^{-1}(t)}; \mathbf{y}_t) \to \mathrm{Mul}_{\mathcal{M}}(\{\mathbf{x}_s\}_{s\in S}; \mathbf{z})$$
for any map $f\colon\ S \to T$ of finite sets and objects $\mathbf{x}_s$, $\mathbf{y}_t$, and $\mathbf{z}$ of $\mathcal{M}$; and
4. identity morphisms $\mathrm{id}_X \in \mathrm{Mul}_{\mathcal{M}}(\mathbf{x}; \mathbf{x})$ for any object $\mathbf{x}$.

These are required to satisfy two conditions:

1. unitality: $\mathrm{id}_{\mathbf{y}} \circ g = g \circ (\mathrm{id}_{\mathbf{x}_s}) = g$ for any $g \in \mathrm{Mul}_{\mathcal{M}}(\{\mathbf{x}_s\}_{s\in S}; \mathbf{Y})$; and
2. associativity: $h \circ (g_u \circ (f_t)) = (h \circ (g_u)) \circ f_t$ for any $S \to T \to U$ of finite sets.

The *underlying ordinary category* of $\mathcal{M}$ is the category with the same objects as $\mathcal{M}$ and $\mathrm{Hom}_{\mathcal{M}}(\mathbf{x}, \mathbf{y}) = \mathrm{Mul}_{\mathcal{M}}(\mathbf{x}; \mathbf{y})$.

If the sets of multimorphisms are given topologies so that composition is continuous, we refer to $\mathcal{M}$ as a *topological multicategory*.

A (topological) *multifunctor* $F\colon \mathcal{M} \to \mathcal{N}$ is a map $F\colon Ob(\mathcal{M}) \to Ob(\mathcal{N})$ on the level of objects, together with (continuous) maps

$$\mathrm{Mul}_{\mathcal{M}}(\mathbf{x}_1, \ldots, \mathbf{x}_d; \mathbf{y}) \to \mathrm{Mul}_{\mathcal{M}}(F\mathbf{x}_1, \ldots, F\mathbf{x}_d; F\mathbf{y})$$

that preserve identity morphisms and composition.

Example 19.3.20. An operad is equivalent to a single-object multicategory. For any object $\mathbf{x}$ in a multicategory $\mathcal{M}$, the full sub-multicategory spanned by $\mathbf{x}$ is an operad called the endomorphism operad of $\mathbf{x}$.

Example 19.3.21. A symmetric monoidal topological category $\mathcal{M}$ can be regarded as a multicategory by defining

$$\mathrm{Mul}_{\mathcal{M}}(X_1, \ldots, X_d; Y) = \mathrm{Map}_{\mathcal{M}}(X_1 \otimes \cdots \otimes X_d, Y).$$

This recovers the definition of the endomorphism operad of an object X.

The notion of an algebra over a multicategory will extend the notion of an algebra over an operad.

Definition 19.3.22. For (topological) multicategories $\mathcal{M}$ and C, the category $\mathsf{Alg}_{\mathcal{M}}(\mathsf{C})$ of $\mathcal{M}$-*algebras* in C is the category of (topological) multifunctors $\mathcal{M} \to \mathsf{C}$ and natural transformations.

For any object $\mathbf{x} \in \mathcal{M}$, the *evaluation* functor $\mathrm{ev}_{\mathbf{x}}\colon \mathsf{Alg}_{\mathcal{M}}(\mathsf{C}) \to \mathsf{C}$ sends an algebra A to the value $A(\mathbf{x})$.

Example 19.3.23. The multicategory Mod parametrizing "ring-module pairs" has two objects, $\mathbf{a}$ and $\mathbf{m}$, and

$$\mathrm{Mul}_{\mathrm{Mod}}(\mathbf{x}_1, \ldots, \mathbf{x}_d; \mathbf{y}) = \begin{cases} * & \text{if } \mathbf{y} = \mathbf{a} \text{ and all } \mathbf{x}_i \text{ are } \mathbf{a}, \\ * & \text{if } \mathbf{y} = \mathbf{m} \text{ and exactly one } \mathbf{x}_i \text{ is } \mathbf{m}, \\ \emptyset & \text{otherwise.} \end{cases}$$

A multifunctor $\mathrm{Mod} \to \mathsf{C}$ is equivalent to a pair (A, M) of a commutative monoid A of C and an object M with an action of A.

Example 19.3.24. A commutative monoid Γ can be regarded as a symmetric monoidal category with no non-identity morphisms, and in the associated multicategory we have

$$\mathrm{Mul}_{\Gamma}(g_1, \ldots, g_d; g) = \begin{cases} * & \text{if } \sum g_s = g, \\ \emptyset & \text{otherwise.} \end{cases}$$

A multifunctor $\Gamma \to \mathsf{C}$ determines objects X_g of C, a map from the unit to X_0, and multiplication maps $X_{g_1} \otimes \cdots \otimes X_{g_d} \to X_{g_1+\cdots+g_d}$: these multiplications are collectively unital, symmetric, and associative. We refer to such an object as a Γ-graded commutative monoid.

Example 19.3.25. The addition of natural numbers makes the partially ordered set $(\mathbb{N}, \geq)$ into a symmetric monoidal category. In the associated multicategory we have

$$\mathrm{Mul}_{\mathbb{N}}(n_1, \ldots, n_d; m) = \begin{cases} * & \text{if } \sum n_i \geq m, \\ \emptyset & \text{otherwise.} \end{cases}$$

A multifunctor $\Gamma \to \mathsf{C}$ determines a sequence of objects

$$\cdots \to X_2 \to X_1 \to X_0$$

of C and multiplication maps $X_{n_1} \otimes \cdots \otimes X_{n_d} \to X_{n_1+\cdots+n_d}$: these multiplications are collectively unital, symmetric, and associative, as well as being compatible with the inverse system. We refer to such an object *strongly filtered* commutative monoid in C.

Remark 19.3.26. If $\mathcal{M}_1$ and $\mathcal{M}_2$ are multicategories, there is a product multicategory $\mathcal{M}_1 \times \mathcal{M}_2$, obtained by taking products of objects and products of multimorphism spaces. Products allow us to extend the above constructions. For example, taking the product of an operad $\mathcal{O}$ with the multicategories of the previous examples, we construct multicategories that parametrize: pairs (A, M) of an $\mathcal{O}$-algebra and an $\mathcal{O}$-module; Γ-graded $\mathcal{O}$-algebras; and strongly filtered $\mathcal{O}$-algebras.

Example 19.3.27. Let $\mathcal{M}$ be the multicategory whose objects are integers, and define $\mathrm{Mul}_{\mathcal{M}}(m_1, \ldots, m_d; n)$ to be the set of natural transformations

$$\theta\colon\ H^{m_1}(X) \times \cdots \times H^{m_d}(X) \to H^n(X)$$

of contravariant functors on the category S of spaces; composition is composition of natural transformations. The category $\mathcal{M}$ is a category of multivariate cohomology operations. Any fixed space X determines an evaluation multifunctor $\mathrm{ev}_X\colon\ \mathcal{M} \to \mathsf{Sets}$, sending n to $H^n(X)$; any homotopy class of map $X \to Y$ of spaces determines a natural transformation of multifunctors in the opposite direction. Stated concisely, this is a functor

$$h\mathsf{S}^{op} \to \mathsf{Alg}_{\mathcal{M}}(\mathsf{Sets})$$

that takes a space to an encoding of its cohomology groups and cohomology operations.

More generally, a category D with a chosen set of functors $\mathsf{D} \to \mathsf{Sets}$ determines a multicategory $\mathcal{M}$ spanned by them: we can define $\mathrm{Mul}(F_1, \ldots, F_d; G)$ to be the set of natural transformations $\prod F_i \to G$, so long as there is always a set (rather than a proper class) of natural transformations. If we view a functor F as assigning an invariant to each object of D, a multimorphism $\prod F_i \to G$ is a natural operation of several variables on such invariants. Evaluation on objects of D takes the form of a functor

$$\mathsf{D} \to \mathsf{Alg}_{\mathcal{M}}(\mathsf{Sets}),$$

encoding both the invariants assigned by these functors and the natural operations on them. These are examples of *multi-sorted algebraic theory* in the sense of Bergner [13], closely related to the work of [17, 88]. We will return to the discussion of this structure in §19.4.4.

Just as with ordinary operads, there are often free-forgetful adjunctions between objects of C and algebras over a multicategory.

Proposition 19.3.28. *Suppose that $\mathcal{M}$ is a small topological multicategory and that C is a symmetric monoidal topological category with compatible colimits in the sense of Definition 19.3.9.*

1. *For objects $\mathbf{x}$ and $\mathbf{y}$ of $\mathcal{M}$, there are extended power functors*

$$\mathrm{Sym}^k_{\mathcal{M},\mathbf{x}\to\mathbf{y}} \colon \mathsf{C}_{\mathbf{x}} \to \mathsf{C}_{\mathbf{y}},$$

given by

$$\mathrm{Sym}^k_{\mathcal{M},\mathbf{x}\to\mathbf{y}}(X) = \mathrm{Mul}_{\mathcal{M}}(\underbrace{\mathbf{x},\mathbf{x},\ldots,\mathbf{x}}_{k};\mathbf{y}) \otimes_{\Sigma_k} X^{\otimes k}.$$

2. *The evaluation functor $\mathrm{ev}_{\mathbf{x}} \colon \mathsf{Alg}_{\mathcal{M}}(\mathsf{C}) \to \mathsf{C}$ has a left adjoint*

$$\mathrm{Free}_{\mathcal{M},\mathbf{x}} \colon \mathsf{C} \to \mathsf{Alg}_{\mathcal{M}}(\mathsf{C}).$$

The value of $\mathrm{Free}_{\mathcal{M},\mathbf{x}}(X)$ on any object $\mathbf{y}$ of $\mathcal{M}$ is

$$\mathrm{ev}_{\mathbf{y}}(\mathrm{Free}_{\mathcal{M},\mathbf{x}}(X)) = \coprod_{k\geq 0} \mathrm{Sym}^k_{\mathcal{M},\mathbf{x}\to\mathbf{y}}(X).$$

Remark 19.3.29. These generalize the constructions of extended powers and free algebras from §19.3.2. If $\mathcal{M}$ has a single object $\mathbf{x}$, encoding an operad $\mathcal{O}$, then $\mathrm{Sym}^k_{\mathcal{M},\mathbf{x}\to\mathbf{x}} = \mathrm{Sym}^k_{\mathcal{O}}$ and $\mathrm{Free}_{\mathcal{M},\mathbf{x}}$ encodes $\mathrm{Free}_{\mathcal{O}}$.

Example 19.3.30. The free $\mathbb{Z}$-graded commutative monoid on an object X in degree $n \neq 0$ is equal to the symmetric product $\mathrm{Sym}^k(X)$ in degree kn for $k \geq 0$. All other gradings are the initial object.

Example 19.3.31. The free strongly filtered commutative monoid on an object X_1 in degree 1 is a filtered object of the form

$$\cdots \to \coprod_{k\geq 2} \mathrm{Sym}^k X_1 \to \coprod_{k\geq 1} \mathrm{Sym}^k X_1 \to \coprod_{k\geq 0} \mathrm{Sym}^k X_1.$$

If we have a strongly filtered commutative algebra $\cdots \to X_2 \to X_1 \to X_0$, then this gives action maps $\mathrm{Sym}^k X_1 \to X_k$. More generally, there are action maps $\mathrm{Sym}^k X_n \to X_{kn}$ that are compatible in n.

19.4 Operations

In this section we will fix a spectrum E, viewed as a coefficient object.

19.4.1 E-homology and E-modules

We can study $\mathcal{O}$-algebras through their E-homology.

Definition 19.4.1. Given a spectrum E, an *E-homology operation* for $\mathcal{O}$-algebras is a natural transformation of functors $\theta\colon E_m(-) \to E_{m+d}(-)$ of functors on the homotopy category of $\mathcal{O}$-algebras.

Such operations can be difficult to classify in general. However, if E has a commutative ring structure then we can do more. In this case, any $\mathcal{O}$-algebra A has an *E-homology object* $E \otimes A$ which is an $\mathcal{O}$-algebra in Mod_E, and any space X has an *E-cohomology object* E^X which is an E_∞-algebra object in Mod_E. By definition, we have

$$E_m(A) = [S^m, E \otimes A]_{\mathsf{Sp}}$$

and

$$E^m(X) = [S^{-m}, E^X]_{\mathsf{Sp}}.$$

Therefore, we can construct natural operations on the E-homology of $\mathcal{O}$-algebras or the E-cohomology of spaces by finding natural operations on the homotopy groups of $\mathcal{O}$-algebras in Mod_E.

Example 19.4.2. If X is an $\mathcal{O}$-algebra in spaces, then $E[X] = E \otimes \Sigma^\infty_+ X$ is an $\mathcal{O}$-algebra in Mod_E.

19.4.2 Multiplicative operations

In this section we will construct our first operations on the homotopy groups of $\mathcal{O}$-algebras over a fixed commutative ring spectrum E.

The functor π_* from the homotopy category of spectra to graded abelian groups is lax symmetric monoidal under the Koszul sign rule. The induced functor π_* from $\mathsf{Alg}_{\mathcal{O}}(\mathsf{Sp})$ or $\mathsf{Alg}_{\mathcal{O}}(\mathrm{Mod}_E)$ to graded abelian groups naturally takes values in the category of graded abelian groups, or graded E_*-modules, with an action of the operad $\pi_0\mathcal{O}$ in sets.

Example 19.4.3. In the case of an E_n-operad, $\pi_0\mathcal{O}$ is isomorphic to the associative operad when $n = 1$ and the commutative operad when $n \geq 2$. The E-homology groups of an E_n-algebra in Sp form a graded E_*-algebra. If $n \geq 2$, this algebra is graded-commutative.

By applying E_* to the action maps in the operad, we stronger information.

Proposition 19.4.4. *The homology groups $E_*\mathcal{O}(k)$ form an operad $E_*\mathcal{O}$ in graded E_*-modules, and the functor π_* from $\mathsf{Alg}_{\mathcal{O}}(\mathrm{Mod}_E)$ to graded abelian groups has a natural lift to the category of graded $E_*\mathcal{O}$-modules.*

Example 19.4.5. The homotopy groups of E_n-algebras have a natural bilinear *Browder bracket*

$$[-,-]\colon \pi_q(A) \otimes \pi_r(A) \to \pi_{q+(n-1)+r}(A).$$

This satisfies the following formulas.

Antisymmetry: $[\alpha, \beta] = -(-1)^{(|\alpha|+n-1)(|\beta|+n-1)}[\beta, \alpha]$.

Leibniz rule: $[\alpha, \beta\gamma] = [\alpha, \beta]\gamma + (-1)^{|\beta|(|\alpha|+n-1)}\alpha[\beta, \gamma]$.

Graded Jacobi identity:

$$\begin{aligned} 0 = \quad & (-1)^{(|\alpha|+n-1)(|\gamma|+n-1)}[\alpha, [\beta, \gamma]] \\ & + (-1)^{(|\beta|+n-1)(|\alpha|+n-1)}[\beta, [\gamma, \alpha]] \\ & + (-1)^{(|\gamma|+n-1)(|\beta|+n-1)}[\gamma, [\alpha, \beta]]. \end{aligned}$$

In the case of E_1-algebras, this reduces to the ordinary bracket

$$[\alpha, \beta] = \alpha\beta - (-1)^{|\alpha||\beta|}\beta\alpha$$

in the graded ring $\pi_*(A)$.

The Browder bracket is defined, just as it was defined in homology [23], using the image of the generating class $\lambda \in \pi_{n-1}\mathcal{E}_n(2) \cong \pi_{n-1}S^{n-1}$ coming from the little cubes operad. The antisymmetry and Jacobi identities are obtained by verifying identities in the graded operad $\pi_*(\Sigma^\infty_+\mathcal{E}_n)$. For example, if σ is the 2-cycle in Σ_2 we have

$$\lambda \circ \sigma = (-1)^n \lambda,$$

and if τ is a 3-cycle in Σ_3 we have

$$\lambda \circ (1 \otimes \lambda) \circ (1 + \tau + \tau^2) = 0.$$

However, the signs indicate that there is some care to be taken. In particular, the Browder bracket of elements $\alpha \in \pi_q(A)$ and $\beta \in \pi_r(A)$ is defined to be the following composite:

$$\begin{aligned} S^q \otimes S^{n-1} \otimes S^r & \to A \otimes \Sigma^\infty_+\mathcal{E}_n(2) \otimes A \\ & \to \Sigma^\infty_+\mathcal{E}_n(2) \otimes A \otimes A \\ & \to A \end{aligned}$$

This order is chosen because it is more consistent with writing the Browder bracket as an inline binary operation $[x, y]$ than with writing it as an operator $\lambda(x, y)$ on the left. The subscript on the range $\pi_{q+(n-1)+r}(A)$ reflects this choice (cf. [73]). This gives us the definition

$$[\alpha, \beta] = (-1)^{(n-1)|\alpha|}\gamma(\lambda \otimes \alpha \otimes \beta),$$

where γ is the action map of the operad $\pi_*(\Sigma^\infty_+\mathcal{E}_n)$ on π_*A. Both the verification of the identities on λ in the stable homotopy groups of configuration spaces, and the verification of the consequent antisymmetry, Leibniz, and Jacobi identities, are reasonable but error-prone exercises from this point; compare [22].

19.4.3 Representability

We will ultimately be interested in natural operations on homotopy and homology groups. However, it is handy to use a more general definition that replaces S^m by a general object. This accounts for the possibility of operations of several variables, and can also help reduce difficulties involving naturality in the input S^m.

Definition 19.4.6. For spectra M and X, we define the M-*indexed homotopy* of X to be

$$\pi_M(X) = [M, X]_{\mathsf{Sp}} \cong [E \otimes M, X]_{\mathrm{Mod}_E}.$$

For spectra M, X, and E we define the M-indexed E-homology of X to be

$$E_M(X) = \pi_M(E \otimes X).$$

If M is S^m, we instead use the more standard notation $\pi_m(-)$ for $\pi_{S^m}(-)$ or $E_m(-)$ for $E_{S^m}(-)$.

Definition 19.4.7. Let E be a commutative ring spectrum. A *homotopy operation* for $\mathcal{O}$-algebras over E is a natural transformation

$$\theta\colon\ \pi_M \to \pi_N$$

of functors on the homotopy category of $\mathsf{Alg}_{\mathcal{O}}(\mathrm{Mod}_E)$. When $\mathcal{O}$ and E are understood, we just refer to such natural transformations as homotopy operations.

We refer to the resulting operation $E_M(-) \to E_N(-)$ on the E-homology groups of $\mathcal{O}$-algebras as the induced E-homology operation.

As in Example 19.3.27, we can assemble operations with varying numbers of inputs into an algebraic structure.

Definition 19.4.8. Fix an operad $\mathcal{O}$ and a commutative ring spectrum E. The multicategory $\mathrm{Op}^E_{\mathcal{O}}$ of *operations* for $\mathcal{O}$-algebras in Mod_E has, as objects, spectra N. For any $M_1, \ldots, M_d$ and N, the group of multimorphisms

$$\mathrm{Op}^E_{\mathcal{O}}(M_1, \ldots, M_d; N)$$

is the group of natural transformations $\prod \pi_{M_i} \to \pi_N$ of functors $h\mathsf{Alg}_{\mathcal{O}}(\mathrm{Mod}_E) \to \mathsf{Sets}$. If E or $\mathcal{O}$ are understood, we drop them from the notation.

In the unary case, we write $\mathrm{Op}^E_{\mathcal{O}}(M; N)$ for the set of homotopy operations $\pi_M \to \pi_N$ for $\mathcal{O}$-algebras in Mod_E.

The free-forgetful adjunction between spectra and $\mathcal{O}$-algebras in Mod_E allows us to exhibit the functor π_M as representable.

Proposition 19.4.9. *Suppose that E is a commutative ring spectrum, $\mathcal{O}$ is an operad with associated free algebra monad $\mathrm{Free}_{\mathcal{O}}$. Then there is a natural isomorphism*

$$\pi_M(A) \cong [E \otimes \mathrm{Free}_{\mathcal{O}}(M), A]_{\mathsf{Alg}_{\mathcal{O}}(\mathrm{Mod}_E)}$$

for A in the homotopy category of $\mathsf{Alg}_{\mathcal{O}}(\mathrm{Mod}_E)$. In particular, the object $E \otimes \mathrm{Free}_{\mathcal{O}}(M)$ is a representing object for the functor π_M.

Proof. The forgetful functor $\mathsf{Alg}_{\mathcal{O}}(\mathrm{Mod}_E) \to \mathsf{Sp}$ can be expressed as a composite $\mathsf{Alg}_{\mathcal{O}}(\mathrm{Mod}_E) \to \mathsf{Alg}_{\mathcal{O}}(\mathsf{Sp}) \to \mathsf{Sp}$, and as such has a composite left adjoint $M \mapsto \mathrm{Free}_{\mathcal{O}}(M) \mapsto E \otimes \mathrm{Free}_{\mathcal{O}}(M)$; this adjunction passes to the homotopy category. Therefore, applying this adjunction we find

$$\begin{aligned} \pi_M(A) &\cong [\mathrm{Free}_{\mathcal{O}}(M), A]_{\mathsf{Alg}_{\mathcal{O}}(\mathsf{Sp})} \\ &\cong [E \otimes \mathrm{Free}_{\mathcal{O}}(M), A]_{\mathsf{Alg}_{\mathcal{O}}(\mathrm{Mod}_E)} \end{aligned}$$

as desired. □

Remark 19.4.10. It is possible to index more generally. Given an E-module L, we also have functors $\pi_L^E(-) = [L, -]_{\mathrm{Mod}_E}$; the free $\mathcal{O}$-algebra $\mathrm{Free}_{\mathcal{O}}(L)$ in the category of E-modules is then a representing object for π_L^E in $\mathsf{Alg}_{\mathcal{O}}(\mathrm{Mod}_E)$. We recover the above case by setting $L = E \otimes M$.

The Yoneda lemma now gives the following.

Corollary 19.4.11. *Let F be a functor from $h\mathsf{Alg}_{\mathcal{O}}(\mathrm{Mod}_E)$ to the category of sets. Natural transformations of functors $\pi_M \to F$ are in bijective correspondence with $F(E \otimes \mathrm{Free}_{\mathcal{O}}(M))$.*

In particular, there is an isomorphism

$$\mathrm{Op}_{\mathcal{O}}^E(M_1, \ldots, M_d; N) \cong E_N(\mathrm{Free}_{\mathcal{O}}(\oplus M_i))$$

from the group of natural transformations $\prod \pi_{M_i} \to \pi_N$ to the E-homology group of the free algebra.

The canonical decomposition of §19.3.2 for the monad $\mathrm{Free}_{\mathcal{O}}$ into extended powers gives us a canonical decomposition of operations.

Definition 19.4.12. For $k \geq 0$, the group of *operations of weight k* is the subgroup

$$\mathrm{Op}_{\mathcal{O}}^E(M_1, \ldots, M_d; N)^{\langle k \rangle} = E_N(\mathrm{Sym}_{\mathcal{O}}^k(\oplus M_i))$$

of $\mathrm{Op}_{\mathcal{O}}^E(M_1, \ldots, M_d; N) \cong E_N(\mathrm{Free}_{\mathcal{O}}(\oplus M_i))$.

A *power operation of weight k* is a unary operation of weight k: an element of the subgroup

$$\mathrm{Op}_{\mathcal{O}}^E(M, N)^{\langle k \rangle} \cong E_N(\mathrm{Sym}_{\mathcal{O}}^k(M))$$

of $\mathrm{Op}_{\mathcal{O}}^E(M, N)$.

Remark 19.4.13. Composition multiplies weight. Furthermore, if the object N is dualizable, the group of all operations is a direct sum: every operation decomposes canonically as a sum of operations of varying weights.

19.4.4 Structure on operations

Even when restricted to ordinary homotopy groups, these operations between the homotopy groups of $\mathcal{O}$-algebras in Mod_E form a rather rich algebraic structure [13], whose characteristics should be discussed; we learned most of this from Rezk [77, 76]. Recall

$$\mathrm{Op}(m_1, \ldots, m_d; n) = \mathrm{Op}^E_{\mathcal{O}}(m_1, \ldots, m_d; n) \cong \pi_n(E \otimes \mathrm{Free}_{\mathcal{O}}(\oplus S^{m_i})).$$

Here are some characteristics of this algebraic theory.

1. We think of the elements in these groups as operators, in the sense that they can *act*. Given $\alpha \in \mathrm{Op}(m_1, \ldots, m_d; n)$, an $\mathcal{O}$-algebra R in Mod_E and $x_i \in \pi_{m_i} R$, we can apply α to get a natural element
$$\alpha \propto (x_1, \ldots, x_d) \in \pi_n R.$$
This action is associative with respect to composition, but only distributes over addition on the left.

2. For each $1 \leq k \leq d$, there is a *fundamental generator* $\iota_k \in \mathrm{Op}(m_1, \ldots, m_d; m_k)$ that acts by projecting:
$$\iota_k \propto (x_1, \ldots, x_d) = x_k.$$

3. These operators can *compose*: given $\alpha \in \mathrm{Op}(m_1, \ldots, m_d; n)$ and $\beta_i \in \mathrm{Op}(\ell_1, \ldots, \ell_c; m_i)$, there is a composite operator
$$\alpha \propto (\beta_1, \ldots, \beta_d) \in \mathrm{Op}(\ell_1, \ldots, \ell_c; n).$$
Composition is unital. It is also associative, both with itself and with acting on elements. Again, it only distributes over addition on the left.

4. Composition respects weight: if α is in weight a and β_i are in weights b_i, then $\alpha \propto (\beta_i)$ is in weight $a \cdot (\sum b_i)$.

Example 19.4.14. Take $E = HR$ for a commutative ring R and let $\mathcal{O}$ to be the associative operad. Then the graded group

$$Op(m_1, \ldots, m_d; *) = \oplus_n \mathrm{Op}(m_1, \ldots, m_d; n) \cong H_*(\mathrm{Free}_{\mathcal{O}}(\oplus S^{m_i}; R))$$

is the free associative graded R-algebra on the fundamental generators $\iota_1 \ldots \iota_d$ with ι_i in degree m_i, and the composition operations are *substitution*. For example, the element $\iota_1 + \iota_2 \in \mathrm{Op}(n, n; n)$ acts by the binary addition operation in degree n; the elements $\iota_1\iota_2$ and $\iota_2\iota_1$ in $\mathrm{Op}(n_1, n_2; n_1 + n_2)$ represent binary multiplication in either order; the element $(\iota_1)^2 \in \mathrm{Op}(n; 2n)$ represents the squaring operation; for $r \in R$ the element $r\iota_1 \in \mathrm{Op}(n; n)$ represents scalar multiplication by r; combinations of these operations are represented by identities such as

$$\iota_1^2 \propto (\iota_1 + \iota_2) = \iota_1^2 + \iota_1\iota_2 + \iota_2\iota_1 + \iota_2^2.$$

In this structure, each monomial has constant weight equal to its degree.

Example 19.4.15. Take $\mathcal{O}$ to be an E_n-operad. Then, for any p and q, the Browder bracket is a natural transformation $\pi_p \times \pi_q \to \pi_{p+(n-1)+q}$, and it is realized by an element $[\iota_1, \iota_2]$ in $\mathrm{Op}(p, q; p + (n-1) + q)$ of weight two. Relations between the product and the Browder bracket are expressed universally by relations between compositions: for example, antisymmetry is expressed by an identity

$$[\iota_1, \iota_2] = -(-1)^{(p+n-1)(q+n-1)}[\iota_2, \iota_1].$$

Remark 19.4.16. Inside the collection of all unary operations, there is a subgroup of *additive* operations: those operations f that satisfy

$$f \propto (\iota_1 + \iota_2) = f \propto \iota_1 + f \propto \iota_2.$$

Composition of such operations is bilinear, and so the collection of objects and additive operations form a category enriched in abelian groups. In some cases, the additive operations can be used to determine the general structure [77].

19.4.5 Power operations

We will begin to narrow our study of power operations and focus on unary operations, of fixed weight, between integer gradings.

Definition 19.4.17. Fix an operad $\mathcal{O}$ and a commutative ring spectrum E. The group of *power operations of weight k on degree m* for $\mathcal{O}$-algebras in Mod_E is the graded abelian group

$$\mathrm{Pow}^E_{\mathcal{O}}(m,k) = \pi_*(F(S^m, E \otimes \mathrm{Sym}^k_{\mathcal{O}}(S^m))) \cong \bigoplus_{r\in\mathbb{Z}} \mathrm{Op}^E_{\mathcal{O}}(m, m+r)^{\langle k\rangle}.$$

If $\mathcal{O}$ or E are understood, we drop them from the notation.

An element of $\mathrm{Pow}(m,k)$ in grading r represents a weight-k natural transformation $\pi_m \to \pi_{m+r}$ on the homotopy category of $\mathcal{O}$-algebras in Mod_E, and induces a natural transformation $E_m \to E_{m+r}$ on the homotopy category of $\mathcal{O}$-algebras. (While we index these group by integers, they depend on a choice of representing object and in particular on an orientation of S^m; making implicit identifications will result in sign issues.)

Remark 19.4.18. These operations, and the relations between them, are still possessed by homotopy $\mathcal{O}$-algebras in the sense of §19.3.2.

Remark 19.4.19. Suppose that Σ_k acts freely and properly discontinuously on $\mathcal{O}(k)$. Let $V \subset \mathbb{R}^k$ be the subspace of elements that sum to 0, with associated vector bundle $\overline{\rho} \to B\Sigma_k$ of dimension $k-1$. For any m there is an associated virtual bundle $\mathbb{R}^m \otimes \overline{\rho}$. If we define

$$P(k) = \mathcal{O}(k)/\Sigma_k,$$

then there is a virtual bundle $m\overline{\rho}$ on $P(k)$. The Thom spectrum $P(k)^{m\overline{\rho}}$ of this virtual bundle is canonically equivalent to the spectrum $\Sigma^{-m}\Sigma^\infty_+\mathcal{O}(k)\otimes_{\Sigma_k}(S^m)^{\otimes k}$ that appears in the definition of $\mathrm{Pow}(m,k)$.

This allows us to give a more concise expression

$$\mathrm{Pow}(m,k) = E_*(P(k)^{m\overline{\rho}}),$$

which is particularly useful in cases where we can apply a Thom isomorphism for E-homology.

Example 19.4.20. Consider the case of operations of weight 2 for E_n-algebras. The space $P(2) = \mathcal{C}_n(2)/\Sigma_2$ is homotopy equivalent to the real projective space $\mathbb{RP}^{n-1}$, the line bundle $\overline{\rho} = \sigma$ is associated to the sign representation of Σ_2, and the Thom spectrum $(\mathbb{RP}^{n-1})^{m\sigma}$ is commonly known as the *stunted projective space* $\mathbb{RP}^{m+n-1}_m$ which has a cell decomposition with one cell in each dimension between m and $m+n-1$. (When $m \geq 0$ this is literally the suspension spectrum of $\mathbb{RP}^{m+n-1}/\mathbb{RP}^{m-1}$.) Therefore, the operations of weight 2 on degree m are parametrized by the E-homology group

$$\mathrm{Op}^E_m(2) = E_*(\mathbb{RP}^{m+n-1}_m).$$

Example 19.4.21. When $E = H\mathbb{F}_2$, we find $H_*(\mathbb{RP}^{m+n-1}_m)$ is $\mathbb{F}_2$ in degrees m through $(m+n-1)$, and so we obtain unique *Dyer–Lashof operations* Q^r for $m \leq r \leq m+n-1$ that send elements in π_m to elements in π_{m+r}.

Example 19.4.22. Consider the cup-1 operad $\mathcal{Q}_1$ defined in Example 19.3.6. Then the weight-2 operations on the E-homology of $\mathcal{Q}_1$-algebras are parametrized by $E_*(\mathbb{RP}^{m+1}_m)$. This stunted projective space is the Thom spectrum of m times the Möbius line bundle over S^1.

For example, we can take E to be the sphere spectrum. If $m = 2k$ there is a splitting

$$\mathbb{RP}^{2k+1}_{2k} \simeq S^{2k} \oplus S^{2k+1}.$$

Chosen generators in $\pi_{2k}(S^{2k}\oplus S^{2k+1})$ and $\pi_{2k+1}(S^{2k}\oplus S^{2k+1})$ give operations that increase degree by $2k$ and $2k+1$, respectively. A choice of splitting $S^{2k+1} \to \mathbb{RP}^{2k+1}_{2k}$ determines an operation $\mathrm{Sq}_1\colon \pi_{2k}(-) \to \pi_{4k+1}(-)$ called the *cup-1 square*. It satisfies $2\,\mathrm{Sq}_1(a) = [a,a]$.

In the case that we have an E_∞ ring spectrum, this has been studied in [20, §V] and [12], and can be chosen in such a way that it satisfies the following addition and multiplication identities on even-degree homotopy elements:

$$2\,\mathrm{Sq}_1(a) = 0$$

$$\mathrm{Sq}_1(a+b) = \mathrm{Sq}_1(a) + \mathrm{Sq}_1(b) + (\tfrac{|a|}{2}+1)ab\eta$$

$$\mathrm{Sq}_1(ab) = a^2\,\mathrm{Sq}_1(b) + \mathrm{Sq}_1(a)b^2 + \tfrac{|ab|}{4}a^2b^2\eta.$$

For example, $\mathrm{Sq}_1(n) = \binom{n}{2}\eta$ for $n \in \mathbb{Z}$. In the absence of higher commutativity, these identities should have correction terms involving the Browder bracket.

19.4.6 Stability

In this section we will consider compatibility relations between operations on different homotopy degrees.

Recall from §19.3.2 that the monad $\mathrm{Free}_{\mathcal{O}}$ decomposed into the homogeneous functors defined by

$$\mathrm{Sym}^k_{\mathcal{O}}(X) = \Sigma^\infty_+ \mathcal{O}(k) \otimes_{\Sigma_k} X^{\otimes k}.$$

In particular, these functors are *continuous*: they induce functions

$$\mathrm{Map}(X, Y) \to \mathrm{Map}(\mathrm{Sym}^k_{\mathcal{O}}(X), \mathrm{Sym}^k_{\mathcal{O}}(Y))$$

between mapping spaces, and for $k > 0$ they have the property that they are *pointed*: $\mathrm{Sym}^k_{\mathcal{O}}(*) = *$ and hence the functor $\mathrm{Sym}^k_{\mathcal{O}}$ induces continuous maps of *pointed* mapping spaces.

Definition 19.4.23. For any spectrum M, any pointed space Z, and any $k > 0$, the *assembly map*

$$\mathrm{Sym}^k_{\mathcal{O}}(M) \otimes \Sigma^\infty Z \to \mathrm{Sym}^k_{\mathcal{O}}(M \otimes \Sigma^\infty Z)$$

is adjoint to the composite map of pointed spaces

$$\begin{aligned} Z &\to \mathrm{Map}_{\mathsf{Sp}}(S^0, \Sigma^\infty Z) \\ &\to \mathrm{Map}_{\mathsf{Sp}}(M, M \otimes \Sigma^\infty Z) \\ &\to \mathrm{Map}_{\mathsf{Sp}}(\mathrm{Sym}^k_{\mathcal{O}}(M), \mathrm{Sym}^k_{\mathcal{O}}(M \otimes \Sigma^\infty Z)). \end{aligned}$$

The *suspension map*

$$\sigma_n\colon\ \mathrm{Pow}(m, k) \to \mathrm{Pow}(m+n, k)$$

is induced by the composite map of function spectra

$$\begin{aligned} F(S^m, E \otimes \mathrm{Sym}^k_{\mathcal{O}}(S^m)) &\to F(S^m \otimes S^n, E \otimes \mathrm{Sym}^k_{\mathcal{O}}(S^m) \otimes S^n) \\ &\to F(S^m \otimes S^n, E \otimes \mathrm{Sym}^k_{\mathcal{O}}(S^m \otimes S^n)). \end{aligned}$$

Remark 19.4.24. The operation $\sigma = \sigma_1$ has a concrete meaning: it is designed for *compatibility with the Mayer–Vietoris sequence*. To illustrate this, first recall that for a homotopy commutative diagram

$$\begin{array}{ccc} A & \longrightarrow & B \\ \downarrow & & \downarrow \\ C & \longrightarrow & D \end{array}$$

of spectra, we have natural maps $A \to P \leftarrow \Sigma^{-1} D$ where P is the homotopy pullback.

Now suppose that we are given a diagram of $\mathcal{O}$-algebras as above that is a homotopy pullback, inducing a boundary map $\partial\colon\ \Sigma^{-1}D \to P \simeq A$. Given maps $\theta\colon\ N \to E \otimes \mathrm{Sym}^k_{\mathcal{O}}(M)$ and $\alpha\colon\ \Sigma M \to D$, we can map in a trivial homotopy pullback diagram to the above, then apply action maps and naturality of the connecting homomorphisms. We get a commuting diagram:

$$\begin{array}{ccccc}
N & \xrightarrow{\theta} & E \otimes \operatorname{Sym}^k_{\mathcal{O}} M & \xrightarrow{\partial\alpha} & A \\
{\scriptstyle\sim}\downarrow & & \downarrow & & \downarrow{\scriptstyle\sim} \\
\Sigma^{-1}\Sigma N & \longrightarrow & P' & \longrightarrow & P \\
{\scriptstyle\sim}\uparrow & & \uparrow & & \uparrow{\scriptstyle\partial} \\
\Sigma^{-1}\Sigma N & \xrightarrow[\sigma\theta]{} & \Sigma^{-1}E \otimes \operatorname{Sym}^k_{\mathcal{O}}(\Sigma M) & \xrightarrow[\Sigma^{-1}\alpha]{} & \Sigma^{-1}D.
\end{array}$$

Therefore, for an operation $\theta\colon [M,-] \to [N,-]$ for $\mathcal{O}$-algebras in Mod_E, we find that

$$\partial \circ \sigma\theta \sim \theta \circ \partial.$$

This description makes implicit choices about the orientation of the circle that appears in the operation Ω when taking homotopy pullbacks, and this can result in sign headaches.

Proposition 19.4.25. *For $k, r > 0$, the suspension $\sigma_r\colon \operatorname{Pow}(m,k) \to \operatorname{Pow}(m+r,k)$ is the map*

$$E_*(P(k)^{m\overline{\rho}}) \to E_*(P(k)^{(m+r)\overline{\rho}})$$

on E-homology induced by the inclusion of virtual bundles $m\overline{\rho} \to m\overline{\rho} \oplus r\overline{\rho}$.

Proof. The assembly map $\operatorname{Sym}^k_{\mathcal{O}}(S^m) \otimes S^n \to \operatorname{Sym}^k_{\mathcal{O}}(S^{m+n})$ is the map

$$(\Sigma^\infty_+ \mathcal{O}(k) \otimes_{\Sigma_k} S^{m\rho}) \otimes S^n \to (\Sigma^\infty_+ \mathcal{O}(k) \otimes_{\Sigma_k} S^{(m+n)\rho}),$$

which is the map

$$P(k)^{m\rho} \otimes S^r \to P(k)^{(m+r)\rho}$$

induced by the direct sum inclusion $m\rho \oplus r \to (m\rho \oplus r) \oplus r\overline{\rho}$ of virtual bundles. The map σ_r is obtained by desuspending both sides $(m+r)$ times, which gives the map induced by the direct sum inclusion $m\overline{\rho} \to m\overline{\rho} \oplus r\overline{\rho}$ of virtual bundles. ☐

Example 19.4.26. The Dyer–Lashof operations for E_n-algebras are explicitly *unstable*. For example, in weight two the n-fold suspension maps $\mathbb{RP}^{m+n-1}_m \to \mathbb{RP}^{(m+n)+n-1}_{m+n}$ are trivial, and so the map $\mathrm{Op}^E_m(2) \to \mathrm{Op}^E_{m+n}(2)$ is trivial. This recovers the well-known fact that all Dyer–Lashof operations for E_n-algebras map to zero under n-fold suspension.

By contrast, the Dyer–Lashof operations for E_∞-algebras are *stable*: the maps $H_*\mathbb{RP}^\infty_m \to H_*\mathbb{RP}^\infty_{m+1}$ are surjections, and so the quadratic operations all lift to elements in the homotopy of

$$\lim_m (H \otimes \mathbb{RP}^\infty_m).$$

By [32, 16.1], this is the desuspended Tate spectrum $(\Sigma^{-1}H)^{t\Sigma_2}$.

Remark 19.4.27. More generally, the fully stable operations of prime weight p on the homotopy of E_∞ E-algebras are detected by the p-localized Tate spectrum

$$(\Sigma^{-1} E_{(p)})^{t\Sigma_p}.$$

See [20, II.5.3] and [30].

19.4.7 Pro-representability

Suppose that $E = \operatorname{colim} E_\alpha$ is an expression of E as a filtered colimit of finite spectra. Then there is an identification

$$E_m A = \operatorname*{colim}_\alpha [S^m, E_\alpha \otimes A] = \operatorname*{colim}_\alpha [S^m \otimes DE_\alpha, A],$$

where D is the Spanier–Whitehead dual. We cannot move the colimit inside, but we can view $\{S^m \otimes DE_\alpha\}$ as a *pro-object* in the category of spectra. This makes the functor E_m representable by embedding the category of spectra into the category of pro-spectra.

For algebras over an operad $\mathcal{O}$, we can go even further and find that

$$E_m(A) = [\{\mathrm{Free}_{\mathcal{O}}(S^m \otimes DE_\alpha)\}, A]_{\text{pro-}\mathcal{O}}$$

is now a representable functor in the homotopy category of pro-$\mathcal{O}$-algebras, and in this category we can determine all the natural operations $E_m \to E_n$:

$$\begin{aligned} Nat_{\text{pro-}\mathcal{O}}(E_m(-), E_n(-)) &= [\{\mathrm{Free}_{\mathcal{O}}(S^n \otimes DE_\alpha)\}, \{\mathrm{Free}_{\mathcal{O}}(S^m \otimes DE_\beta)\}]_{\text{pro-}\mathcal{O}} \\ &= \pi_0 \lim_\beta \operatorname*{colim}_\alpha \mathrm{Map}_{\mathcal{O}}(\mathrm{Free}_{\mathcal{O}}(S^n \otimes DE_\alpha), \mathrm{Free}_{\mathcal{O}}(S^m \otimes DE_\beta)) \\ &= \pi_0 \lim_\beta \operatorname*{colim}_\alpha \mathrm{Map}_{\mathsf{Sp}}(S^n \otimes DE_\alpha, \mathrm{Free}_{\mathcal{O}}(S^m \otimes DE_\beta)) \\ &= \pi_0 \lim_\beta \mathrm{Map}_{\mathsf{Sp}}(S^n, E \otimes \mathrm{Free}_{\mathcal{O}}(S^m \otimes DE_\beta)) \\ &= \pi_n \lim_\beta E \otimes (\mathrm{Free}_{\mathcal{O}}(S^m \otimes DE_\beta)). \end{aligned}$$

The algebra of natural transformations has natural maps in from the group

$$[S^m \otimes E, S^n \otimes E]$$

of cohomology operations (and these maps are isomorphisms if $\mathcal{O}$ is trivial), and it has a natural map to the limit

$$\lim_\beta E_n(\mathrm{Free}_{\mathcal{O}}(S^n \otimes DE_\beta)).$$

This map to the limit is an isomorphism if no higher derived functors intrude. We can think of this as the algebra of *continuous* operations on E-homology.

19.5 Classical operations

19.5.1 E_n Dyer–Lashof operations at $p = 2$

We will now specialize to the case of ordinary mod-2 homology. When we do so, we have Thom isomorphisms for many bundles and we have explicit computations of the homology of configuration spaces due to Cohen [23]. Similar results with more complicated identities hold at odd primes.

Proposition 19.5.1. *Let $H = H\mathbb{F}_2$ be the mod-2 Eilenberg–Mac Lane spectrum. Then the group $\mathrm{Op}^H_m(2)$ of weight-2 operations for E_n-algebras has exactly one nonzero operation in each degree between m and $m+n-1$, and no others.*

Proof. By Remark 19.4.19, this is a calculation $H_*(\mathbb{RP}^{n+m-1}_m)$ of the mod-2 homology of stunted projective spaces. □

Theorem 19.5.2. [20, III.3.1, III.3.2, III.3.3] *Let $H = H\mathbb{F}_2$ be the mod-2 Eilenberg–Mac Lane spectrum. Then E_n-algebras in Mod_H have* Dyer–Lashof operations

$$Q_i\colon \pi_m \to \pi_{2m+i}$$

for $0 \le i \le n-1$. These satisfy the following formulas.

- **Additivity:** $Q_r(x+y) = Q_r(x) + Q_r(y)$ *for* $r < n-1$.
- **Squaring:** $Q_0 x = x^2$.
- **Unit:** $Q_j 1 = 0$ *for* $j > 0$.
- **Cartan formula:** $Q_r(xy) = \sum_{p+q=r} Q_p(x)Q_q(y)$ *for* $r < n-1$.
- **Adem relations:** $Q_r Q_s(x) = \sum \binom{j-s-1}{2j-r-s} Q_{r+2s-2j}Q_j(x)$ *for* $r > s$.
- **Stability:** $\sigma Q_0 = 0$, *and* $\sigma Q_r = Q_{r-1}$ *for* $r > 0$.
- **Extension:** *If an E_n-algebra structure extends to an E_{n+1}-algebra structure, the operations Q_r for E_{n+1}-algebras coincide with the operations Q_r for E_n-algebras.*

There is also a bilinear Browder bracket

$$[-,-]\colon \pi_r \otimes \pi_s \to \pi_{r+(n-1)+s}$$

satisfying the following formulas.

- **Antisymmetry:** $[x,y] = [y,x]$ *and* $[x,x] = 0$.
- **Unit:** $[x,1] = 0$.
- **Leibniz rule:** $[x,yz] = [x,y]z + y[x,z]$.
- **Jacobi identity:** $[x,[y,z]] + [y,[z,x]] + [z,[x,y]] = 0$.
- **Dyer–Lashof vanishing:** $[x, Q_r y] = 0$ *for* $r < n-1$.

Top additivity: $Q_{n-1}(x+y) = Q_{n-1}x + Q_{n-1}y + [x,y]$.

Top Cartan formula: $Q_{n-1}(xy) = \sum_{p+q=n-1} Q_p(x)Q_q(y) + x[x,y]y$.

Adjoint identity: $[x, Q_{n-1}y] = [y,[y,x]]$.

Extension: *If an E_n-algebra structure extends to an E_{n+1}-algebra structure, the bracket is identically zero.*

E_1-bracket: $[x,y] = xy + yx$ *if* $n = 1$.

Remark 19.5.3. There are two common indexing conventions for the Dyer–Lashof operations. This lower-indexing convention is designed to emphasize the range where the operations are defined, and is especially useful for E_n-algebras. The upper-indexing convention defines $Q^s x = Q_{s-|x|}x$ so that Q^s is always a natural transformation $\pi_m \to \pi_{s+m}$, with the understanding that $Q^s x = 0$ for $s < |x|$.

Example 19.5.4. Suppose that X is an n-fold loop space, so that $H[X]$ is an E_n-algebra in left H-modules. Then we recover the classical Dyer–Lashof operations

$$Q_r\colon\ H_n(X) \to H_{2n+r}(X)$$

in the homology of iterated loop spaces.

Theorem 19.5.5. [20, IX.2.1], [23, III.3.1] *For any spectrum X and any $1 \le n \le \infty$, $H_*(\mathrm{Free}_{E_n}(X))$ is the free object $\mathbb{Q}_{E_n}(H_*X)$ in the category of graded $\mathbb{F}_2$-algebras with Dyer–Lashof operations and Browder bracket satisfying the identities of Theorem 19.5.2.*

Remark 19.5.6. This theorem is the analogue of the calculation of the cohomology of Eilenberg–Mac Lane spaces as free algebras in a category of algebras with Steenrod operations. As such, it means that we have a *complete* theory of homotopy operations for E_n-algebras over H.

Example 19.5.7. In the case $n < \infty$ we can give a straightforward description of $\mathbb{Q}_{E_n}V$ if V has a basis with a single generator e. In this case, the antisymmetry, unit, and Dyer–Lashof vanishing axioms can be used to show that the free algebra has trivial Browder bracket, and so the free algebra $\mathbb{Q}_{E_n}(V)$ is a graded polynomial algebra

$$\mathbb{F}_2[Q_J e]$$

as we range over generators $Q_J e = (Q_1)^{j_1}(Q_2)^{j_2}\dots(Q_{n-1})^{j_{n-1}}e$.

19.5.2 E_∞ Dyer–Lashof operations at $p = 2$

When $n = \infty$, the results of the previous section become significantly simpler, and it is worth expressing using the upper indexing for Dyer–Lashof operations.

Theorem 19.5.8. [20, III.1.1] *Let $H = H\mathbb{F}_2$ be the mod-2 Eilenberg–Mac Lane spectrum. Then E_∞-algebras in Mod_H have* Dyer–Lashof operations

$$Q^r\colon\ \pi_m \to \pi_{m+r}$$

for $r \in \mathbb{Z}$. These satisfy the following formulas.

Additivity: $Q^r(x + y) = Q^r(x) + Q^r(y)$.

Instability: $Q^r x = 0$ *if* $r < |x|$.

Squaring: $Q^r x = x^2$ *if* $r = |x|$.

Unit: $Q^r 1 = 0$ *for* $r \neq 0$.

Cartan formula: $Q^r(xy) = \sum_{p+q=r} Q^p(x)Q^q(y)$.

Adem relations: $Q^r Q^s = \sum \binom{i-s-1}{2i-r} Q^{s+r-i} Q^i$ *for* $r > 2s$.

Stability: $\sigma Q^r = Q^r$.

Example 19.5.9. For any space X, H^X is an E_∞-algebra in the category of left H-modules, and hence it has Dyer–Lashof operations

$$Q^i \colon\ H^n(X) \to H^{n-i}(X).$$

It turns out that these are precisely the *Steenrod operations*:

$$Sq^i = Q^{-i}.$$

From this point of view, the identity $Q^0 x = x$ is not obvious. In fact, Mandell has shown that this identity is characteristic of algebras that come from spaces: the functor $X \mapsto (H\overline{\mathbb{F}}_p)^X$ from spaces to E_∞-algebras over the Eilenberg–Mac Lane spectrum $H\overline{\mathbb{F}}_p$ is fully faithful, and the essential image is detected in terms of the coefficient ring being generated by classes that are annihilated by the analogue at arbitrary primes of the identity $(Q^0 - 1)$ [56].

Example 19.5.10. In the case $n = \infty$ there is always a straightforward basis for the free algebra. If $\{e_i\}$ is a basis of a graded vector space V over $\mathbb{F}_2$, then the free algebra $\mathbb{Q}_{E_\infty}(V)$ is a graded polynomial algebra

$$\mathbb{F}_2[Q^J e_i]$$

as we range over generators $Q^J e_i = Q^{j_1} \cdots Q^{j_p} e_i$ such that $j_i \leq 2j_{i+1}$ and $j_1 - j_2 - \cdots - j_p > |e_i|$.

19.5.3 Iterated loop spaces

The following is an unpointed group-completion theorem for E_n-spaces.

Theorem 19.5.11. [23, III.3.3] *For any space X and any $1 \leq n \leq \infty$, the map $X \to \Omega^n \Sigma^n X_+$ induces a map $\mathrm{Free}_{E_n}(X) \to \Omega^n \Sigma^n X_+$, and the resulting ring map*

$$\mathbb{Q}_{E_n}(H_* X) = H_*(\mathrm{Free}_{E_n}(X)) \to H_*(\Omega^n \Sigma^n X_+)$$

is a localization that inverts the images of $\pi_0(X)$.

Remark 19.5.12. A pointed version of the group-completion theorem, involving $\Omega^n \Sigma^n X$, is much more standard and implies this one. This theorem holds for $\Omega^n \Sigma^n$ if we replace Free_{E_n} with a version that takes the basepoint to a unit and we replace $\mathbb{Q}_{E_n}(H_* X)$ with either $\mathbb{Q}_{E_n}(\widetilde{H}_*X)$ a reduced version $\widetilde{\mathbb{Q}}_{E_n}$ that sends a chosen element to the unit. However, we wanted to give a version that de-emphasizes implicit basepoints for comparison with §19.8.2.

Proposition 19.5.13. *Suppose Y is a pointed space. Then the suspension map*

$$\sigma\colon\ \widetilde{H}_*(\Omega^n Y) \to \widetilde{H}_{*+1}(\Omega^{n-1}Y),$$

induced by the map $\Sigma\Omega^n Y \to \Omega^{n-1}Y$, is compatible with the Dyer–Lashof operations and the Browder bracket:

$$\sigma(Q^r x) = Q^r(\sigma x)$$
$$\sigma[x, y] = [\sigma x, \sigma y].$$

In particular, in the bar spectral sequence

$$\mathrm{Tor}^{H_*\Omega^n Y}_{**}(\mathbb{F}_2, \mathbb{F}_2) \Rightarrow H_*\Omega^{n-1}Y,$$

the operations on the image $\widetilde{H}_*\Omega^n Y \twoheadrightarrow \mathrm{Tor}_1^{H_*\Omega^n Y}(\mathbb{F}_2, \mathbb{F}_2)$ are representatives for the operations on $H_*\Omega^{n-1}Y$.

This provides some degree of conceptual interpretation for the bracket and the Dyer–Lashof operations. Since $H_*\Omega^2 Y$ is commutative, the Tor-algebra is also commutative even though it is converging to the possibly noncommutative ring $H_*\Omega Y$, and so the noncommutativity is tracked by multiplicative extensions in the spectral sequence [73]. The Browder bracket in $H_*\Omega^n Y$ exists to remember that, after $n-1$ deloopings, there are commutators $xy \pm yx$ in $H_*\Omega Y$.

Similarly, elements in positive filtration in the Tor-algebra of a commutative ring always satisfy $x^2 = 0$, even though this may not be the case in $H_*\Omega^{n-1}Y$. The element $Q_0 x$ is x^2; the elements $Q_1 x, Q_2 x, \ldots, Q_{n-1}x$ determine the line of succession for the property of being x^2 as the delooping process is iterated.

Remark 19.5.14. The group-completion theorem allows us to relate the homology of a delooping to certain nonabelian derived functors [68]. Similar spectral sequences computing E_n-homology of chain complexes have been studied by Richter and Ziegenhagen [78].

Associated to the n-fold loop space $\Omega^n Y$ of an $(n-1)$-connected space, which is an E_n-algebra (or an infinite loop space $\Omega^\infty Y$ associated to a connective spectrum), we can construct three augmented simplicial objects:

$$\cdots \mathrm{Free}_{E_n}\mathrm{Free}_{E_n}\mathrm{Free}_{E_n}\Omega^n Y \Rrightarrow \mathrm{Free}_{E_n}\mathrm{Free}_{E_n}\Omega^n Y \rightrightarrows \mathrm{Free}_{E_n}\Omega^n Y \to \Omega^n Y$$
$$\cdots \Omega^n\Sigma^n_+\mathrm{Free}_{E_n}\mathrm{Free}_{E_n}\Omega^n Y \Rrightarrow \Omega^n\Sigma^n_+\mathrm{Free}_{E_n}\Omega^n Y \rightrightarrows \Omega^n\Sigma^n_+\Omega^n Y \to \Omega^n Y$$
$$\cdots \Sigma^n_+\mathrm{Free}_{E_n}\mathrm{Free}_{E_n}\Omega^n Y \Rrightarrow \Sigma^n_+\mathrm{Free}_{E_n}\Omega^n Y \rightrightarrows \Sigma^n_+\Omega^n Y \to Y$$

These are, respectively, two-sided bar constructions: $B(\mathrm{Free}_{E_n}, \mathrm{Free}_{E_n}, \Omega^n Y)$, $B(\Omega^n\Sigma^n_+, \mathrm{Free}_{E_n}, \Omega^n Y)$, and $B(\Sigma^n_+, \mathrm{Free}_{E_n}, \Omega^n Y)$.

The first augmented bar construction $B(\mathrm{Free}_{E_n}, \mathrm{Free}_{E_n}, \Omega^n Y)$ has an extra degeneracy, and so its geometric realization is homotopy equivalent to $\Omega^n Y$ as E_n-spaces. Therefore, it is a group-complete E_n-space.

There is a natural map

$$B(\mathrm{Free}_{E_n}, \mathrm{Free}_{E_n}, \Omega^n Y) \to B(\Omega^n \Sigma^n_+, \mathrm{Free}_{E_n}, \Omega^n Y)$$

which is, levelwise, a group-completion map [29, Appendix Q], [66], and induces a group-completion map on geometric realization. However, the source is already group-complete, and so this map is an equivalence on geometric realizations. Thus, the augmentation $|B(\Omega^n \Sigma^n_+, \mathrm{Free}_{E_n}, \Omega^n Y)| \to \Omega^n Y$ is an equivalence.

The bar construction $B(\Sigma^n_+, \mathrm{Free}_{E_n}, \Omega^n Y)$ is a simplicial diagram of $(n-1)$-connected pointed spaces, and so by a theorem of May [62] we can commute Ω^n across geometric realization. The natural augmentation

$$\Omega^n |B(\Sigma^n_+, \mathrm{Free}_{E_n}, \Omega^n Y)| \to |B(\Omega^n \Sigma^n_+, \mathrm{Free}_{E_n}, \Omega^n Y)| \to \Omega^n Y$$

is an equivalence. By assumption, Y is $(n-1)$-connected and so $|B(\Sigma^n_+, \mathrm{Free}_{E_n}, \Omega^n Y)| \to Y$ is also an equivalence. Therefore, the simplicial object $B(\Sigma^n_+, \mathrm{Free}_{E_n}, \Omega^n Y)$ can be used to compute $H_* Y$.

Let $A = H_*(\Omega^n Y)$. The reduced homology of $B(\Sigma^n_+, \mathrm{Free}_{E_n}, \Omega^n Y)$ is

$$\cdots \Sigma^n \mathbb{Q}_{E_n} \mathbb{Q}_{E_n} A \Rrightarrow \Sigma^n \mathbb{Q}_{E_n} A \Rightarrow \Sigma^n A,$$

which is a bar complex $\Sigma^n B(Q, \mathbb{Q}_{E_n}, A)$ computing nonabelian derived functors. These are specifically the derived functors of an *indecomposables* functor Q, which takes an augmented $\mathbb{Q}_{E_n}$-algebra $A \to \mathbb{F}_2$ and returns the quotient of the augmentation ideal by all products, brackets, and Dyer–Lashof operations. The result is a Miller spectral sequence that begins with nonabelian derived functors of Q on $H_*(\Omega^n Y)$ and converges to $\widetilde{H}_* Y$.

19.5.4 Classical groups

The Dyer–Lashof operations on the homology of the spaces BO and BU, and hence on the homology of the Thom spectra MO and MU, was determined by work of Kochman [43]; here we will state a form due to Priddy [74].

Theorem 19.5.15. *The ring $H_* MO \cong H_* BO$ is a polynomial algebra on classes a_i in degree i. The Dyer–Lashof operations are determined by the identities of formal series*

$$\sum_j Q^j a_k = \left(\sum_{n=k}^{\infty} \sum_{u=0}^{k} \binom{n-k+u-1}{u} a_{n+u} a_{k-u} \right) \left(\sum_{n=0}^{\infty} a_n \right)^{-1},$$

where $a_0 = 1$ by convention. In particular, $Q^n a_k \equiv \binom{n-1}{k} a_{n+k}$ mod decomposable elements.

The ring $H_*MU \cong H_*BU$ is a polynomial algebra on classes b_i in degree $2i$, The Dyer–Lashof operations are determined by the identities of formal series

$$\sum_j Q^j b_k = \left(\sum_{n=k}^{\infty}\sum_{u=0}^{k}\binom{n-k+u-1}{u} b_{n+u}b_{k-u}\right)\left(\sum_{n=0}^{\infty} b_n\right)^{-1},$$

where $b_0 = 1$ by convention. In particular, $Q^{2n}b_k \equiv \binom{n-1}{k} b_{n+k}$ mod decomposable elements, and $Q^{2n+1}b_k = 0$.

Remark 19.5.16. Implicit in this calculation is the fact that the Thom isomorphisms $H_*MO \cong H_*BO$ and $H_*MU \cong H_*BU$ preserve Dyer–Lashof operations. Lewis showed that, for an E_n-map $f\colon X \to BGL_1(\mathbb{S})$, the Thom isomorphism $H_*X \cong H_*Mf$ lifts to an equivalence of E_n ring spectra

$$H[X] \to H \otimes Mf$$

called the Thom diagonal [49, 7.4]. As a result, the Thom isomorphism is automatically compatible with Dyer–Lashof operations for H-algebras.

Example 19.5.17. We have explicit calculations of the first few Dyer–Lashof operations in H_*MO:

$$\begin{aligned}
Q^2 a_1 &= a_1^2\\
Q^4 a_1 &= a_3 + a_1a_2 + a_1^3\\
Q^6 a_1 &= a_1^4\\
Q^8 a_1 &= a_5 + a_1a_4 + a_2a_3 + a_1^2a_3 + a_1a_2^2 + a_1^3a_2 + a_1^5\\
Q^6 a_2 &= a_5 + a_1a_4 + a_2a_3 + a_1a_2^2
\end{aligned}$$

These same formulas hold for the b_i in H_*MU.

19.5.5 The Nishida relations and the dual Steenrod algebra

Recall that, if R is an E_n-algebra in Sp, $H\otimes R$ is an E_n-algebra in Mod_H whose homotopy groups are the homology groups of R. As a result, there are two types of operations on H_*R:

- The E_n-algebra structure gives $H_*(R)$ Dyer–Lashof operations $Q_0, \ldots, Q_{n-1}$ and a Browder bracket.
- The property of being homology gives $H_*(R)$ Steenrod operations $P_d\colon H_mR \to H_{m-d}R$. To make these dual to the Steenrod operations Sq^d in cohomology, $P_d(x)$ is defined as a composite

  $$S^{m-d} \xrightarrow{\Sigma^{-d}x} (\Sigma^{-d}H)\otimes R \xrightarrow{\chi Sq^d} H \otimes R.$$

  This implicitly reverses multiplication order: for example, the Adem relation $Sq^3 = Sq^1Sq^2$ becomes $P_3 = P_2P_1$.

The Nishida relations express how these structures interact.

Theorem 19.5.18. [20, III.1.1, III.3.2] *Suppose that R is an E_n-algebra in Sp. Then the Steenrod operations in homology satisfy relations as follows.*

Cartan formula: $P_r(xy) = \sum_{p+q=r} P_p(x)P_q(y)$.

Nishida relations: $P_rQ^s = \sum \binom{s-r}{r-2i} Q^{s-r+i}P_i$.

Browder Cartan formula: $P_r[x,y] = \sum_{p+q=r}[P_px, P_qy]$.

Remark 19.5.19. By contrast with the Adem relations, the Nishida relations behave very differently if we use lowerindexing. We find

$$P_rQ_s(x) = \sum \binom{|x|+s-r}{r-2i} Q_{s-r+i}P_i(x).$$

In particular, the lower-indexed Nishida relations depend on the degree of x [21].

Remark 19.5.20. If we use the pro-representability of homology as in §19.4.7, we can obtain a combined algebraic object that encodes both the Q^r and the P_d together with the Nishida relations.

19.5.6 Eilenberg–Mac Lane objects

If the homology H_*R is easily described a module over the Steenrod algebra, the Nishida relations can completely determine the Dyer–Lashof operations. This was applied by Steinberger to compute the Dyer–Lashof operations in the dual Steenrod algebra explicitly. (Conversely, Baker showed that the Nishida relations themselves are completely determined by the Dyer–Lashof operation structure of the dual Steenrod algebra [5].)

Theorem 19.5.21. [20, III.2.2, III.2.4] *Let A_* be the dual Steenrod algebra*

$$\mathbb{F}_2[\xi_1, \xi_2, \dots]$$

where $|\xi_i| = 2^i - 1$, with conjugate generators $\overline{\xi}_i$ (here ξ_i is denoted by ζ_i in [70]). Then the Dyer–Lashof operations on the generators are determined by the following formulas.

1. *There is an identity of formal series*

$$(1 + \xi_1 + Q^1\xi_1 + Q^2\xi_1 + Q^3\xi_1 + \dots) = (1 + \xi_1 + \xi_2 + \xi_3 + \dots)^{-1}.$$

2. *For any i, we have*

$$Q^s\overline{\xi}_i = \begin{cases} Q^{s+2^i-2}\xi_1 & \text{if } s \equiv 0, -1 \mod 2^i, \\ 0 & \text{otherwise.} \end{cases}$$

3. *In particular, $Q^{2^i-2}\xi_1 = \overline{\xi}_i$, and $Q_1\overline{\xi}_i = \overline{\xi}_{i+1}$.*

Remark 19.5.22. This allows us to say that the dual Steenrod algebra can be re-expressed as follows:

$$A_* \cong \mathbb{F}_2[x, Q_1x, (Q_1)^2x, \dots]$$

This is the same as the homology of $\Omega^2 S^3$: both are identified with the homology of the free E_2-algebra on a generator $x = \xi_1$ in degree 1. Mahowald showed that it was possible to realize this isomorphism of graded algebras: he constructed a Thom spectrum over $\Omega^2 S^3$ such that the Thom isomorphism realizes the isomorphism $A_* \cong H_*\Omega^2 S^3$ [55]. This has a rather remarkable interpretation: there exists a construction of the Eilenberg–Mac Lane spectrum H as the free E_2-algebra R such that the unit map $\mathbb{S} \to R$ has a chosen nullhomotopy of the image of 2. This result has been extended to odd primes by Blumberg–Cohen–Schlichtkrull [14].

Proposition 19.5.23. *Let Hk be the Eilenberg–Mac Lane spectrum for an algebra k over $\mathbb{F}_2$. Then there is an isomorphism*

$$H_* H\mathbb{F} \cong A_* \otimes k$$

of graded rings, and under this identification the Dyer–Lashof operation Q^r on H_*Hk is given by $Q^r \otimes \varphi$, where φ is the Frobenius on k.

Proof. For any H-module N, the action $H \otimes N \to N$ induces an isomorphism $H_*H \otimes \pi_* N \to H_* N$. We already know $Q^0(1 \otimes \alpha) = 1 \otimes \alpha^2$, and so by the Cartan formula it suffices to show that $Q^s(1 \otimes \alpha) = 0$ for $s > 0$.

We now proceed inductively by applying the Nishida relations. If we know $Q^t(1 \otimes \alpha) = 0$ for $0 < t < s$, we find that for all $r > 0$ we have

$$\begin{aligned} P_r Q^s(1 \otimes \alpha) &= \sum \binom{s-r}{r-2i} Q^{s-r+i} P_i(1 \otimes \alpha) \\ &= \binom{s-r}{r} Q^{s-r}(1 \otimes \alpha). \end{aligned}$$

By the inductive hypothesis, this vanishes unless $s = r$, but in the case $s = r$ the binomial coefficient vanishes. However, the only elements in H_*Hk that are acted on trivially by all the Steenrod operations are the elements in the image of $\pi_* Hk$, and those are concentrated in degree zero. Thus $Q^s(1 \otimes \alpha) = 0$. □

Remark 19.5.24. The same proof can be used to show that the Browder bracket is trivial on H_*Hk.

Example 19.5.25. The composite map $MU \to MO \to H$, on homology, is given in terms of the generators of Theorem 19.5.15 by $b_1 \mapsto a_1^2 \mapsto \xi_1^2$ and $b_2 \mapsto 0$. The image of H_*MU in A_* is $\mathbb{F}_2[\xi_1^2, \xi_2^2, \dots]$, the homology of the Brown–Peterson spectrum BP.

In H_*MU, Example 19.5.17 implies we have the identities

$$Q^6 b_2 = b_5 + b_1 b_4 + b_2 b_3 + b_1 b_2^2 = Q^8 b_1 + b_1^2 Q^4(b_1).$$

By contrast, in the dual Steenrod algebra we have the identity $0 = Q^8(\xi_1^2) + \xi_1^4 Q^4(\xi_1^2)$. Even though the map $H_*MU \to H_*BP$ splits as a map of algebras, and the target is closed under the Dyer–Lashof operations, we have

$$Q^8 b_1 + b_1^2 Q^4(b_1) = Q_6(b_1) + b_1^2 Q_2(b_1) \neq 0$$

but its image is zero. This implies that the map $H_*MU \to H_*BP$ does not have a splitting that respects the Dyer–Lashof operations for E_7-algebras. As a result, there exists no map $BP \to MU_{(2)}$ of E_7-algebras. This result, and its analogue at odd primes, is due to Hu–Kriz–May [39].

19.5.7 Nonexistence results

The tremendous amount of structure present in the homology of a ring spectrum allows us to produce a rather large number of nonexistence results. The following is a generalization of the classical result that the mod-2 Moore spectrum does not admit a multiplication due to the existence of a nontrivial Steenrod operation Sq^2 in its cohomology; we learned this line of argument from Charles Rezk.

Proposition 19.5.26. *Suppose that R is a homotopy associative ring spectrum containing an element u in nonzero degree such that $P_k(u)$ vanishes either in the range $k > |u|$ or in the range $0 < k < |u|$. Then either $P_{|u|}(u)$ is nilpotent or H_*R is nonzero in infinitely many degrees.*

Proof. We find, by the Cartan formula, that

$$P_{d|u|}(u^d) = (P_{|u|}u)^d.$$

Therefore, either the elements u^d are nonzero for all d or the element $P_{|u|}u$ is nilpotent. □

Corollary 19.5.27. *Suppose that R is a connective homotopy associative ring spectrum such that $H_0(R) = \pi_0(R)/2$ has no nilpotent elements. If any nonzero element in $H_0(R)$ is in the image of the Steenrod operations, then H_*R must be nonzero in infinitely many degrees.*

Corollary 19.5.28. *Suppose that R is a homotopy associative ring spectrum and that some Hopf invariant element 2, η, ν, or σ maps to zero under the unit map $\mathbb{S} \to R$. Then either $H_*R = 0$ or H_*R is infinite-dimensional.*

Proof. Writing h for Hopf invariant element in degree $2^k - 1$ with trivial image, the unit $\mathbb{S} \to R$ extends to a map $f\colon C(h) \to R$ from the mapping cone. The homology of $C(h)$ has a basis of elements 1 and v with one nontrivial Steenrod operation acting via $P_{2^k}v = 1$, and $u = f_*(v)$ has the desired properties. □

Recall from §19.3.3 that, for R connective, a map $\pi_0 R \to A$ of commutative rings automatically extends to a map $R \to HA$ compatible with the multiplicative structure that exists on R; e.g., if R is homotopy commutative then the map $R \to HA$ has the structure of a map of $\mathcal{Q}_1$-algebras. This has the following consequence.

Proposition 19.5.29. *Suppose that R is a connective ring spectrum with a ring homomorphism $\pi_0 R \to k$ where k is an $\mathbb{F}_2$-algebra (equivalently, a map $H_0 R \to k$). Then there is a map $R \to Hk$ that induces a homology map $H_*R \to A_* \otimes k$ with the following properties.*

1. *The map $H_*R \to A_* \otimes k$ is a map of rings that is surjective in degree zero.*
2. *If R is homotopy commutative, then there is an operation Q_1 on H_*R that is compatible with the operation Q_1 on $A_* \otimes k$.*
3. *If R has an E_n-algebra structure, the map $R \to Hk$ is a map of E_n-algebras and so $H_*R \to A_* \otimes k$ is compatible with the Dyer–Lashof operations $Q_0, \dots, Q_{n-1}$.*

In particular, the image of H_*R in $A_* \otimes k$ is a subalgebra $B_* \subset A_*$ closed under multiplication and some number of Dyer–Lashof operations.

Example 19.5.30. For $n > 0$ there are connective Morava K-theories $k(n)$, with coefficient ring $\mathbb{F}_2[v_n]$, that have homology

$$\mathbb{F}_2[\bar{\xi}_1, \dots, \bar{\xi}_n, \bar{\xi}_{n+1}^2, \bar{\xi}_{n+2}, \dots]$$

as a subalgebra of the dual Steenrod algebra. This subring is not closed under the Dyer–Lashof operation Q_1 unless $n = 0$, and so the connective Morava K-theories are not homotopy–commutative. (By convention we often define the connective Morava K-theory $k(0)$ to be $H\mathbb{Z}_2$, which is commutative.)

Similarly, for $n > 0$ the integral connective Morava K-theories $k_{\mathbb{Z}}(n)$, with coefficient ring $\mathbb{Z}_2[v_n]$, have homology

$$\mathbb{F}_2[\bar{\xi}_1^2, \bar{\xi}_2, \dots, \bar{\xi}_n, \bar{\xi}_{n+1}^2, \bar{\xi}_{n+2}, \dots]$$

as a subalgebra of the dual Steenrod algebra. This subring is not closed under the Dyer–Lashof operation Q_1 unless $n = 1$, and so the only possible homotopy-commutative integral Morava K-theory is $k_{\mathbb{Z}}(1)$—the connective complex K-theory spectrum.

There are obstruction-theoretic proofs which show that all of these have A_∞ structures [3, 48].

Example 19.5.31. The Dyer–Lashof operations satisfy $Q_2(\bar{\xi}_i^2) = \bar{\xi}_{i+1}^2$, and so the smallest possible subring of A_* that contains $\xi_1^2 = \bar{\xi}_1^2$ and is closed under Q_2 is an infinite polynomial algebra $\mathbb{F}_2[\bar{\xi}_i^2] = \mathbb{F}_2[\xi_i^2]$. If R is a connective ring spectrum with a quotient map $\pi_0 R \to \mathbb{F}$ such that the Hopf element $\eta \in \pi_1(\mathbb{S})$ maps to zero in $\pi_* R$, then there is a commutative diagram

$$\begin{array}{ccc} C(\eta) & \longrightarrow & R \\ \downarrow & & \downarrow \\ H\mathbb{Z}/2 & \longrightarrow & H\mathbb{F}. \end{array}$$

We conclude that ξ_1^2 is in the image of the map $H_*R \to H_*H\mathbb{F}$.

The spectra $X(n)$ appearing in the nilpotence and periodicity theorems of Devinatz–Hopkins–Smith fit into a sequence

$$X(1) \to X(2) \to X(3) \to \dots$$

of Thom spectra on the spaces $\Omega SU(n)$. They have E_2-ring structures, and each ring $H_*X(n)$ is a polynomial algebra $\mathbb{F}_2[x_1, \dots, x_{n-1}]$ on finitely many generators. For $n = 2$ the map

$H_*X(2) \to A_*$ is the map $\mathbb{F}_2[\xi_1^2] \to A_*$, and this implies that each $X(n)$ has ξ_1^2 in the image of its homology. As $H_*X(n)$ is finitely generated as an algebra, its image in the dual Steenrod algebra is too small to be closed under the operation Q_2. This excludes the possibility that $X(n)$ has an E_3-structure.

19.5.8 Ring spaces

Associated to an E_∞ ring spectrum E, there is a sequence of infinite loop spaces $\{E_n\}_{n\in\mathbb{Z}}$ in an Ω-spectrum representing E. These spaces are extremely strongly structured: they inherit both *additive* structure from the spectrum structure on E, and *multiplicative* structure from the E_∞ ring structure. In the case of the sphere spectrum, these operations were investigated in-depth in relationship to surgery theory [67, 54, 61]. Ravenel and Wilson discussed the structure coming from a ring spectrum E extensively in [75], encoding it in the structure of a *Hopf ring*, and the interaction between additive and multiplicative operations is developed in-depth in [23, §II]. These structures are very tightly wound.

1. Because the E_n are spaces, the diagonals $E_n \to E_n \times E_n$ gives rise to a *coproduct*

$$\Delta\colon H_*(E_n) \to H_*(E_n) \otimes H_*(E_n),$$

   For an element x we write $\sum x' \otimes x''$ for its coproduct. The path components $E^n = \pi_0 E_n$ also give rise to elements $[\alpha] \in H_0 E_n$.

2. The homology groups H_*E_n have Steenrod operations P_r.

3. The suspension maps $\Sigma E_n \to E_{n+1}$ in the spectrum structure give stabilization maps

$$H_m(E_n) \to H_{m+1}(E_{n+1}),$$

4. The infinite loop space structure on E_n gives H_*E_n an *additive* Pontryagin product

$$\#\colon H_*(E_n) \otimes H_*(E_n) \to H_*(E_n)$$

   making it into a Hopf algebra, and it has *additive* Dyer–Lashof operations

$$Q^r\colon H_m(E_n) \to H_{m+r}(E_n).$$

5. If E has a ring spectrum structure, the multiplication $E \otimes E \to E$ gives *multiplicative* Pontryagin products

$$\circ\colon H_*(E_n) \otimes H_*(E_m) \to H_*(E_{n+m}).$$

   These are appropriately unital, associative, or graded-commutative if E has these properties.

6. If E has an E_∞ ring spectrum structure, there are *multiplicative* Dyer–Lashof operations

$$\widetilde{Q}^r\colon H_m(E_0) \to H_{m+r}(E_0)$$

on the homology of the 0'th space. In general, we cannot say more. If E has further structure—an H_∞^d-structure—there are also multiplicative Dyer–Lashof operations outside degree zero [47, §4.1].

These are subject to a large number of identities discussed in [75, 1.12, 1.14], [23, II.1.5, II.1.6, II.2.5], and [45, 1.5]. Here are the most fundamental identities:

Distributive rule: $(x \# y) \circ z = \sum (x \circ z') \# (y \circ z'')$

Projection formula: $x \circ Q^s y = \sum Q^{s+k}(P_k x \circ y)$

Mixed Cartan formula:

$$\widetilde{Q}^n(x \# y) = \sum_{p+q+r=n} \widetilde{Q}^p(x') \# Q^q(x'' \circ y') \# \widetilde{Q}^r(y'')$$

Mixed Adem relations:

$$\widetilde{Q}^r Q^s x = \sum_{i+j+k+l=r+s} \binom{r-i-2l-1}{j+s-i-l} Q^i \widetilde{Q}^j x' \# Q^k \widetilde{Q}^l x''$$

Example 19.5.32. There is an identity

$$Q^1[a] \# [-2a] = \eta \cdot a$$

which allows us to determine information about the multiplication-by-η map $\pi_0 R \to \pi_1 R \to H_1 \Omega^\infty R$ from the additive Dyer–Lashof structure. Similarly $\widetilde{Q}^1$ determines information about its multiplicative version $\eta_m \colon \pi_0(R) \to \pi_1(R)$. For example, the mixed Cartan formula implies that

$$\eta_m(x+y) = \eta_m(x) + \eta_m(y) + \eta \cdot xy$$

in $H_1(R)$. In particular, $\widetilde{Q}^1[n] = \binom{n}{2}\eta \# [n^2]$ for $n \in \mathbb{Z}$ (cf. Example 19.3.11).

19.6 Higher-order structure

19.6.1 Secondary composites

Secondary operations, at their core, arise when there are relations between relations. Suppose that we are a sequence $X_0 \xrightarrow{f} X_1 \xrightarrow{g} X_2 \xrightarrow{h} X_3$ of maps such that the double composites are nullhomotopic. Then hgf is nullhomotopic for two reasons. Choosing nullhomotopies of gf and hg, we can glue the nullhomotopies together to determine a loop in the space of maps $X \to W$: a *value* of the associated secondary operation. Because we must make choices of nullhomotopy, there is some natural indeterminacy in this construction, and so it typically takes a set of values $\langle h, g, f \rangle$. To construct secondary operations, we minimally need to work in a category C with mapping spaces; we also need canonical basepoints of the spaces $\mathrm{Map}_{\mathsf{C}}(X_i, X_j)$ for $j \geq i+2$ that are preserved under composition [47, §2].

Example 19.6.1. Suppose that A is a subspace of X and $\alpha \in H^n(X, A)$ is a cohomology element that restricts to zero in $H^n(X)$. Then the long exact sequence in cohomology implies that we can lift α to an element in $H^{n-1}(A)$, but there are multiple choices of lift. This can be represented by a sequence of maps

$$A \to X \to X/A \to K(\mathbb{Z}, n)$$

where the double composites are nullhomotopic; the secondary operation is then a map $A \to \Omega K(\mathbb{Z}, n) = K(\mathbb{Z}, n-1)$.

Secondary operations enrich the homotopy category $h\mathsf{C}$ with extra structure.

1. Every test object $T \in \mathsf{C}$ represents a functor $[T, -] = \pi_0 \mathrm{Map}_{\mathsf{C}}(T, -)$ on $h\mathsf{C}$, and if T has an augmentation $T \to 0$ to an initial object then this functor has a canonical null element. If the values of $[T, -]$ differ on X and Y, X and Y cannot be equivalent in $h\mathsf{C}$.

2. Every map of test objects $\Theta\colon S \to T$ determines an operation: a natural transformation of functors $\theta\colon [T, -] \to [S, -]$ on $h\mathsf{C}$. If S and T are augmented and the map Θ is compatible with the augmentations, then θ preserves the null element. If θ has different behaviour for X and Y, X and Y cannot be equivalent.

3. Given an augmented map $\Phi\colon R \to S$ and a map $\Theta\colon S \to T$ such that the double composite $\Theta\Phi\colon R \to T$ is trivial, we get an identity $\varphi\theta = 0$ of associated operations. There is an associated secondary operation $\langle -, \Theta, \Phi \rangle$. It is only defined on those elements $\alpha \in [T, X]$ with $\theta(\alpha) = 0$; it takes values in $\pi_1 \mathrm{Map}_{\mathsf{C}}(R, X)$; it is only well-defined up to indeterminacy.

4. We can also associate information to maps in the same way. Suppose we have an augmented map $\Theta\colon S \to T$ of test objects representing an operation θ. Given any map $f\colon X \to Y$, there is an associated functional operation $\langle f, -, \Theta \rangle$. It is only defined on those elements $\alpha \in [T, X]$ such that $f(\alpha) = 0$ and $\theta(\alpha) = 0$; it takes values in $\pi_1 \mathrm{Map}_{\mathsf{C}}(S, Y)$; it is only well-defined up to indeterminacy.

Applying this to the test objects S^n in the category of pointed spaces, we get Toda's bracket construction that enriches the homotopy groups of spaces with secondary composites. Applying this to the test objects $K(A, n)$ in the opposite of the category of spaces, we get Adams' secondary operations that enrich the cohomology groups of spaces with secondary cohomology operations.

19.6.2 Secondary operations for algebras

We recall from §19.4.3 that, for a spectrum M and an $\mathcal{O}$-algebra in Mod_E, we have

$$\pi_M(A) = [M, A]_{\mathsf{Sp}} \cong [E \otimes \mathrm{Free}_{\mathcal{O}}(M), A]_{\mathsf{Alg}_{\mathcal{O}}(\mathrm{Mod}_E)}.$$

Using free algebras as our test objects, we already used this representability of homotopy groups to classify the natural operations on the homotopy groups of $\mathcal{O}$-algebras in Mod_E. The space of maps now means that we can construct secondary operations.

Proposition 19.6.2. *Suppose that we have zero-preserving operations* $\theta\colon \pi_M \to \pi_N$ *and* $\varphi\colon \pi_N \to \pi_P$ *on the homotopy category of* $\mathcal{O}$*-algebras in* Mod_E*, and that there is a relation* $\varphi \circ \theta = 0$*. Then there exists a secondary operation*

$$\langle -, \Theta, \Phi \rangle\colon\ \pi_M(A) \supset \mathrm{Ker}\theta \to \pi_{P+1}(A)/Im(\sigma\varphi),$$

where $\sigma(\varphi)$ *is a suspended operation (see §19.4.6).*

Such a secondary operation is constructed from a sequence

$$E \otimes \mathrm{Free}_{\mathcal{O}}(P) \xrightarrow{\Phi} E \otimes \mathrm{Free}_{\mathcal{O}}(N) \xrightarrow{\Theta} E \otimes \mathrm{Free}_{\mathcal{O}}(M) \to A$$

where the double composites are null; the nullhomotopy of $\Theta \circ \Phi$ is chosen once and for all, while the second nullhomotopy is allowed to vary. This produces elements in

$$\pi_1 \mathrm{Map}_{\mathtt{Alg}_{\mathcal{O}}(\mathrm{Mod}_E)}(E \otimes \mathrm{Free}_{\mathcal{O}}(P), A) \cong \pi_1 \mathrm{Map}_{\mathtt{Sp}}(P, A) \cong [\Sigma P, A].$$

Example 19.6.3. Every Adem relation between Dyer–Lashof operations produces a secondary Dyer–Lashof operation. For example, the relation $Q^{2n+2}Q^n + Q^{2n+1}Q^{n+1} = 0$ produces a natural transformation

$$\pi_m(A) \supset \mathrm{Ker}(Q^n, Q^{n+1}) \to \pi_{m+2n+3}(A)/Im(Q^{2n+2}, Q^{2n+1})$$

on the homotopy of H-algebras.

Example 19.6.4. Relations involving operations other than composition and addition can also produce secondary operations, and the canonical examples of these are *Massey products*. An $\mathcal{A}_2$-algebra R has a binary multiplication operation $R \otimes R \to R$, and if R is an $\mathcal{A}_3$-algebra it has a chosen associativity homotopy. As a result, if we have elements x, y, and z in $\pi_* R$ such that $xy = yz = 0$, then we can glue together two nullhomotopies of xyz to obtain a *bracket* $\langle x, y, z \rangle$ that specializes to definitions of Massey products or Toda brackets.

In trying to express nonlinear relations as secondary operations, however, we rapidly find that we want to move into a *relative* situation. A Massey product is defined on the kernel of the map $\pi_p \times \pi_q \times \pi_r \to \pi_{p+q} \times \pi_{q+r}$ sending (x, y, z) to (xy, yz). However, the relation $x(yz) = (xy)z$ is not expressible solely as some operation on xy and yz: we need to remember x and z as well, but we *do not* want to enforce that they are zero.

We find that the needed expression is homotopy commutativity of the following diagram:

$$\begin{array}{ccc}
\mathrm{Free}_{\mathcal{A}_3}(S^p \oplus S^{p+q+r} \oplus S^r) & \xrightarrow{\Phi} & \mathrm{Free}_{\mathcal{A}_3}(S^p \oplus S^{p+q} \oplus S^{q+r} \oplus S^r) \\
\downarrow & & \downarrow \Theta \\
\mathrm{Free}_{\mathcal{A}_3}(S^p \oplus S^r) & \longrightarrow & \mathrm{Free}_{\mathcal{A}_3}(S^p \oplus S^q \oplus S^r)
\end{array}$$

The right-hand map classifies the operation $\theta(x, y, z) = (x, xy, yz, z)$, and the top map classifies the operation $\varphi(x, u, v, z) = (x, xv - uz, z)$. The bottom-left object is not the initial object in the category of $\mathcal{A}_3$-algebras, so we *enforce* this by switching to the category $\mathtt{C}$ of $\mathcal{A}_3$-algebras under $\mathrm{Free}_{\mathcal{A}_3}(S^p \oplus S^r)$. In this category, we genuinely have augmented objects with a nullhomotopic double composite

$$\mathrm{Free}_{\mathsf{C}}(S^{p+q+r}) \to \mathrm{Free}_{\mathsf{C}}(S^{p+q} \oplus S^{q+r}) \to \mathrm{Free}_{\mathsf{C}}(S^q)$$

that defines a Massey product.

19.6.3 Juggling

Secondary operations are part of the homotopy theory of C, and there is typically no method to determine secondary operations purely in terms of the homotopy category. However, there are many composition-theoretic tools that use one secondary operation to determine information about another: typically, one starts with a 4-fold composite

$$X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} U \xrightarrow{k} V,$$

with some assortment of double-composites being nullhomotopic, and relates various associated secondary operations. This process is called *juggling*, and learning to juggle secondary operations is one of the main steps in applying them. For instance, one of the main juggling formulas—the Peterson–Stein formula—asserts that the sets $\langle k, h, g\rangle f$ and $k\langle h, g, f\rangle$ are inverse in π_1 when both sides make sense.

Example 19.6.5. The Adem relations $Q^{2n+1}Q^n$ and $Q^{4n+3}Q^{2n+1}$ give rise to a secondary operation $\langle \mathbf{Q}^n, \mathbf{Q}^{2n+1}, \mathbf{Q}^{4n+3}\rangle$, an element of $\pi_{7n+5+m}(H \otimes \mathrm{Free}_{E_\infty}(S^m))$ representing an operation that increases degree by $7n+5$. The juggling formula says that, for any element $\alpha \in \pi_m(A)$ with $Q^n(A) = 0$, we have

$$Q^{4n+3}\langle \alpha, \mathbf{Q}^n, \mathbf{Q}^{2n+1}\rangle = \langle \mathbf{Q}^n, \mathbf{Q}^{2n+1}, \mathbf{Q}^{4n+3}\rangle(\alpha).$$

In other words, this secondary composite of operations gives a universal formula for how to apply Q^{4n+3} to this secondary operation.

19.6.4 Application to the Brown–Peterson spectrum

In this section we will give a brief account of the main result of [47], which uses secondary operatons to show that the 2-primary Brown–Peterson spectrum BP does not admit the structure of an E_{12} ring spectrum. These results have been generalized by Senger to show that, at the prime p, BP does not have an E_{2p^2+4} ring structure [86].

As in §19.3.3, if the Brown–Peterson spectrum has an E_n-algebra structure then the map

$$BP \to H\mathbb{Z}_{(2)} \to H$$

can be given the structure of a map of E_n-algebras. On homology, this would then induce a monomorphism

$$\mathbb{F}_2[\xi_1^2, \xi_2^2, \dots] \to \mathbb{F}_2[\xi_1, \xi_2, \dots]$$

of algebras equipped with E_n Dyer–Lashof operations and secondary Dyer–Lashof operations. The dual Steenrod algebra, on the right, has operations that are completely forced.

Therefore, if we can calculate enough to show that the subalgebra H_*BP is not closed under secondary operations for E_n-algebras, we arrive at an obstruction to giving BP the structure of E_n ring spectrum.

The calculation of secondary operations in H_*H is accomplished with judicious use of juggling formulas, ultimately reducing questions about secondary operations to questions about primary ones.

- There is a pushout diagram of E_∞ ring spectra

$$\begin{array}{ccc} H \otimes MU & \xrightarrow{i} & H \otimes H \\ \downarrow & & \downarrow{\scriptstyle j} \\ H & \longrightarrow & H \otimes_{MU} H. \end{array}$$

  This makes $H \otimes MU$ into an augmented H-algebra, and gives a nullhomotopy of the composite $H \otimes MU \to H \otimes H \to H \otimes_{MU} H$. The elements α in H_*MU that map to zero in H_*H are then candidates for secondary operations: we can construct $\langle j, i, \alpha \rangle$ in the MU-dual Steenrod algebra $\pi_*(H \otimes_{MU} H)$.

- These elements are concretely detected: they have explicit representatives on the 1-line of a two-sided bar spectral sequence

$$\operatorname{Tor}^{H_*MU}(H_*H, \mathbb{F}_2) \Rightarrow \pi_*(H \otimes_{MU} H).$$

- If we can determine *primary* operations $\theta(\langle i, j, \alpha \rangle)$ in the MU-dual Steenrod algebra, the juggling formulas of §19.6.3 tell us about functional operations $\langle j, \alpha, \Theta \rangle$ in the ordinary dual Steenrod algebra.

- Steinberger's calculations of primary operations φ in the dual Steenrod algebra then allow us to determine the values of $\varphi\langle j, \alpha, \Theta \rangle$, and juggling formulas again allow us to determine information about secondary operations $\langle \alpha, \Theta, \Phi \rangle$ in the dual Steenrod algebra.

This method, then, reduces us to carrying out some key computations.

We must determine primary operations in the MU-dual Steenrod algebra. Some of these, by work of Tilson [92], are determined by Kochman's calculations from Theorem 19.5.15: the Künneth spectral sequence

$$\operatorname{Tor}^{\pi_*MU}(\mathbb{F}_2, \mathbb{F}_2) \Rightarrow \pi_*(H \otimes_{MU} H)$$

calculating the MU-dual Steenrod algebra is compatible with Dyer–Lashof operations. However, there are remaining extension problems in the Tor, and these turn out to be precisely what we are interested in when juggling.

The MU-dual Steenrod algebra is an exterior algebra, whose generators correspond to the indecomposables in π_*MU. The extension problems in the Tor spectral sequence arise because some generators in π_*MU have nontrivial image in H_*MU and are detected by Tor_1, while others have trivial image in H_*MU and are detected by Tor_0. The solution is to find an algebra R mapping to MU that does not have this problem. If we can find one so that the map $\pi_*R \to \pi_*MU$ is surjective, the map from the R-dual Steenrod algebra to the MU-dual Steenrod algebra is surjective. If the generators of π_*R have nontrivial image in H_*R, then the spectral sequence

$$\mathrm{Tor}^{H_*R}(H_*H, \mathbb{F}_2) \Rightarrow \pi_*(H \otimes_R H)$$

detects all needed classes with Tor_1 and hence eliminates the extension problem.

For this purpose, we used the spherical group algebra $\mathbb{S}[SL_1(MU)]$. The Dyer–Lashof operations in $H_*SL_1(MU)$ are derived from the multiplicative Dyer–Lashof operations $\widetilde{Q}^n$ in $\Omega^\infty MU$. This is a lengthy calculation of power operations in the Hopf ring, and it is ultimately determined by calculations of Johnson–Noel of power operations in the formal group theory of MU [40].

Finally, we must determine a candidate secondary operation in H_*H to which we can apply this procedure—there are many candidate operations and many dead ends. The secondary operation is rather large: it was found using a calculation in Goerss–Hopkins obstruction theory that is detailed at length in [46].

19.7 Coherent structures

In §19.3 we discussed algebras over an operad in a general topological category, or more generally algebras over a multicategory $\mathcal{M}$, including extended power and free algebra functors. The definitions we used made heavy use of a strict symmetric monoidal structure on the category of spectra.

In this section we will discuss the coherent viewpoint on these constructions that makes use of the machinery of Lurie [53] and with the goal of connecting different strata in the literature. To begin, we should point some of the problems that this discussion is meant to solve.

We would like to demonstrate that our constructions are model-independent. There are several different symmetric monoidal categories of spectra [27, 38, 82] with several different model structures, and there is a nontrivial amount of work involved in showing that an equivalence between two different categories of spectra gives an equivalence between categories of algebras [84]. These issues are compounded when we attempt to relate notions of commutative algebras in different categories, even if they have equivalent homotopy theory [93].

We would also like to allow weaker structure than a symmetric monoidal structure. For example, given a fixed E_n-algebra R we will use this to discuss the classification of power operations on E_n-algebras under R. Our natural home for this discussion will be the category of E_n R-modules (as in Example 19.6.4).

19.7.1 Structured categories

As discussed in §19.1, classical symmetric monoidal categories are analogues of commutative monoids with the difference that they require natural isomorphisms to express associativity, commutativity, and the like. We can express this structure using simplicial operads. For any categories C and D, there is a groupoid $\mathrm{Fun}(\mathsf{C},\mathsf{D})^{\simeq}$ of functors and natural isomorphisms. Taking the nerve, we get a simplicially enriched category $\mathcal{C}at$, and it makes sense to ask whether C has the structure of an algebra over a simplicial operad $\mathcal{O}$.

Example 19.7.1. A symmetric monoidal category can be expressed as an algebra over the Barratt–Eccles operad [6].

Example 19.7.2. In classical category theory, a *braided monoidal category* in the sense of [41] can be encoded by a sequence of maps

$$NP_n \to \mathrm{Fun}(\mathsf{C}^n,\mathsf{C})$$

from the nerves of the pure braid groups to the categories of functors $\mathsf{C}^n \to \mathsf{C}$. The required compatibilities between these maps can be concisely expressed by noting that these nerves assemble into an E_2-operad, and that a braided monoidal category is an algebra over this operad.

We would like to discuss E_n-analogues of these structures in the context of categories with morphism spaces. We will give some definitions in this section, on the point-set level, with the purpose of interpolating the older and newer definitions. We would like to say that an $\mathcal{O}$-monoidal category is an algebra over the operad $\mathcal{O}$ in $\mathcal{C}at$, but this requires us to be clever enough to have a well-behaved definition of a *space* of functors between two enriched categories; the failure of enriched categories to have a well-behaved enriched functor category is a principal motivation for the use of quasicategories.

Until further notice, all categories and multicategories are assumed to be enriched in spaces and all functors are functors of enriched categories.

Definition 19.7.3. Suppose that $p\colon \mathsf{C} \to \mathcal{M}$ is a multifunctor, and write $\mathsf{C}_{\mathbf{x}}$ for the category $p^{-1}(\mathbf{x})$. Given objects $X_i \in \mathsf{C}_{\mathbf{x}_i}$ and a map

$$\alpha\colon\ A \to \mathrm{Mul}_{\mathcal{M}}(\mathbf{x}_1,\dots,\mathbf{x}_d;\mathbf{y})$$

of spaces, an *α-twisted product* is an object $Y \in \mathsf{C}_{\mathbf{y}}$ and a map $A \to \mathrm{Mul}_{\mathsf{C}}(X_1,\dots,X_d;Y)$ such that, for any $Z \in \mathsf{C}$ with $p(Z) = \mathbf{z}$, the diagram

$$\begin{array}{ccc} \mathrm{Map}_{\mathsf{C}}(Y,Z) & \longrightarrow & \mathrm{Map}(A,\mathrm{Mul}_{\mathsf{C}}(X_1,\dots,X_d;Z)) \\ \downarrow & & \downarrow \\ \mathrm{Map}_{\mathcal{M}}(\mathbf{y},\mathbf{z}) & \longrightarrow & \mathrm{Map}(A,\mathrm{Mul}_{\mathcal{M}}(\mathbf{x}_1,\dots,\mathbf{x}_d;\mathbf{z})) \end{array}$$

is a pullback. If it exists, we denote it by $A \ltimes_\alpha (X_1,\dots,X_d)$.

Definition 19.7.4. A *weakly $\mathcal{M}$-monoidal category* is a multifunctor $p\colon \mathsf{C} \to \mathcal{M}$ that has α-twisted product for any inclusion

$$\alpha\colon \{f\} \subset \mathrm{Mul}_{\mathcal{M}}(\mathbf{x}_1, \ldots, \mathbf{x}_d; \mathbf{y}).$$

A *strongly $\mathcal{M}$-monoidal category* is a category that has α-twisted products for all α.

Remark 19.7.5. In particular, for any point $f \in \mathrm{Mul}_{\mathcal{M}}(\mathbf{x}_1, \ldots, \mathbf{x}_d; \mathbf{y})$, this universal property can be used to produce a functor

$$\{f\} \ltimes (-)\colon \mathsf{C}_{\mathbf{x}_1} \times \cdots \times \mathsf{C}_{\mathbf{x}_d} \to \mathsf{C}_{\mathbf{y}},$$

and these are compatible with composition (up to natural isomorphism). A weakly $\mathcal{M}$-monoidal category determines, up to natural equivalence, a multifunctor $\mathcal{M} \to \mathcal{C}at$.

Example 19.7.6. Every multicategory C has a multifunctor to the one-object multicategory $\mathcal{C}omm$ associated to the commutative operad. The multicategory is $\mathcal{C}omm$-monoidal if and only if multimorphisms $(X_1, \ldots, X_d) \to Y$ are always representable by an object $X_1 \otimes \cdots \otimes X_d$, which is precisely when C comes from a symmetric monoidal category. It is strongly $\mathcal{C}omm$-monoidal only if it is also tensored over spaces in a way compatible with the monoidal structure as in Definition 19.3.9.

Example 19.7.7. Associated to a monoidal category C we can build a multicategory: multimorphisms $(X_1, \ldots, X_d) \to Y$ are pairs of a permutation $\sigma \in \Sigma_d$ and a map $f\colon X_{\sigma(1)} \otimes \cdots \otimes X_{\sigma(d)} \to Y$. There is a multifunctor from this category to the multicategory $\mathcal{A}ssoc$ corresponding to the associative operad: it sends all objects to the unique object, and sends each multimorphism (σ, f) as above to the permutation σ. Conversely, an $\mathcal{A}ssoc$-monoidal category comes from a monoidal category.

Example 19.7.8. Suppose that A is a commutative ring and B is an A-algebra. Then there is a multicategory C as follows.

1. An object of C is either an A-module or a right B-module.
2. The set $\mathrm{Mul}_{\mathsf{C}}(M_1, \ldots, M_d; N)$ of multimorphisms is

$$\begin{cases} \mathrm{Hom}_A(M_1 \otimes_A \cdots \otimes_A M_d, N) & \text{if } N \text{ and all } M_i \text{ are } A\text{-modules},\\ \mathrm{Hom}_B(M_1 \otimes_A \cdots \otimes_A M_d, N) & \text{if } N \text{ and exactly one } M_i \text{ are } B\text{-modules},\\ \emptyset & \text{otherwise.} \end{cases}$$

This comes equipped with a functor from C to the multicategory Mod from Example 19.3.23 that parametrizes ring-module pairs: any A-module is sent to $\mathbf{a}$ and any B-module is sent to $\mathbf{m}$. This makes C into a Mod-monoidal category, expressing the fact that Mod_A has a tensor product and that objects of RMod_B can be tensored with objects of Mod_A. This makes RMod_B *left-tensored* over Mod_A.

Example 19.7.9. Fiberwise homotopy theory studies the category $\mathsf{S}_{/B}$ of spaces over B. Let $\mathcal{O}$ be an operad and B be a space with the structure of an $\mathcal{O}$-algebra. Then $\mathsf{S}_{/B}$ has the structure of a strongly $\mathcal{O}$-monoidal category in the following way. For spaces $X_1, \ldots, X_d$ and Y over B, the space of multimorphisms is the pullback

$$\begin{array}{ccccc} \mathrm{Mul}_{/B}(X_1, \ldots, X_d; Y) & \longrightarrow & & & \mathrm{Map}(X_1 \times \cdots \times X_d, Y) \\ \downarrow & & & & \downarrow \\ \mathcal{O}(d) & \longrightarrow & \mathrm{Map}(B^d, B) & \longrightarrow & \mathrm{Map}(X_1 \times \cdots \times X_d, B). \end{array}$$

That is, a multimorphism consists of a point $f \in \mathcal{O}(d)$ and a commutative diagram

$$\begin{array}{ccc} X_1 \times \cdots \times X_d & \longrightarrow & Y \\ \downarrow & & \downarrow \\ B^d & \xrightarrow[f]{} & B. \end{array}$$

With this definition, it is straightforward to verify that for $\alpha\colon A \to \mathcal{O}(d)$, the α-twisted product $A \ltimes_\alpha (X_1, \ldots, X_d)$ is the following space over B:

$$A \times X_1 \times \cdots \times X_d \to \mathcal{O}(d) \times B^d \to B.$$

In general, this should not be expected to be part of a symmetric monoidal structure on the category of spaces over B, even up to equivalence.

Example 19.7.10. Let $\mathcal{L}$ be the category of *universes*: an object is a countably infinite dimensional inner product space U. These objects have an associated multicategory: the space $\mathrm{Mul}_{\mathcal{L}}(U_1, \ldots, U_d; V)$ of multimorphisms is the (contractible) space of linear isometric embeddings $U_1 \oplus \cdots \oplus U_d \hookrightarrow V$. Over $\mathcal{L}$, there is a category $\mathsf{Sp}_{\mathcal{L}}$ of *indexed spectra*. An object is a pair (U, X) of a universe U and a spectrum X (in the Lewis–May–Steinberger sense [49]) indexed on U; a multimorphism $((U_1, X_1), \ldots, (U_d, X_d)) \to (V, Y)$ is a pair of a linear isometric embedding $i\colon U_1 \oplus \cdots \oplus U_d \to V$ and a map $i_*(X_1 \wedge \cdots \wedge X_d) \to Y$ of spectra indexed on V.

This does not describe the topology on the multimorphisms in this category. Given a map $A \to \mathcal{L}(U_1, \ldots, U_d; V)$ and spectra X_i indexed on U_i, there is a *twisted half-smash product* $A \ltimes (X_1, \ldots, X_d)$ indexed on V [49, §VI], equivalent to the smash product $A_+ \wedge X_1 \wedge \cdots \wedge X_d$. There exists a topology on the multimorphisms so that a continuous map in from A is equivalent to a map $A \to \mathcal{L}(U_1, \ldots, U_d; V)$ and a map $A \ltimes (X_1, \ldots, X_d) \to Y$. By design, then, the projection $\mathsf{Sp}_{\mathcal{L}} \to \mathcal{L}$ makes the category of indexed spectra strongly $\mathcal{L}$-monoidal.

Example 19.7.11. Fix an E_n-algebra A in Sp, and consider the category of E_n-algebras R with a factorization $A \to R \to A$ of the identity map. This has an associated *stable category*, serving as the natural target for Goodwillie's calculus of functors: the category of E_n A-modules [28]. This category should also not be expected to have a symmetric monoidal structure, but the tensor product over R does give it the structure of an E_n-monoidal category. For example, for an associative algebra A in Sp, the tensor product over A gives the category of A-bimodules a monoidal structure.

19.7.2 Multi-object algebras

Just as we cannot make sense of a commutative monoid in a nonsymmetric monoidal category, we need relationships between an operad $\mathcal{O}$ and any multiplicative structure on a category C before $\mathcal{O}$ can act on objects.

Definition 19.7.12. Suppose $p\colon \mathsf{C} \to \mathcal{N}$ and $\mathcal{M} \to \mathcal{N}$ are multifunctors. An $\mathcal{M}$-*algebra in* C is a lift in the diagram

$$\begin{array}{ccc} & & \mathsf{C} \\ & \nearrow & \downarrow p \\ \mathcal{M} & \longrightarrow & \mathcal{N} \end{array}$$

of multifunctors. We write $\mathsf{Alg}_{\mathcal{M}/\mathcal{N}}(\mathsf{C})$ for this category of $\mathcal{M}$-algebras.

Example 19.7.13. If C and $\mathcal{M}$ are arbitrary multicategories, then using the unique maps from C and $\mathcal{M}$ to the terminal multicategory $\mathcal{C}omm$ we recover the definition of $\mathsf{Alg}_{\mathcal{M}}(\mathsf{C})$, the category of $\mathcal{M}$-algebras in C from Definition 19.3.22.

Example 19.7.14. Let the space B be an algebra over an operad $\mathcal{O}$ and consider the fiberwise category $\mathsf{S}_{/B}$ of spaces over B with the strongly $\mathcal{O}$-monoidal structure from Example 19.7.9. An $\mathcal{O}$-algebra in $\mathsf{S}_{/B}$ is an $\mathcal{O}$-algebra X with a map of $\mathcal{O}$-algebras $X \to B$.

Example 19.7.15. Consider the category of indexed spectra $\mathsf{Sp}_{\mathcal{L}}$ from Example 19.7.10. The fact that the external smash product $(X_1 \wedge \cdots \wedge X_n)$ is naturally indexed on the direct sum of the associated universes obstructed making the category of spectra indexed on any individual universe Sp strictly symmetric monoidal, and so we cannot ask about commutative monoids in $\mathsf{Sp}_{\mathcal{L}}$—but the structure available is still enough to do multiplicative homotopy theory. An $\mathcal{L}$-algebra in $\mathsf{Sp}_{\mathcal{L}}$ recovers the classical definition of an E_∞ ring spectrum from [63]. Similarly we can define $\mathcal{O}$-algebras for any operad $\mathcal{O}$ with an augmentation to $\mathcal{L}$ [49, VII.2.1].

Proposition 19.7.16. *Suppose that* C *is strongly* $\mathcal{N}$*-monoidal and that* $\mathcal{M} \to \mathcal{N}$ *is a map of multicategories. In addition, suppose that* C *has enriched colimits and that formation of* α*-twisted products preserves enriched colimits in each variable.*

1. *For objects* $\mathbf{x}$ *and* $\mathbf{y}$ *of* $\mathcal{M}$*, there are extended power functors*

$$\mathrm{Sym}^k_{\mathcal{M},\mathbf{x}\to\mathbf{y}}\colon \mathsf{C}_{\mathbf{x}} \to \mathsf{C}_{\mathbf{y}},$$

given by

$$\mathrm{Sym}^k_{\mathcal{M},\mathbf{x}\to\mathbf{y}}(X) = \mathrm{Mul}_{\mathcal{M}}(\underbrace{\mathbf{x},\mathbf{x},\ldots,\mathbf{x}}_{k};\mathbf{y}) \ltimes (X, X, \ldots, X)/\Sigma_k.$$

2. *The evaluation functor* $\mathrm{ev}_{\mathbf{x}}\colon \mathsf{Alg}_{\mathcal{M}}(\mathsf{C}) \to \mathsf{C}_{\mathbf{x}}$ *has a left adjoint*

$$\mathrm{Free}_{\mathcal{M},\mathbf{x}}\colon \mathsf{C}_{\mathbf{x}} \to \mathsf{Alg}_{\mathcal{M}}(\mathsf{C}).$$

The value of $\mathrm{Free}_{\mathcal{M},\mathbf{x}}(X)$ *on any object* $\mathbf{y}$ *of* $\mathcal{M}$ *is*

$$\mathrm{ev}_{\mathbf{y}}(\mathrm{Free}_{\mathcal{M},\mathbf{x}}(X)) = \coprod_{k\geq 0} \mathrm{Sym}^k_{\mathcal{M},\mathbf{x}\to\mathbf{y}}(X).$$

Example 19.7.17. Let B be a space with an action of an operad $\mathcal{O}$, and let X a space over B. Then the extended powers are

$$\mathrm{Sym}^k_{\mathcal{O}}(X) = \left(\mathcal{O}(k) \times_{\Sigma_k} X^k \to \mathcal{O}(k) \times_{\Sigma_k} B^k \to B\right).$$

Example 19.7.18. Suppose that Γ is a commutative monoid and that X is a Γ-graded E_∞ ring spectrum, as in Example 19.3.24. Then there are action maps $\mathrm{Sym}^k X_g \to X_{kg}$. These give rise to Dyer–Lashof operations $Q^i\colon H_* X_g \to H_{*+i}(X_{2g})$.

Example 19.7.19. Suppose that $\cdots \to X_2 \to X_1 \to X_0$ is a strongly filtered E_∞ ring spectrum, as in Example 19.3.25. Then there are action maps $\mathrm{Sym}^k X_n \to X_{kn}$ that are compatible. These give rise to power operations $Q^i\colon H_* X_n \to H_{*+i} X_{2n}$ that are compatible as n varies, and there are induced power operations on the associated spectral sequence.

Example 19.7.20. Given a spectrum X indexed on a universe U as in Example 19.7.10, the extended powers are modeled by twisted half-smash products:

$$\mathrm{Sym}^k_{U\to U}(X) \simeq E\Sigma_k \ltimes_{\Sigma_k} (X^{\wedge k}).$$

This recovers the machinery that was put to effective use in the 1970s and 1980s for studying E_∞ ring spectra and H_∞-ring spectra, before the development of strictly monoidal categories of spectra.

19.7.3 ∞-operads

The point-set discussion of the previous sections provides a library of examples. As the basis for a theory it relies on the existence of rigid models and preservation of colimits.

Example 19.7.21. Consider the fiberwise category of spaces over a fixed base space B. This category has a symmetric monoidal fiber product $X \times_B Y$. The fiber product typically needs fibrant input to represent the homotopy fiber product; the fiber product typically does not produce cofibrant output. This makes it difficult to use the standard machinery to study algebras and modules in this category. These problems have received significant attention in the setting of parametrized stable homotopy theory [64, 51, 52].

Example 19.7.22. The category of nonnegatively graded chain complexes over a commutative ring R is equivalent to the category of simplicial R-modules via the Dold–Kan correspondence. This correspondence is lax symmetric monoidal in one direction, but only lax monoidal in the other. Moreover, while both sides have morphism spaces, the Dold–Kan correspondence only preserves these up to weak equivalence, even for fibrant-cofibrant objects.

Example 19.7.23. In the standard models of equivariant stable homotopy theory the notion of strict G-commutativity is equivalent to one encoded by equivariant operads rather than ordinary ones [58, 35, 15]. This means that an E_∞-algebra A (in the sense of an ordinary E_∞ operad) may not have a strictly commutative model [65, 34], and this makes it more difficult to construct a symmetric monoidal model for the category of A-modules.

The framework of ∞-operads [53] (or, alternatively, that of dendroidal sets [71]) is one method to express coherent multiplicative structures. Here are some of the salient points.

- This generalization takes place in the theory of ∞-categories (specifically quasicategories), equivalent to the study of categories enriched in spaces. Every category enriched in spaces gives rise to an ∞-category; every ∞-category has morphism spaces between its objects.

- In this framework, for ∞-categories C and D there is a space $\mathrm{Fun}(\mathsf{C}, \mathsf{D})$ encoding the structure of functors and natural equivalences.

- In an ∞-category, homotopy limits and colimits are intrinsic notions rather than arising from a particular construction. Many common constructions produce presentable ∞-categories, which have all homotopy limits and colimits.

- Multicategories generalize to so-called ∞-*operads*. These have an underlying ∞-category, and there are spaces of multimorphisms to an object from a tuple of objects. Every topological multicategory gives rise to an ∞-operad; every ∞-operad can be realized by a topological multicategory. The precise definitions are similar in spirit to Segal's encoding of E_∞-spaces [85].

- An ∞-operad $\mathcal{O}$ has an associated notion of an $\mathcal{O}$-monoidal ∞-category. An $\mathcal{O}$-monoidal ∞-category is expressed in terms of maps $\mathsf{C} \to \mathcal{O}$ of ∞-operads with properties analogous to that from Definition 19.7.4, with the main difference that spaces of morphisms are respected. An $\mathcal{O}$-monoidal ∞-category is also equivalent to a functor from $\mathcal{O}$ to a category of categories: each object $\mathbf{x}$ of $\mathcal{O}$ has an associated category $\mathsf{C}_{\mathbf{x}}$, and one can associate a map
$$\mathrm{Mul}_{\mathcal{O}}(\mathbf{x}_1, \ldots, \mathbf{x}_d; \mathbf{y}) \to \mathrm{Fun}(\mathsf{C}_{\mathbf{x}_1}, \ldots, \mathsf{C}_{\mathbf{x}_d}; \mathsf{C}_{\mathbf{y}})$$
of spaces.

- We can discuss algebras and modules in terms of sections, just as in Definition 19.7.12.

All of this structure is systematically invariant under equivalence. Equivalent ∞-operads give rise to equivalent notions of an $\mathcal{O}$-algebra structure on C; ∞-categories equivalent to C have equivalent notions of $\mathcal{O}$-algebra structures to those on C; equivalent $\mathcal{O}$-monoidal ∞-categories have equivalent categories of $\mathcal{M}$-algebras for any map $\mathcal{M} \to \mathcal{O}$ of ∞-operads.

Example 19.7.24. An E_n-operad has an associated ∞-operad $\mathcal{O}$, and as a result we can define an E_n-monoidal ∞-category C to be an $\mathcal{O}$-monoidal ∞-category. When $n = 1$, 2, or ∞ we can recover monoidal, braided monoidal, and symmetric monoidal structures.

19.7.4 Modules

Mandell's theorem (19.3.7), which is about structure on the homotopy category of left modules over an E_n-algebra, is a reflection of higher structure on the category of left modules itself.

Theorem 19.7.25. [53, 5.1.2.6, 5.1.2.8] *Suppose that* C *is an* E_k*-monoidal* ∞*-category that has geometric realization of simplicial objects, and such that the tensor product preserves such geometric realizations in each variable separately. Then the category of left modules over an* E_k*-algebra* A *is* E_{k-1}*-monoidal, and has all colimits that exist in* C.

As previously discussed, the category of left modules over an associative algebra R is not made monoidal under the tensor product over R, but the category of bimodules is. The generalization of this result to E_n-algebras is the following.

Theorem 19.7.26. [53, 3.4.4.2] *Suppose* C *is an* E_n*-monoidal presentable* ∞*-category such that the monoidal structure preserves homotopy colimits in each variable separately. Then for any* E_n*-algebra* R *in* C*, there is a category* $\mathrm{Mod}_R^{E_n}(\mathsf{C})$ *of* E_n R*-modules. This is a presentable* E_n*-monoidal* ∞*-category whose underlying monoidal operation is the tensor product over* R.

In particular, if C *is a presentable* ∞*-category with a symmetric monoidal structure that preserves colimits in each variable, and* R *is an* E_n*-algebra in* C*, the category of* E_n R*-modules in* C *has an* E_n*-monoidal structure that preserves colimits in each variable.*

Roughly, an E_n R-module M has multiplication operations $R^{\otimes k} \otimes M \to M$ parametrized by $(k+1)$-tuples of points of configuration space, where one point is marked by M and the rest by R. This has the more precise description of E_n-modules as left modules.

Theorem 19.7.27. ([53, 5.5.4.16], [28]) *Suppose that* C *is a symmetric monoidal* ∞*-category and that the monoidal product preserves colimits in each variable separately. For an* E_n*-algebra* R *in* C*, the factorization homology* $\int_{D^n \setminus 0} R$ *has the structure of an* E_1*-algebra, and the category of* E_n R*-modules is equivalent to the category of left modules over* $\int_{D^n \setminus 0} R$.

Remark 19.7.28. In the category of spectra, this could be regarded as a consequence of the Schwede–Shipley theorem [83] or its generalizations. There is a free-forgetful adjunction between E_n R-modules and Sp, and the image $\mathrm{Free}_{E_n\text{-}R}(\mathbb{S})$ of the sphere spectrum under the left adjoint is a compact generator for the category of E_n R-modules. Therefore, E_n R-modules are equivalent to the category of modules over the endomorphism ring

$$F_{E_n\text{-}R}(\mathrm{Free}_{E_n\text{-}R}(\mathbb{S}), \mathrm{Free}_{E_n\text{-}R}(\mathbb{S})) \simeq \mathrm{Free}_{E_n\text{-}R}(\mathbb{S}).$$

This theorem, then, is an identification of the free E_n R-module.

Example 19.7.29. When $n = 1$, the category of E_1 R-modules is the category of left modules over $R \otimes R^{op}$. When $n = 2$, the category of E_2-R-modules is the category of left modules over the topological Hochschild homology $\mathrm{THH}(R)$.

19.7.5 Coherent powers

In the classical case, we described an $\mathcal{O}$-algebra structure on A in terms of action maps

$$\mathrm{Sym}^k_{\mathcal{O}}(A) = \mathcal{O}(k) \otimes_{\Sigma_k} A^{\otimes k} \to A$$

from extended power constructions to A, and gave a formula

$$\mathrm{Free}_{\mathcal{O}}(X) = \coprod_{k\geq 0} \mathrm{Sym}^k_{\mathcal{O}}(A)$$

for the free $\mathcal{O}$-algebra on an object in the case where the monoidal structure is compatible with enriched colimits; we also discussed the multi-object analogue in §19.7.2. The analogous constructions for ∞-operads are carried out in [53, §3.1.3], and we will sketch these results here.

Fix an ∞-operad $\mathcal{O}$. For any objects $\mathbf{x}_1, \ldots, \mathbf{x}_d, \mathbf{y}$ of $\mathcal{O}$, we can construct a space

$$\mathrm{Mul}_{\mathcal{O}}(\mathbf{x}_1, \ldots, \mathbf{x}_d; \mathbf{y})$$

of multimorphisms in $\mathcal{O}$; if the $\mathbf{x}_i$ are equal, this further can be given a natural action of the symmetric group.

Let C be an $\mathcal{O}$-monoidal ∞-category C. In particular, C encodes categories $\mathsf{C}_{\mathbf{x}}$ parametrized by the objects $\mathbf{x}$ of $\mathcal{O}$, and functors $f\colon \mathsf{C}_{\mathbf{x}_1} \times \cdots \times \mathsf{C}_{\mathbf{x}_d} \to \mathsf{C}_{\mathbf{y}}$ parametrized by the multimorphisms $f\colon (\mathbf{x}_1, \ldots, \mathbf{x}_d) \to \mathbf{y}$ of $\mathcal{O}$. Suppose that the categories $\mathsf{C}_{\mathbf{x}}$ have homotopy colimits and the functors preserve homotopy colimits in each variable. Then there exist *extended power functors*

$$\mathrm{Sym}^k_{\mathcal{O},\mathbf{x}\to\mathbf{y}}\colon \mathsf{C}_{\mathbf{x}} \to \mathsf{C}_{\mathbf{y}},$$

whose value on $X \in \mathsf{C}_{\mathbf{u}}$ is a homotopy colimit

$$\left(\operatorname*{hocolim}_{\alpha\in\mathrm{Mul}_{\mathcal{O}}(\mathbf{x},\ldots,\mathbf{x};\mathbf{y})} \alpha(X \oplus \cdots \oplus X)\right)_{h\Sigma_k}.$$

These extended powers have the property that an $\mathcal{O}$-algebra A has natural maps $\mathrm{Sym}^k_{\mathcal{O},\mathbf{x}\to\mathbf{y}}(A(\mathbf{x})) \to A(\mathbf{y})$. Moreover, there is a free-forgetful adjunction between $\mathcal{O}$-algebras and $\mathsf{C}_{\mathbf{x}}$, and the free object $\mathrm{Free}_{\mathcal{O},\mathbf{x}}(X)$ on $X \in \mathsf{C}_U$ has the property that its value on $\mathbf{y}$ is exhibited as the coproduct

$$\mathrm{ev}_{\mathbf{y}}(\mathrm{Free}_{\mathcal{O},\mathbf{x}}(X)) \simeq \coprod_{k\geq 0} \mathrm{Sym}^k_{\mathcal{O},\mathbf{x}\to\mathbf{y}}(X).$$

Remark 19.7.30. Composing with the diagonal $\mathsf{C}_{\mathbf{x}} \to \prod \mathsf{C}_{\mathbf{x}}$ gives a Σ_k-equivariant map

$$\mathrm{Mul}_{\mathcal{O}}(\underbrace{\mathbf{x}, \ldots, \mathbf{x}}_{k}; \mathbf{y}) \to \mathrm{Fun}(\mathsf{C}_{\mathbf{x}} \times \cdots \times \mathsf{C}_{\mathbf{x}}, \mathsf{C}_{\mathbf{y}}) \to \mathrm{Fun}(\mathsf{C}_{\mathbf{x}}, \mathsf{C}_{\mathbf{y}})$$

that factors through the homotopy orbit space

$$P(k) = \mathrm{Mul}_{\mathcal{O}}(\mathbf{x}, \ldots, \mathbf{x}; \mathbf{y})_{h\Sigma_k}.$$

This space $P(k)$ then serves as a parameter space for tensor-power functors $\mathsf{C}_{\mathbf{x}} \to \mathsf{C}_{\mathbf{y}}$.

In the case of an ordinary single-object ∞-operad $\mathcal{O}$ such as an E_n-operad, we can rephrase in terms of $P(k)$. Such an ∞-operad $\mathcal{O}$ is equivalent to an ordinary operad in spaces and an $\mathcal{O}$-monoidal ∞-category is equivalent to an ∞-category C with a map $\mathcal{O} \to \mathrm{End}(\mathsf{C})$.

We recover a formula

$$\mathrm{Free}_{\mathcal{O}}(X) \simeq \coprod_{k\geq 0} \operatorname*{hocolim}_{\alpha\in P(k)} \alpha(X,\ldots,X)$$

for the free algebra on X. When $X = S^m$, this is the Thom spectrum

$$\coprod_{k\geq 0} P(k)^{m\rho},$$

closely related to Remark 19.4.19.

When $\mathcal{O}$ is an E_n-operad, the space $P(k)$ is equivalent to the space $\mathcal{C}_n(k)/\Sigma_k$, a model for the space of unordered configurations of k points in $\mathbb{R}^n$. When $n = \infty$ the space $P(k)$ is a model for $B\Sigma_k$, and we find that the we recover the ordinary homotopy symmetric power:

$$\mathrm{Sym}^k_{E_\infty}(X) \simeq (X^{\otimes k})_{h\Sigma_k}.$$

Example 19.7.31. Fix a space B and consider the fiberwise category $\mathsf{S}_{/B}$. The homotopy fiber product $X \times^h_B Y$ gives this the structure of a symmetric monoidal ∞-category, breaking up independently over the components of B. If B is path-connected, then the extended power and free functors on $(X \to B)$ are those obtained by applying the extended power and free functors to the fiber.

Example 19.7.32. Given an E_n R-module M, the free E_n R-algebra on an E_n R-module M is

$$\coprod_{k\geq 0} \operatorname*{hocolim}_{\alpha\in C_n(k)/\Sigma_k} M^{\otimes_\alpha k},$$

where each point α of configuration space determines a functor $M^{\otimes_\alpha k} \simeq M \otimes_R \cdots \otimes_R M$.

More can be said under the identification between E_n-modules and modules over factorization homology. If M is the free E_n R-module on S^m, then we obtain an identification of the free E_n-algebra under R on S^m:

$$R \amalg^{E_n} \mathrm{Free}_{E_n}(S^m) \simeq \coprod_{k\geq 0} \left(\int_{\mathbb{R}^k\setminus\{p_1,\ldots,p_k\}} R \right) \otimes_{\Sigma_k} S^{m\rho_k}.$$

Remark 19.7.33. The interaction between connective objects and their Postnikov truncations from §19.3.3 generalizes to the case where we have an $\mathcal{O}$-monoidal ∞-category C with a *compatible t-structure* in the sense of [53, 2.2.1.3]. This means that the categories $\mathsf{C}_{\mathbf{x}}$ indexed by the objects $\mathbf{x}$ of $\mathcal{O}$ all have t-structures, and the functors induced by the morphisms in $\mathcal{O}$ are all additive with respect to connectivity. Then [53, 2.2.1.8] implies that connective $\mathcal{O}$-algebras have Postnikov towers: the collection of truncation functors $\tau_{\leq n}$ is compatible with the $\mathcal{O}$-monoidal structure on $\mathsf{C}_{\geq 0}$.

19.8 Further invariants

19.8.1 Units and Picard spaces

Definition 19.8.1. For an E_n-monoidal ∞-category C with unit $\mathbb{I}$, the *Picard space* $\mathrm{Pic}(\mathsf{C})$ is the full subgroupoid of C spanned by the *invertible objects*: objects X for which there exists an object Y such that $Y \otimes X \simeq X \otimes Y \simeq \mathbb{I}$.

Remark 19.8.2. The classical Picard group of the homotopy category $h\mathsf{C}$ is the set $\pi_0 \mathrm{Pic}(\mathsf{C})$ of path components.

In particular, $\mathrm{Pic}(\mathsf{C})$ is closed under the E_n-monoidal structure on C, giving it a canonical E_n-space structure. Moreover, by construction $\pi_0 \mathrm{Pic}(\mathsf{C}) = (\pi_0 \mathsf{C})^\times$ is a group, and so $\mathrm{Pic}(\mathsf{C})$ is an n-fold loop space. The loop space $\Omega \mathrm{Pic}(\mathsf{C})$ is the space of homotopy self-equivalences of the unit $\mathbb{I}$; in the case of the category LMod_R of left modules, it is homotopy equivalent to the unit group $GL_1(R)$ of R.

Proposition 19.8.3. [2, §7] *If R is an E_n ring spectrum, then the space $GL_1(R)$ of homotopy self-equivalences of the left module R has an n-fold delooping. If $n \geq 2$, the space* $\mathrm{Pic}(R) = \mathrm{Pic}(\mathsf{LMod}_R)$ *has an $(n-1)$-fold delooping.*

19.8.2 Topological André–Quillen cohomology

Topological André-Quillen homology and cohomology are invariants of ring spectra developed by Kriz and Basterra [44, 7]. For a fixed map of E_∞ ring spectra $A \to B$, we can define a topological André–Quillen homology object $\mathrm{TAQ}(A \to R \to B)$ for any object R in the category of E_∞ rings between A and B. This is characterized by the following properties [8]:

1. It naturally takes values in the category of B-modules.
2. It takes homotopy colimits of E_∞ ring spectra between A and B to homotopy colimits of B-modules.
3. There is a natural map $B \otimes_A (R/A) \to \mathrm{TAQ}(A \to R \to B)$.
4. For a left A-module X with a map $X \to B$, the composite natural map

$$B \otimes_A X \to B \otimes_A \mathrm{Free}^A_{E_\infty}(X) \to \mathrm{TAQ}(A \to \mathrm{Free}^A_{E_\infty}(X) \to B)$$

   of B-modules is an equivalence.
5. Under the above equivalence, the natural map

$$\mathrm{TAQ}(A \to \mathrm{Free}^A_{E_\infty} \mathrm{Free}^A_{E_\infty}(X) \to B) \to \mathrm{TAQ}(A \to \mathrm{Free}^A_{E_\infty}(X) \to B)$$

   is equivalent to the map

$$B \otimes_A \mathrm{Free}^A_{E_\infty}(X) \to B \otimes_A X$$

   that collapses $B \otimes_A (\amalg \mathrm{Sym}^k(X))$ to the factor with $k = 1$.

Topological André-Quillen homology measures how difficult it is to build R as an A-algebra: any description of R as an iterated pushout along maps of free of E_∞-algebras, starting from A, determines a description of the topological André–Quillen cohomology of R as an iterated pushout of B-modules. Basterra showed that TAQ-cohomology groups

$$\mathrm{TAQ}^n(R;M) = [\mathrm{TAQ}(\mathbb{S} \to R \to R), \Sigma^n M]_{\mathrm{Mod}_R}$$

plays the role for Postnikov towers of E_∞ ring spectra that ordinary cohomology does for spectra.

From this point of view, TAQ also has natural generalizations to $\mathrm{TAQ}^{\mathcal{O}}$ for algebras over an arbitrary operad [8, 33], although there may be a choice of target category that takes more work to describe. In particular, for E_n-algebras these are related to an iterated bar construction [9].

Topological André–Quillen homology also enjoys the following properties, proved in [7, 8].

Base-change: For a map $B \to C$, we have a natural equivalence

$$C \otimes_B \mathrm{TAQ}(A \to R \to B) \simeq \mathrm{TAQ}(A \to R \to C).$$

In particular, if we define $\Omega_{R/A} = \mathrm{TAQ}(A \to R \to R)$, then

$$\mathrm{TAQ}(A \to R \to B) = B \otimes_R \Omega_{R/A}.$$

Transitivity: For a composite $A \to R \to S \to B$, there is a natural cofiber sequence

$$\mathrm{TAQ}(A \to R \to B) \to \mathrm{TAQ}(A \to S \to B) \to \mathrm{TAQ}(R \to S \to B).$$

In particular, for $A \to R \to S$ we have cofiber sequences

$$S \otimes_R \Omega_{R/A} \to \Omega_{S/A} \to \Omega_{S/R}.$$

Representability: Suppose that there is a functor h^* from the category of pairs $(R \to S)$ of E_∞ ring spectra between A and B to the category of graded abelian groups. Suppose that this is a cohomology theory on the category of E_∞ ring spectra between A and B: it satisfies homotopy invariance, has a long exact sequence, satisfies excision for homotopy pushouts of pairs, and takes coproducts to products. Then there is a B-module M with a natural isomorphism

$$\begin{aligned} h^n(S,R) &\cong \mathrm{TAQ}^n(S,R;M) \\ &= [\mathrm{TAQ}(R \to S \to B), \Sigma^n M]_{\mathrm{Mod}_B} \end{aligned}$$

of abelian groups.

For any E_∞ ring spectrum B, algebras mapping to B have TAQ-homology TAQ($\mathbb{S} \to R \to B$), valued in the category of B-modules. The square-zero algebras

$$B \oplus M$$

are representing objects for TAQ-cohomology $\mathrm{TAQ}^*(R; M)$.

Representability allows us to construct and classify operations in TAQ-cohomology by B-algebra maps between such square-zero extensions.

Proposition 19.8.4. *Any element in* $[\Sigma \operatorname{Sym}^2 M, N]_{\mathrm{Mod}_B}$ *has a naturally associated map* $B \oplus M \to B \oplus N$ *of augmented commutative* B*-algebras and hence gives rise to a natural* TAQ*-cohomology operation* $\mathrm{TAQ}(-; M) \to \mathrm{TAQ}(-; N)$ *for commutative algebras mapping to* B*.*

Proof. By viewing B as concentrated in grading 0 and M as concentrated in grading 1, we can give a $\mathbb{Z}$-graded construction (as in Example 19.3.24) of $B \oplus M$ as an iterated sequence of pushouts along maps of free algebras. The first such pushout is

$$\mathrm{Free}^B_{E_\infty}(M) \leftarrow \mathrm{Free}^B_{E_\infty}(\operatorname{Sym}^2 M) \to B.$$

Further pushouts only alter gradings 3 and higher.

We now view $B \oplus N$ as graded by putting N in grading 2. We find that homotopy classes of maps of graded algebras $B \oplus M \to B \oplus N$ are equivalent to maps $\Sigma \operatorname{Sym}^2 M \to N$. □

Example 19.8.5. Letting $M = B \otimes S^m$, we have

$$\Sigma \operatorname{Sym}^2(M) \simeq B \otimes \Sigma^{m+1} \mathbb{RP}^\infty_m.$$

Therefore, we get a map from the B-cohomology $B^n(\Sigma^{m+1}\mathbb{RP}^\infty_m)$ of stunted projective spaces to the group of natural cohomology operations $TAQ^m(-; B) \to TAQ^n(-; B)$.

Remark 19.8.6. The fact that elements in the B-homology of stunted projective spaces produce homotopy operations while elements in their B-cohomology produce TAQ-cohomology operations with a shift is a reflection of Koszul duality.

Example 19.8.7. Letting $M = (B \otimes S^q) \oplus (B \otimes S^r)$, and using the projection

$$\begin{aligned}\Sigma \operatorname{Sym}^2(B \otimes (S^q \oplus S^r)) &\simeq \Sigma \operatorname{Sym}^2(B \otimes S^q) \oplus \Sigma \operatorname{Sym}^2(B \otimes S^r) \oplus \Sigma(B \otimes S^q \otimes S^r)\\ &\to B \otimes S^{q+1+r},\end{aligned}$$

we get a binary operation

$$[-,-]\colon\ \mathrm{TAQ}^q(-; B) \times \mathrm{TAQ}^r(-; B) \to \mathrm{TAQ}^{q+1+r}(-; B)$$

that (up to a normalization factor) we call the TAQ-*bracket*.

Example 19.8.8. If $B = H\mathbb{F}_2$, then there are TAQ-cohomology operations

$$R^a\colon\ \mathrm{TAQ}^m(-; H\mathbb{F}_2) \to \mathrm{TAQ}^{m+a}(-; H\mathbb{F}_2)$$

for $a \geq m+1$, and a bracket

$$\mathrm{TAQ}^q(-;H\mathbb{F}_2) \times \mathrm{TAQ}^r(-;H\mathbb{F}_2) \to \mathrm{TAQ}^{q+1+r}(-;H\mathbb{F}_2).$$

In this form, the operation R^{a+1} is Koszul dual to Q^a, in the sense that nontrivial values of R^{a+1} in TAQ-cohomology detect relations on the operator Q^a in homology. Similarly, the bracket in TAQ is Koszul dual to the multiplication.

The operations were constructed by Basterra–Mandell [10]. In further unpublished work, they showed that these operations (and their odd-primary analogues) generate all the natural operations on TAQ-cohomology with values in $H\mathbb{F}_p$ and determined the relations between them. In particular, the operations R^a above satisfy the same Adem relations that the Steenrod operations Sq^a do; the TAQ-bracket has the structure of a shifted restricted Lie bracket, whose restriction is the bottommost defined operation R^a.

Basterra–Mandell's proof uses a variant of the Miller spectral sequence from [68]. We will close out this section with a sketch of how such spectral sequences are constructed, parallel to the delooping spectral sequence from Remark 19.5.14.

Proposition 19.8.9. *Suppose that R is an E_∞ ring spectrum with a chosen map $R \to H\mathbb{F}_p$. Then there is a Miller spectral sequence*

$$AQ^{DL}_*(\pi_*(H\mathbb{F}_p \otimes R)) \Rightarrow \mathrm{TAQ}_*(\mathbb{S} \to R \to H\mathbb{F}_p),$$

where the left-hand side are the nonabelian derived functors of an indecomposables functor Q that sends an augmented graded-commutative $\mathbb{F}_p$-algebra with Dyer–Lashof operations to the quotient of the augmentation ideal by all products and Dyer–Lashof operations.

Proof. We construct an augmented simplicial object:

$$\cdots \mathrm{Free}_{E_\infty} \mathrm{Free}_{E_\infty} \mathrm{Free}_{E_\infty} R \Rrightarrow \mathrm{Free}_{E_\infty} \mathrm{Free}_{E_\infty} R \rightrightarrows \mathrm{Free}_{E_\infty} R \to R.$$

If U is the forgetful functor, from commutative ring spectra mapping to $H\mathbb{F}_p$ to spectra mapping to $H\mathbb{F}_p$, this is the bar construction $B(\mathrm{Free}_{E_\infty}, U\,\mathrm{Free}_{E_\infty}, UR)$. The underlying simplicial spectrum $B(U\,\mathrm{Free}_{E_\infty}, U\,\mathrm{Free}_{E_\infty}, UR)$ has an extra degeneracy, so its geometric realization is equivalent to R. Moreover, the forgetful functor from E_∞ rings to spectra preserves sifted homotopy colimits, and hence geometric realization because the simplicial indexing category is sifted. Therefore, applying the homotopy colimit preserving functor $\mathrm{TAQ} = \mathrm{TAQ}(\mathbb{S} \to (-) \to H\mathbb{F}_p)$ and the natural equivalence $\mathrm{TAQ} \circ \mathrm{Free}_{E_\infty}(R) \simeq H\mathbb{F}_p \otimes R$, we get an equivalence

$$|B(H\mathbb{F}_p \otimes (-), U\,\mathrm{Free}_{E_\infty}, UR)| \simeq \mathrm{TAQ}(R).$$

However, this bar construction is a simplicial object of the form

$$\cdots H\mathbb{F}_p \otimes \mathrm{Free}_{E_\infty} \mathrm{Free}_{E_\infty} R \Rrightarrow H\mathbb{F}_p \otimes \mathrm{Free}_{E_\infty} R \rightrightarrows H\mathbb{F}_p \otimes R.$$

Taking homotopy groups, we get a simplicial object

$$\mathbb{Q}_{E_\infty}\mathbb{Q}_{E_\infty}H_*R \Rrightarrow \mathbb{Q}_{E_\infty}H_*R \rightrightarrows H_*R.$$

Moreover, the structure maps make this the bar construction

$$B(Q, \mathbb{Q}_{E_\infty}, H_*R)$$

that computes derived functors of Q on graded-commutative algebras with Dyer–Lashof operations. Therefore, the spectral sequence associated to the geometric realization computes $\mathrm{TAQ}_*(\mathbb{S} \to R \to H\mathbb{F}_p)$ and has the desired E_2-term. □

Remark 19.8.10. We can also apply cohomology rather than homology and get a spectral sequence computing topological André–Quillen cohomology.

This leaves open a hard algebraic part of Basterra–Mandell's work: actually calculating these derived functors, and in particular finding relations amongst the operations R^a and the bracket $[-,-]$ that give a complete description of TAQ-cohomology operations.

19.9 Further questions

We will close this paper with some problems that we think are useful directions for future investigation.

Problem 19.9.1. Develop useful obstruction theories which can determine the existence of or maps between E_n-algebras in a wide variety of contexts.

The obstruction theory due to Goerss–Hopkins [31] is the prototype for these results. In unpublished work [87], Senger has given a development of this theory for E_∞-algebras where the obstructions occur in nonabelian Ext-groups calculated in the category of graded-commutative rings with Dyer–Lashof operations and Steenrod operations satisfying the Nishida relations, and provided tools for calculating with them. This played a critical role in [47, 46].

In closely related situations, the tools available remain rudimentary. For example, there is essentially no workable obstruction theory for the construction of commutative rings of any type in equivariant stable theory. Tools arising from the Steenrod algebra have been essential in most of the deep results in homotopy theory, such as the Segal conjecture [50] and the Sullivan conjecture [69]. Without the analogues, there is a limit to how much structure can be revealed.

Problem 19.9.2. Give a modern redevelopment of homology operations for E_∞ ring spaces and E_n ring spaces.

The observant reader may have noticed that, despite the rich structure present, the principal material that we have referenced for E_∞ ring spaces is several decades old. Several major advances have happened in multiplicative stable homotopy theory since then, and the author feels that there is still a great deal to be mined. Having this material accessible to modern toolkits would be extremely useful.

For one example, the theory of E_∞ ring spaces from the point of view of symmetric spectra has been studied in detail by Sagave and Schlichtkrull [81, 79, 80]. For another, the previous emphasis on E_∞ ring spaces should be tempered by the variety of examples that we now know only admit A_∞ or E_n ring structures.

Problem 19.9.3. Give a unified theory of graded Hopf algebras and Hopf rings, capable of encoding some combination of non-integer gradings, power operations, group-completion theorems, and the interaction with the unit.

Ravenel–Wilson's theory of Hopf rings is integer-graded. We now know many examples—motivic homotopy theory, equivariant homotopy theory, $K(n)$-local theory, modules over E_n ring spectra—that may have natural gradings of a much wider variety than this, such as a Picard group. Moreover, multiplicative theory should involve much more structure: we should have a sequence of spaces graded not just by a Picard group, but by the Picard space that also encodes structure nontrivial higher interaction between gradings and the unit group.

Problem 19.9.4. Give a precise general description of the Koszul duality relationship between homotopy operations and TAQ-cohomology operations. Give a complete construction of the algebra of operations on TAQ-cohomology for E_n-algebras with coefficients in Hk, for k a commutative ring. Give complete descriptions of the TAQ-cohomology for a large library of Eilenberg–Mac Lane spectra Hk and Morava's forms of K-theory.

Because TAQ-cohomology governs the construction of ring spectra via their Postnikov tower, essentially any information that we can provide about these objects is extremely useful.

Problem 19.9.5. Determine an algebro-geometric expression for power operations and their relationship to the Steenrod operations.[1] Do the same for the operations that appear in the Hopf ring associated to an E_∞ ring space.

At the prime 2, it has been known for some time that the action of the Steenrod algebra can be concisely packaged as a coaction of the dual Steenrod algebra, a Hopf algebra corresponding to the group scheme of automorphisms of the additive formal group over $\mathbb{F}_2$. The Dyer–Lashof operations on infinite loop spaces generate an algebra analogous to the Steenrod algebra, and its dual was described by Madsen [54]; the result is closely related to Dickson invariants. However, the full action of the Dyer–Lashof operations or the interaction between the Dyer–Lashof algebra and the Steenrod algebra do not yet have a geometric packaging.

Conjecture 19.9.6. *For Lubin–Tate cohomology theories E and F of height n, there is a natural algebraic structure parametrizing operations from continuous E-homology to continuous F-homology for certain E_∞ ring spectra, expressed in terms of the algebraic geometry of* isogenies *of formal groups.*

[1] See the chapter by Nathaniel Stapleton in this Handbook for related material.

This is complete: there is an obstruction theory for the construction of and mapping between $K(n)$-local E_∞ ring spectra whose algebraic input is completed E-homology equipped with these operations.

In this paper we have not really touched on the extensive study of power operations in chromatic homotopy theory, but see the chapters in this Handbook by Nathaniel Stapleton and by Tobias Barthel and Agnès Beaudry. Given Lubin–Tate cohomology theories E and F associated to formal groups of height n at the prime p, we have both cohomology operations and power operations. In [37] the algebra of cohomology operations is expressed in terms of isomorphisms of formal groups. Extensive work of Ando, Strickland, and Rezk has shown that power operations are expressed in terms of quotient operations for subgroups of the formal groups. It has been known for multiple decades [91, §28] that the natural home combining these two types of operations is the theory of *isogenies* of formal groups. However, there are important details about formal topologies which have never been resolved.[2]

Problem 19.9.7. Determine the natural instability relations for operations in *unstable* elliptic cohomology and in unstable Lubin–Tate cohomology.

Strickland states that isogenies are a natural interpretation for unstable cohomology operations in E-theory. However, isogenies encode the analogue of the cohomological Steenrod operations, the multiplicative Dyer–Lashof operations, and the Nishida relations between them. They do *not* encode any analogue of the instability relation $Sq^n = Q^{-n}$ that we see in the cohomology of spaces.

In chromatic theory, our only accessible example so far is K-theory. For p-completed K-theory, the cohomology operations are generated by the Adams operations ψ^k for $k \in \mathbb{Z}_p^\times$. For torsion-free algebras, the power operations are controlled by the operation ψ^p and its congruences [36, 77]. The *unstable* operations in the K-theory of spaces, by contrast, arise from the algebra of symmetric polynomials and are essentially governed by the ψ^n for $n \in \mathbb{N}$; the fact that the other ψ^k are determined by these enforces some form of continuity. This is also closely tied to the question of whether there are geometric interpretations of some type for elliptic cohomology theories or Lubin–Tate cohomology theories.[3]

Problem 19.9.8. Determine a useful way to encode secondary operation structures on E_∞ or E_n rings.

In the case of secondary Steenrod operations, there is a useful formulation due to Baues of an extension of the Steenrod algebra that can be used to encode all of the secondary operation structure [11, 72]. No such systematic descriptions are known for secondary Dyer–Lashof operations, especially since the Dyer–Lashof operations are expressed in a more complicated way than the action of an algebra on a module.

Problem 19.9.9. Determine useful relationships between the homotopy types of an E_n ring spectrum, the unit group GL_1 and the Picard space $\mathrm{Pic}(R)$, and the spaces $BGL_n(R)$.

[2] The reader should be advised that, even at height 1, there are difficult issues with E-theory here involving left-derived functors of completion.

[3] One possible viewpoint is that we could interpret $\mathbb{N}$ as the monoid of endomorphisms of the *multiplicative monoid* M_1, which contains the unit group GL_1.

This is closely tied to orientation theory, algebraic K-theory, and the study of spaces involved in surgery theory.

Investigations in these directions due to Mathew–Stojanoska revealed that there is a nontrivial relationship between the k-invariants for R and the unit spectrum $gl_1(R)$ at the edge of the stable range at the prime 2 [59], and forthcoming work of Hess has shown that this relation can be recovered from the mixed Cartan formula. The odd-primary analogues of this are not yet known.

Problem 19.9.10. Find an odd-primary formula for the mixed Adem relations similar to the Kuhn–Tsuchiya formula.

There is a description of the mixed Adem relations [23], valid at any prime, but it is difficult to apply in concrete examples. The 2-primary formula described in §19.5.8 is much more direct; it was originally stated by Tsuchiya and proven by Kuhn [45]. There is no known odd-primary analogue of this formula.

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