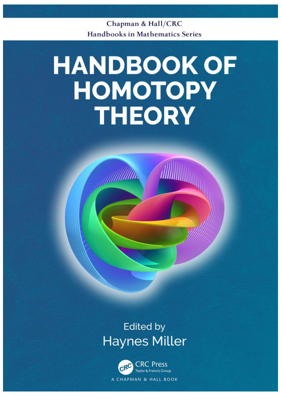

Handbook of Homotopy Theory

Haynes Miller

A short course on -categories

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14

A short course on ∞-categories

Moritz Groth

14.1 Introduction

The aim of this short course is to give a non-technical account of some ideas in the theory of ∞-categories (a.k.a. quasi-categories, inner Kan complexes, weak Kan complexes, Boardman complexes, or quategories), as originally introduced by Boardman–Vogt [23, p. 102] in their study of homotopy-invariant algebraic structures. Recently, ∞-categories have been studied intensively by Joyal [67, 68, 69], Lurie [82, 79, 80, 81], and others (this includes the Riehl–Verity program which was started in [97]). ∞-categories have applications in many areas of pure mathematics (and some of them are taken up in various chapters of this Handbook). In this chapter we are more modest. We simply try to emphasize the philosophy and some of the main ideas of ∞-category theory, and we sketch the lines along which the theory is developed. *In particular, this means that there is no claim of originality.*

Category theory is an important mathematical discipline in that it provides us with a convenient language which applies whenever we put into practice the following slogan: 'In order to study a collection of objects one should also consider suitably defined morphisms between such objects.' Many classes of mathematical objects like groups, modules over a ring, manifolds, or schemes can be organized into a category and from typical constructions one frequently abstracts the categorical character behind them. Let us recall that a category consists of objects, morphisms and a composition law which is suitably associative and unital. This allows us, in particular, to speak about *isomorphisms*, and all functorial constructions trivially preserve isomorphisms.

Category theory is a very powerful and useful language. However, it also has its limitations. To put it as a slogan, in many areas of pure mathematics we would like to identify two objects which are, while possibly not isomorphic in the purely categorical sense, 'equivalent' from a more homotopy theoretic perspective. To illustrate this, the desire of identifying different resolutions of objects in abelian categories leads us to study chain complexes up to quasi-isomorphisms. Similarly, in homotopy theory we would like to think of weak homotopy equivalences between topological spaces as actual isomorphisms. And even in category theory, often we do not have to distinguish two categories as long as they are equivalent.

These are only three examples for the fairly common situation in which we start with a pair $(\mathcal{C}, W)$ consisting of a category $\mathcal{C}$ and a class W of so-called *weak equivalences*, a class of morphisms that we would like to treat as isomorphisms. In such situations, functo-

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rial constructions are only 'meaningful' if they preserve weak equivalences. The search for convenient languages to study such situations has already quite some history and various different approaches have been considered. This includes triangulated categories [90], model categories [60], relative categories [9], derivators [48], simplicial categories [14], topological categories [64], and ∞-categories [82, 83].

The three last named approaches belong to a fairly large zoo of different models all of which realize 'a theory of $(\infty,1)$-categories'. While an $(\infty,1)$-category is not a well-defined mathematical notion, there is the general agreement that such a category-like concept should enjoy the following features.

(i) As part of the structure there is a class of objects.

(ii) There should be morphisms between objects, 2-morphisms between morphisms (like chain homotopies, homotopies, and natural isomorphisms), as well as 3-morphisms, 4-morphisms, and so forth, explaining the parameter '∞' in '$(\infty,1)$-categories'.

(iii) Morphisms can be composed in a suitably associative and unital way.

(iv) Higher morphisms, i.e., 2-morphisms, 3-morphisms, and so on, are supposed to be invertible in a certain sense. (All morphisms above dimension one are invertible, explaining the parameter '1' in '$(\infty,1)$-categories'.)

References for survey articles on the zoo of different axiomatizations of $(\infty,1)$-categories include [16] and [1].

The aim of these notes instead is to focus essentially on one of those different models and to describe how a good deal of classical category theory can be extended to this particular model. In these notes we follow Lurie [82, 83] in his choice of terminology and refer to these particular models for $(\infty,1)$-categories as *∞-categories*. We refrain from giving a more detailed introduction here and instead refer the reader to the titles as well as to the short introductions of the individual sections. To conclude this introduction there are the following few remarks.

(i) In these notes we ignore essentially all set-theoretic issues (with the exception of the discussion of locally presentable categories where some care is needed).

(ii) For many of the mathematical concepts to be introduced below, there are at least two different terminologies (most frequently, one due to Joyal and one due to Lurie). Since we do not want to cause further confusion, we have to stick to one of these possible choices. As the expanded version of Lurie's thesis [82, 83] is our main reference, we stick to Lurie's terminology.

(iii) In these notes we use the environment 'Perspective' in order to include some sketches of 'the larger picture' and also to give more references to the literature.

(iv) Finally, the main prerequisites for these notes are some acquaintance with key concepts from category theory [85] as well as basics concerning simplicial sets. References for simplicial sets include the monographs [43, 46] and the more introductory account [41].

14.2 Two models for $(\infty, 1)$-categories

In this section we define ∞-*categories* as simplicial sets satisfying certain horn extension properties. These extension properties are a common generalization of extension properties enjoyed by singular complexes of topological spaces and nerves of ordinary categories. We indicate why this notion gives us a model for the theory of $(\infty, 1)$-categories. While ∞-categories have mapping spaces which can be endowed with a coherently associative and unital composition law, there is also the more rigid approach to $(\infty, 1)$-categories based on *simplicial categories* coming with strictly associative and unital composition laws. These two approaches are respectively organized by means of model structures, the *Joyal model structure* in the case of ∞-categories and the *Bergner model structure* in the case of simplicial categories. The coherent nerve construction of Cordier can be shown to be part of a Quillen equivalence between these two approaches.

14.2.1 Basics on ∞-categories

Before we give the central definition of an ∞-category, we consider two classes of examples, which one definitely wants to be covered by the definition. The actual definition of an ∞-category will then be a common generalization of these two classes of examples. The first class of examples comes from spaces.

Example 14.2.1. Given a topological space X, recall that associated to X there is the *fundamental groupoid* $\pi_1(X)$[1] of X. The objects of $\pi_1(X)$ are the points of X, and morphisms from x to y are homotopy classes of paths from x to y relative to the boundary points. Note that this is a groupoid, i.e., that all morphisms are invertible, since every path admits an inverse up to homotopy. This fundamental groupoid only depends on the 1-type of X, and hence discards a lot of information. A refined version is given by the *fundamental* ∞-*groupoid* $\pi_{<\infty}(X)$ which is roughly constructed as follows: objects are given by points of X, morphisms are paths in X, 2-morphisms are homotopies between paths, and higher morphisms are given by higher homotopies.

Before giving a precise definition of the fundamental groupoid, we include some heuristic comments. Note that $\pi_{<\infty}(X)$ seems to be an ∞-groupoid in that all morphisms are equivalences, i.e., invertible in a certain weak sense — this justifies that we refer to it as fundamental ∞-*groupoid* as opposed to as fundamental ∞-*category*. The following is a generally accepted principle of higher category theory.

'*Spaces and ∞-groupoids should be the same.*'

This principle is referred to as the *Grothendieck homotopy hypothesis*. Instead of working with topological spaces, one could also consider 'simplicial models for spaces', more specifically *Kan complexes* (see Definition 14.2.5). If one formalizes ∞-categories in the framework

[1] We deviate from the common notation $\Pi(X)$ since $\pi_1(X)$ matches better with the notation in Example 14.2.9.

of simplicial sets (by way of Definition 14.2.6), then the above principle tells us that ∞-groupoids should be the same as Kan complexes. Corollary 14.2.18 turns this principle into an actual mathematical statement. Using this approach to ∞-categories, a model for the fundamental ∞-groupoid of a space X is the usual singular complex $\mathrm{Sing}(X)$, which is well-known to be a Kan complex.

Let us establish some basic notation related to simplicial sets. Recall that the category $s\mathcal{S}et$ of simplicial sets is defined as the category of functors $\Delta^{\mathrm{op}} \to \mathcal{S}et$ where Δ is the category of finite ordinals

$$[n] = (0 < \cdots < n), \quad n \geq 0,$$

and order-preserving maps. Prominent maps in Δ are the coface maps d^k and codegeneracy maps s^k. The map $d^k\colon [n-1] \to [n]$, $0 \leq k \leq n$, is the unique injective map that does not have k in its image, while $s^k\colon [n+1] \to [n], 0 \leq k \leq n$, is the unique surjective map that hits k twice. We follow the usual convention and write $X_n = X([n])$ for the values of a simplicial set X, $d_k = X(d^k)$ for the face maps, and $s_k = X(s^k)$ for the degeneracy maps.

Obviously, ordinary categories should be examples of ∞-categories: we imagine all higher morphisms to be identities. To make this precise, we recall the *nerve construction* which associates a simplicial set to a category.

Example 14.2.2. Given a category $\mathcal{C}$, one can form the simplicial set $N(\mathcal{C}) \in s\mathcal{S}et$, called the *nerve* of $\mathcal{C}$. By definition, we have $N(\mathcal{C})_n = \mathrm{Fun}([n], \mathcal{C})$, where $[n]$ also denotes the ordinal number $0 < \cdots < n$ considered as a category and $\mathrm{Fun}(-,-)$ is the set of functors. Thus $N(\mathcal{C})_n$ is essentially the set of strings of n composable morphisms in $\mathcal{C}$. As a special case we have $N([n]) \cong \Delta^n, n \geq 0$, where $\Delta^n \in s\mathcal{S}et$ denotes the usual simplicial n-simplex, i.e., $\Delta^n = \hom_\Delta(-, [n])$ is the simplicial set represented by $[n] \in \Delta$.

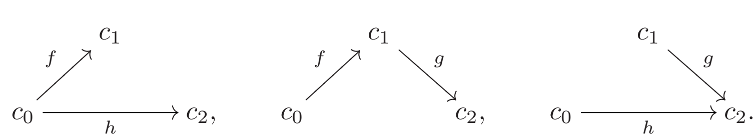
Lemma 14.2.3. *The nerve functor $N\colon \mathcal{C}at \to s\mathcal{S}et$ is fully faithful and hence induces an equivalence onto its essential image.*

In order to describe the essential image one observes that nerves of categories enjoy certain *horn extension properties*. Recall that the k-th n-*horn* $\Lambda^n_k \subseteq \partial\Delta^n$ for $n \geq 1, 0 \leq k \leq n$, is obtained from $\partial\Delta^n$ by removing the k-th face $\partial_k\Delta^n$, i.e., the face opposite to vertex k. More formally, the horn Λ^n_k is defined as the following coequalizer

$$\coprod_{0\leq i<j\leq n} \Delta^{n-2} \rightrightarrows \coprod_{i\neq k} \Delta^{n-1} \to \Lambda^n_k.$$

(See for example [46, p.9] which is a great reference for many more advanced aspects of simplicial homotopy theory.)

In dimension $n = 2$, horns $\lambda\colon \Lambda^2_k \to N(\mathcal{C})$ for $0 \leq k \leq 2$ hence respectively look like



Using the composition $h = g \circ f$ we see that one can uniquely extend any horn $\lambda\colon \Lambda^2_1 \to N(\mathcal{C})$ to an entire 2-simplex $\sigma\colon \Delta^2 \to N(\mathcal{C})$, i.e., there is a unique dashed arrow making the diagram

$$\begin{array}{ccc} \Lambda^2_1 & \longrightarrow & N(\mathcal{C}) \\ \downarrow & \nearrow_{\exists ! \sigma} & \\ \Delta^2 & & \end{array}$$

commute. The composition is given by the new face $d_1(\sigma)\colon \Delta^1 \to N(\mathcal{C})$. If instead we consider a horn $\lambda\colon \Lambda^2_0 \to N(\mathcal{C})$ in the special case that $h = \mathrm{id}$ is an identity morphism, then the existence of an extension to a 2-simplex is equivalent to the existence of a left inverse to f. Similar observations can be made for horns $\lambda\colon \Lambda^2_2 \to N(\mathcal{C})$. This different behavior extends to higher dimensions and motivates the following terminology: the horns $\Lambda^n_k, 0 < k < n$, are *inner horns* while the extremal cases Λ^n_0 and Λ^n_n are *outer horns*. It turns out that these horn extension properties are suitable to describe the essential image of the nerve functor. We denote by $\mathcal{G}rpd$ the category of groupoids.

Proposition 14.2.4. *Let X be a simplicial set.*

(i) *X is isomorphic to the nerve of a category if and only if* every inner horn $\Lambda^n_k \to X, 0 < k < n$, *can be* uniquely *extended to an n-simplex $\Delta^n \to X$.*

(ii) *X is isomorphic to the nerve of a groupoid if and only if* every horn $\Lambda^n_k \to X, 0 \le k \le n$, *can be* uniquely *extended to an n-simplex $\Delta^n \to X$.*

The characterization of the essential image of the nerve functor $N\colon \mathcal{G}rpd \to s\mathcal{S}et$ reminds us of the notion of a Kan complex whose definition will be recalled here for convenience.

Definition 14.2.5. A simplicial set X is a *Kan complex* if every horn $\Lambda^n_k \to X$ for $0 \le k \le n$ can be extended to an n-simplex $\Delta^n \to X$.

Denoting by $\mathcal{K}an \subseteq s\mathcal{S}et$ the full subcategory spanned by the Kan complexes, we thus have the following commutative diagram of fully faithful functors

$$\begin{array}{ccc} \mathcal{G}rpd & \longrightarrow & \mathcal{C}at \\ {\scriptstyle N}\downarrow & & \downarrow{\scriptstyle N} \\ \mathcal{K}an & \longrightarrow & s\mathcal{S}et. \end{array} \tag{14.2.1}$$

As a common generalization of Kan complexes and nerves of small categories there is the following definition.

Definition 14.2.6. A simplicial set $\mathcal{C}$ is an ∞*-category* if every inner horn $\Lambda^n_k \to \mathcal{C}$, $0 < k < n$, can be extended to an n-simplex $\Delta^n \to \mathcal{C}$.

Thus, by the above, spaces and ordinary categories give rise to ∞-categories. We will soon see that there are interesting ∞-categories that are not of this form and hence provide some first 'honest examples'. In particular, every simplicial model category has an underlying ∞-category but there are also other examples (see Corollary 14.2.30, Examples 14.2.31, and Perspective 14.3.7).

We begin by introducing some basic terminology. Given an ∞-category $\mathcal{C}$, the *objects* are the vertices $x \in \mathcal{C}_0$ and the *morphisms* are the 1-simplices $f \in \mathcal{C}_1$. The face map $s = d_1\colon \mathcal{C}_1 \to \mathcal{C}_0$ is the *source map*, and $t = d_0\colon \mathcal{C}_1 \to \mathcal{C}_0$ is the *target map*. As in ordinary category theory, we write $f\colon x \to y$ if $s(f) = x$ and $t(f) = y$. Slightly more formally, we define the set of morphisms $\hom_{\mathcal{C}}(x, y)$ from x to y as the pullback

$$\begin{array}{ccc} \hom_{\mathcal{C}}(x,y) & \longrightarrow & \mathcal{C}_1 \\ \downarrow & \lrcorner & \downarrow{\scriptstyle (s,t)} \\ * & \xrightarrow[(x,y)]{} & \mathcal{C}_0 \times \mathcal{C}_0. \end{array}$$

It turns out that associated to two objects in an ∞-category there is an entire *space of morphisms* (see Remark 14.2.13).

The degeneracy map $s_0\colon \mathcal{C}_0 \to \mathcal{C}_1$ sends an object x to the *identity map* $\mathrm{id}_x = s_0(x)$ of x. The simplicial identities $d_0 s_0 = d_1 s_0 = \mathrm{id}_{\mathcal{C}_0}$ imply that id_x is an endomorphism, and the terminology and notation will be justified by Proposition 14.2.12.

As for compositions, let us consider morphisms $f\colon x \to y$ and $g\colon y \to z$ in an ∞-category $\mathcal{C}$. These morphisms together define an inner horn in $\mathcal{C}$,

$$\lambda = (g, \bullet, f)\colon \Lambda^2_1 \to \mathcal{C},$$

such that $d_0\lambda = g$ and $d_2\lambda = f$. Any such horn can be *non-uniquely* extended to a 2-simplex $\sigma\colon \Delta^2 \to \mathcal{C}$. The new face $d_1(\sigma)$ opposite to vertex 1 is then a *candidate composition* of g and f. To re-emphasize, this is one of the central points in which ∞-category theory differs from ordinary category theory: one does not ask for uniquely determined compositions. Instead one demands only that there is a way to compose arrows and that any choice of such a composition is equally good: the space of all such choices is to be contractible (see the discussion of Theorem 14.2.14 for a precise statement).

We now want to describe the homotopy category of an ∞-category. This can be done in a more straightforward way, but we prefer to include a short digression in category theory as this allows us to mention a general fact which is in the background of a later construction anyhow.

Digression 14.2.7. (Yoneda extension.) Let A be a small category and let us consider the associated presheaf category $\mathrm{Fun}(A^{\mathrm{op}}, \mathcal{S}et)$, i.e., the category of contravariant set-valued functors on A. Moreover, let $\mathcal{C}$ be a cocomplete category and let us be given a functor $Q\colon A \to \mathcal{C}$. Thus, we are in the situation

$$\begin{array}{ccc} A & \xrightarrow{Q} & \mathcal{C} \\ {\scriptstyle y}\downarrow & \nearrow & \\ \mathrm{Fun}(A^{\mathrm{op}}, \mathcal{S}et) & & \end{array}$$

where y denotes the Yoneda embedding of A. Recall from [85, p.76] that every presheaf on a small category is canonically a colimit of representable ones (see also §14.4.1). The cocompleteness of $\mathcal{C}$ hence allows us to extend Q to a colimit-preserving functor

$$|-|_Q\colon \mathrm{Fun}(A^{\mathrm{op}}, \mathcal{S}et) \to \mathcal{C}.$$

(At a more conceptual level, we are thus forming the left Kan extension of Q along y.) Moreover, associated to Q, there is also a functor in the opposite direction

$$\mathrm{Sing}_Q(-)\colon \mathcal{C} \to \mathrm{Fun}(A^{\mathrm{op}}, \mathcal{S}et),$$

that is defined by $\mathrm{Sing}_Q(c)_a = \hom_{\mathcal{C}}(Qa, c)$. One observes now that this pair of functors defines an adjunction

$$(|-|_Q, \mathrm{Sing}_Q(-))\colon \mathrm{Fun}(A^{\mathrm{op}}, \mathcal{S}et) \rightleftarrows \mathcal{C}$$

with $|-|_Q$ as left adjoint and $\mathrm{Sing}_Q(-)$ as right adjoint. (Related to this see also the general discussion in §14.4.1 and in particular Theorem 14.4.5).

But even more is true: For every cocomplete category $\mathcal{C}$ the assignment that sends Q to the adjunction $(|-|_Q, \mathrm{Sing}_Q(-))$ defines an equivalence of categories

$$\mathrm{Fun}(A, \mathcal{C}) \xrightarrow{\simeq} \mathrm{Adj}\left(\mathrm{Fun}(A^{\mathrm{op}}, \mathcal{S}et), \mathcal{C}\right).$$

Here, $\mathrm{Adj}(-,-)$ is the category of adjunctions where objects are adjunctions and morphisms are, say, natural transformations of left adjoints. This construction is sometimes referred to as *Yoneda extension* since it is essentially given by left Kan extension along the Yoneda embedding. A more detailed discussion of this can for example be found in [72, pp.62-64].

In what follows we are only interested in the special case where $A = \Delta$, the category of finite ordinals. Thus, we conclude that a cosimplicial object $\Delta \to \mathcal{C}$ in a cocomplete category is equivalently specified by an adjunction $s\mathcal{S}et \rightleftarrows \mathcal{C}$. The notation employed in Digression 14.2.7 is, of course, motivated by the following example.

Example 14.2.8. Let $\mathcal{C} = \mathcal{T}op$ be the category of topological spaces and let us consider the standard cosimplicial space $|\Delta^\bullet|\colon \Delta \to \mathcal{T}op$. The associated adjunction is the usual adjunction given by the geometric realization and the singular complex functor,

$$(|-|, \mathrm{Sing})\colon s\mathcal{S}et \rightleftarrows \mathcal{T}op.$$

Example 14.2.9. Let $\mathcal{C} = \mathcal{C}at$ be the cocomplete category of small categories (see Remark 14.2.22 for the fact that $\mathcal{C}at$ is cocomplete). The inclusion $\Delta \to \mathcal{C}at$ obtained by considering a finite ordinal as a category induces by Digression 14.2.7 an adjunction

$$(\tau_1, N)\colon s\mathcal{S}et \rightleftarrows \mathcal{C}at$$

with right adjoint the nerve functor. The left adjoint τ_1 is the *fundamental category functor* or the *categorical realization functor*. The motivation for the first terminology and the notation τ_1 stems from the following. Composition with the groupoidification $\mathcal{C}at \to \mathcal{G}rpd$, i.e., the left adjoint to the forgetful functor $\mathcal{G}rpd \to \mathcal{C}at$ yields the usual *fundamental groupoid functor* $\pi_1\colon s\mathcal{S}et \to \mathcal{G}rpd$. Thus associated to the cosimplicial object $\Delta \to \mathcal{G}rpd$ which sends $[n]$ to the free groupoid on $[n]$ we obtain the adjunction

$$(\pi_1, N)\colon s\mathcal{S}et \rightleftarrows \mathcal{G}rpd.$$

If the simplicial set happens to be an ∞-category, then there is a simplification of the description of the fundamental category. It turns out that morphisms can be represented by actual 1-simplices as we shall discuss now (Proposition 14.2.12). For that purpose, we need the following definition.

Definition 14.2.10. Two morphism $f, g\colon x \to y$ in an ∞-category $\mathcal{C}$ are *homotopic* (notation: $f \simeq g$) if there is a 2-simplex $\sigma\colon \Delta^2 \to \mathcal{C}$ whose boundary $\partial\sigma = (d_0\sigma, d_1\sigma, d_2\sigma)$ is given by (g, f, id_x), i.e., the boundary looks like

$$\begin{array}{ccccc} & & x & & \\ & {\scriptstyle \mathrm{id}_x}\nearrow & & \searrow{\scriptstyle g} & \\ x & & \xrightarrow[f]{} & & y. \end{array} \tag{14.2.2}$$

Any such 2-simplex σ is a *homotopy* from f to g, denoted $\sigma\colon f \to g$.

There is a similar notion of homotopies where the identity morphism sits on the face opposite to vertex zero. However, using the inner horn extension property both notions can be shown to be the same. Moreover, there is the following result.

Proposition 14.2.11. *Let $\mathcal{C}$ be an ∞-category and let $x, y \in \mathcal{C}$. The homotopy relation is an equivalence relation on* $\hom_{\mathcal{C}}(x, y)$*. The* homotopy class *of a morphism $f\colon x \to y$ is denoted by* $[f]$.

We include a partial proof in order to give an idea of how this works. Associated to a morphism $f\colon x \to y$, let us consider $\kappa_f = s_0 f\colon \Delta^2 \to \mathcal{C}$. The simplicial identities imply that $d_0\kappa_f = d_1\kappa_f = f$ and also $d_2\kappa_f = d_2 s_0 f = s_0 d_1 f = \mathrm{id}_x$. Thus, the boundary of κ_f is $\partial\kappa_f = (f, f, \mathrm{id}_x)$ and we receive a homotopy $\kappa_f\colon f \to f$, the *constant homotopy* of f, which establishes the reflexivity of $\simeq$. For the symmetry one way to proceed is as follows. Given a homotopy $\sigma\colon f \to g$, let us form the inner horn

$$(\sigma, \kappa_g, \bullet, \kappa_{\mathrm{id}_x})\colon \Lambda^3_2 \to \mathcal{C} \tag{14.2.3}$$

in $\mathcal{C}$. By definition of an ∞-category this horn can be extended to a 3-simplex $\tau\colon \Delta^3 \to \mathcal{C}$. The new face $\tilde{\sigma} = d_2\tau \in \mathcal{C}_2$ defines a homotopy $\tilde{\sigma}\colon g \to f$, an *inverse homotopy* of σ, showing that $\simeq$ is symmetric.

With the homotopy relation at our disposal, we would like to define the *homotopy category* $\mathrm{Ho}(\mathcal{C})$ of an ∞-category $\mathcal{C}$ by passing to homotopy classes of morphisms. The composition law in $\mathrm{Ho}(\mathcal{C})$ is obtained by representing homotopy classes by morphisms in $\mathcal{C}$, *choosing candidate compositions* of the representatives, and then passing to homotopy classes again. Of course, in order to get a well-defined category there are a lot of things to be checked, but we content ourselves by showing that all candidate compositions are homotopic. Let us consider morphisms $f\colon x \to y$ and $g\colon y \to z$ together with 2-simplices $\sigma_1, \sigma_2\colon \Delta^2 \to \mathcal{C}$ witnessing that $h_1 = d_1(\sigma_1)$ and $h_2 = d_1(\sigma_2)$ are candidate compositions of g and f. Then we can form the inner horn

$$(\sigma_1, \sigma_2, \bullet, \kappa_f)\colon \Lambda^3_2 \to \mathcal{C}$$

in $\mathcal{C}$. Again, we can find an extension to a 3-simplex $\tau\colon \Delta^3 \to \mathcal{C}$ and the new face $d_2\tau\colon \Delta^2 \to \mathcal{C}$ gives us the desired homotopy $h_2 \to h_1$. Using similar arguments, one can establish the following result which can already be found in [23].

Proposition 14.2.12. *Let $\mathcal{C}$ be an ∞-category. There is an ordinary category* $\mathrm{Ho}(\mathcal{C})$*, the* homotopy category *of $\mathcal{C}$, with the same objects as $\mathcal{C}$ and morphisms the homotopy classes of morphisms in $\mathcal{C}$. Composition and identities are given by*

$$[g] \circ [f] := [g \circ f] \qquad \textit{and} \qquad \mathrm{id}_x := [\mathrm{id}_x] = [s_0 x],$$

where $g \circ f$ is an arbitrary *candidate composition of g and f. Furthermore, there is a natural isomorphism of categories* $\mathrm{Ho}(\mathcal{C}) \cong \tau_1(\mathcal{C})$.

Remark 14.2.13. (i) One guiding principle for the theory of ∞-categories is that there should be a way to compose arrows and that the space of all such choices is contractible. Using the extension property for inner 2-horns, it is immediate that the space is non-empty. By means of the extension property for inner horns up to dimension three, we just checked that two candidate compositions are homotopic, i.e., that the space of all choices is connected. But this is only a π_0-statement of something much stronger: The extension property with respect to higher-dimensional inner horns can be thought of as guaranteeing the higher connectivity of the space of all such choices, giving finally that it is weakly contractible. See the discussion of Theorem 14.2.14 for a precise statement.

(ii) A second guiding principle for the theory of ∞-categories is that there should be morphisms of arbitrary dimensions. Let $\mathcal{C}$ be an ∞-category and let x, y be objects in $\mathcal{C}$. Then a morphism $f \colon x \to y$ is given by

$$f \colon \Delta^1 \to \mathcal{C} \qquad \text{such that} \qquad f|_{\Delta^{\{0\}}} = x \qquad \text{and} \qquad f|_{\Delta^{\{1\}}} = y.$$

Here and in the sequel, the notation is as follows: for vertices $i_0, \ldots, i_k$ in Δ^n, $\Delta^{\{i_0,\ldots,i_k\}} \subseteq \Delta^n$ denotes the k-simplex of Δ^n spanned by the given vertices. Moreover, given a vertex $x \in \mathcal{C}$ we write $x \colon \Delta^{\{i_0,\ldots,i_k\}} \to \mathcal{C}$ for the constant map with value x. A homotopy between two parallel morphisms $x \to y$ in $\mathcal{C}$ can be interpreted as a 2-*morphism* from x to y. Recall that a homotopy is given by

$$\sigma \colon \Delta^2 \to \mathcal{C} \qquad \text{such that} \qquad \sigma|_{\Delta^{\{0,1\}}} = x \qquad \text{and} \qquad \sigma|_{\Delta^{\{2\}}} = y.$$

This can be generalized to higher dimensions: an *n-morphism* from x to y is a map of simplicial sets

$$\tau \colon \Delta^{n+1} \to \mathcal{C} \qquad \text{such that} \qquad \tau|_{\Delta^{\{0,\ldots,n\}}} = x \qquad \text{and} \qquad \tau|_{\Delta^{\{n+1\}}} = y.$$

For varying n, the sets of n-morphisms can be assembled in a *space of morphisms* $\mathrm{Map}^R_{\mathcal{C}}(x, y) \in s\mathcal{S}et$ which can be shown to be a Kan complex.

We already mentioned that there is a variant to our definition of a homotopy (obtained by choosing the identity morphism in (14.2.2) to sit opposite to vertex zero). More generally, there is an obvious dual way to define a space of morphisms $\mathrm{Map}^L_{\mathcal{C}}(x, y)$ which turns out to be a weakly equivalent Kan complex [82, Cor. 4.2.1.8]. Thus, the homotopy type of the mapping space is well-defined.

(iii) A third guiding principle for the theory of ∞-categories is that they should give a model for $(\infty,1)$-categories, i.e., all higher morphisms should be invertible in some weak sense. To indicate that we succeeded in establishing such a framework, let us consider a homotopy σ in an ∞-category $\mathcal{C}$,

$$\sigma\colon f \simeq g\colon x \to y.$$

In order to establish the symmetry of the homotopy relation, we considered the inner horn (14.2.3) which can be extended to a 3-simplex $\tau\colon \Delta^3 \to \mathcal{C}$. The new face $\widetilde{\sigma} = d_2\tau$ then gives us the intended homotopy $\widetilde{\sigma}\colon g \simeq f$. Note that τ satisfies $\tau|_{\Delta^{\{0,1,2\}}} = x$ and τ is hence a *3-morphism*, which can be interpreted as a 2-homotopy

$$\tau\colon \kappa_g \simeq \widetilde{\sigma} \circ \sigma.$$

Thus, every homotopy has (up to a 2-homotopy) a left inverse and a similar observation can be made for right inverses. Taking for granted that the horn extension property for higher dimensional horns allows us to deduce similar observations for higher homotopies, we are reassured that ∞-categories really provide a model for $(\infty,1)$-categories.

The following result due to Joyal [69, Prop. 2.24] makes precise that we succeeded in finding an axiomatic framework for categories with compositions determined up to contractible choices. Let $i\colon \Lambda^2_1 \to \Delta^2$ be the obvious inclusion. Moreover, let us denote by $\mathrm{Map}(-,-)\colon s\mathcal{S}et^{\mathrm{op}} \times s\mathcal{S}et \to s\mathcal{S}et$ the simplicial mapping space functor,

$$\mathrm{Map}(X,Y)_\bullet = \hom_{s\mathcal{S}et}(\Delta^\bullet \times X, Y), \tag{14.2.4}$$

so that vertices are maps, edges are homotopies, and higher dimensional simplices are 'higher homotopies' (see e.g. [46, p.20]).

Theorem 14.2.14. *A simplicial set X is an ∞-category if and only if the restriction map $i^*\colon \mathrm{Map}(\Delta^2, X) \to \mathrm{Map}(\Lambda^2_1, X)$ is an acyclic Kan fibration.*

We can think of $\mathrm{Map}(\Lambda^2_1, X)$ as the *space of composition problems* and similarly of $\mathrm{Map}(\Delta^2, X)$ as the *space of solutions to composition problems*. The theorem then tells us that the *defining feature* of an ∞-category is that these two spaces are the same from a homotopical perspective, and this motivates us to henceforth suppress the 'candidate' in 'candidate composition'.

We now turn to equivalences in an ∞-category.

Definition 14.2.15. A morphism $f\colon x \to y$ in an ∞-category $\mathcal{C}$ is an *equivalence* if $[f]\colon x \to y$ is an isomorphism in $\mathrm{Ho}(\mathcal{C})$.

It is immediate that identities are equivalences and that for two homotopic morphisms $f_1 \simeq f_2$ we have that f_1 is an equivalence if and only if f_2 is one. Moreover, it turns out that a morphism $f\colon x \to y$ in $\mathcal{C}$ is an equivalence if and only if there is a morphism $g\colon y \to x$

in $\mathcal{C}$ such that there are 2-simplices with boundaries as in

$$\begin{array}{ccc} & y & \\ {\scriptstyle f}\nearrow & & \searrow{\scriptstyle g} \\ x & \xrightarrow[\mathrm{id}_x]{} & x, \end{array} \qquad \begin{array}{ccc} & x & \\ {\scriptstyle g}\nearrow & & \searrow{\scriptstyle f} \\ y & \xrightarrow[\mathrm{id}_y]{} & y. \end{array}$$

In a way one could be surprised that we can characterize equivalences in an ∞-category by these two conditions. Since ∞-category theory is some sort of 'coherent category theory' one might have expected that also higher coherence data would be necessary to characterize equivalences in an ∞-category. For a precise statement and a proof of the equivalence of these two potentially different invertibility conditions we refer to [68, Corollary 1.6] and [34, Proposition 2.2].

We mentioned already the accepted principle that all ∞-*groupoids* should come from spaces. In order to make this explicit let us give the following definition.

Definition 14.2.16. An ∞-category is an ∞-*groupoid* if the homotopy category is a groupoid.

Thus an ∞-category is an ∞-groupoid if and only if all morphisms are equivalences. In the motivation of the definition of an ∞-category, we saw that, in general, one should only demand the horn extension property for inner horns in order to obtain a good generalization of arbitrary categories (and not just of groupoids!). Joyal established the following result, saying that outer horns can be extended as soon as certain maps are equivalences.

Proposition 14.2.17. *Let $\mathcal{C}$ be an ∞-category. Any horn $\lambda\colon \Lambda^n_0 \to \mathcal{C}$, $n \geq 2$, such that $\lambda|_{\Delta^{\{0,1\}}}$ is an equivalence can be extended to a simplex $\Delta^n \to \mathcal{C}$.*

There is of course a similar statement using the horns Λ^n_n instead. This allows us to turn the principle that all ∞-groupoids should be given by spaces into the following precise statement (see [68, Corollary 1.4] or [82, p.35]).

Corollary 14.2.18. *An ∞-category is an ∞-groupoid if and only if it is a Kan complex.*

With this result at hand, diagram (14.2.1) consisting of fully faithful functors can be refined to

$$\begin{array}{ccccc} \mathcal{G}rpd & \longrightarrow & \mathcal{C}at & \overset{N}{\searrow} & \\ {\scriptstyle N}\downarrow & & \downarrow{\scriptstyle N} & & \\ \mathcal{K}an = \mathcal{G}rpd_\infty & \longrightarrow & \mathcal{C}at_\infty & \longrightarrow & s\mathcal{S}et. \end{array} \tag{14.2.5}$$

Perspective 14.2.19. As in Corollary 14.2.18, for low values of $n \in \mathbb{N}$ there are statements using n-*types of spaces* and n-*groupoids* in a certain precise sense. The statements that higher homotopy types should be classified by higher groupoids is frequently also referred to as the *homotopy hypothesis*.

In the case of $n = 1$, a precise statement can for example be found in [58] where it is shown that such a classification is induced by the adjunction $(\pi_1, N)\colon s\mathcal{S}et \rightleftarrows \mathcal{G}rpd$. In fact, this adjunction can be seen to be a Quillen adjunction with respect to the Kan–Quillen model structure on $s\mathcal{S}et$ and the so-called *natural model stucture* on $\mathcal{G}rpd$ (related to this see

[70] or the nice short account in [93]). The slogan that 'groupoids do not carry any higher homotopical information' can be made precise as follows: the Quillen adjunction (π_1, N) induces a Quillen equivalence between the S^2-nullification of $s\mathcal{S}et$ and $\mathcal{G}rpd$. Related results in the cases of $n = 2, 3$, more precisely, in the context of *bicategories* [13] and *Gray categories* [47, 77], are made explicit in [76, §6].

14.2.2 Simplicial categories and the relation to ∞-categories

There are many alternative approaches to a theory of $(\infty, 1)$-categories including simplicial categories [14], Segal categories [57], and complete Segal spaces [94]. Besides in the original references, more details can for example be found in [15, 16, 107, 1] and in Bergner's chapter in this Handbook. Here we include a short discussion of *simplicial categories* or, more precisely, *simplicially enriched categories*. Given two objects x, y in a simplicial category $\mathcal{C}$, we write $\mathrm{Map}_{\mathcal{C}}(x, y)$ for the associated simplicial mapping space. This more rigid approach — coming with a specified strictly associative and unital composition law — gives us, by definition, a notion of a category with morphisms of arbitrary dimensions. Building on work of Joyal and Bergner, Lurie has shown that this approach and the one using ∞-categories are equivalent in a very precise sense (see Theorem 14.2.29).

We begin by describing a relation between simplicial sets and simplicial categories. First, let us recall that the nerve $N(\mathcal{C})$ of an ordinary category $\mathcal{C}$ is the simplicial set

$$N(\mathcal{C})_\bullet = \hom_{\mathcal{C}at}([\bullet], \mathcal{C}),$$

where $[\bullet]\colon \Delta \to \mathcal{C}at$ is obtained by considering the finite ordinals $[n]$ as categories. Given a *simplicial* category $\mathcal{C}$, we could simply forget the simplicial enrichment and form the nerve of the underlying ordinary category. More precisely, if we denote by $s\mathcal{C}at$ the category of (small) simplicial categories and simplicial functors, then there is the forgetful functor $s\mathcal{C}at \to \mathcal{C}at$ which we could compose with the ordinary nerve functor $\mathcal{C}at \to s\mathcal{S}et$. But this approach obviously discards too much information and instead one should proceed differently.

A better way is given by replacing $[n] \in \mathcal{C}at$ by *simplicially thickened versions* $C[\Delta^n] \in s\mathcal{C}at$ and then building the simplicial set

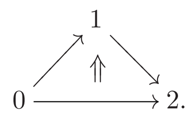
$$N_\Delta(\mathcal{C})_\bullet = \hom_{s\mathcal{C}at}(C[\Delta^\bullet], \mathcal{C}),$$

where $\hom_{s\mathcal{C}at}(-,-)$ denotes the set of simplicial functors. The idea behind this simplicial thickening is that $C[\Delta^n]$ encodes as objects the vertices of the standard simplex Δ^n, as morphisms all paths in increasing direction, as 2-morphisms all homotopies, and so on in higher dimensions. More conceptual comments about this construction can be found in Perspective 14.2.25. Before we give a precise definition of $C[\Delta^\bullet]$ let us describe what we expect to obtain in low dimensions.

Example 14.2.20. In dimensions 0 and 1 nothing new happens, and the simplicial categories $C[\Delta^0]$ and $C[\Delta^1]$ are just the ordinary categories [0] and [1], respectively, considered as simplicial categories with discrete mapping spaces. Thus, the pictures we have in mind are

$$C[\Delta^0]\colon\quad 0 \qquad\text{and}\qquad C[\Delta^1]\colon\quad 0 \to 1.$$

But from dimension 2 the simplicial picture is richer. In Δ^2, there are two ways to pass from 0 to 2, namely the straight path and the path passing through 1. These paths should be encoded in $C[\Delta^2]$ together with a homotopy between them. The simplicial category $C[\Delta^2]$ can hence be depicted by



(14.2.6)

We now give a precise definition of $C[\Delta^n]$. The objects of $C[\Delta^n]$ are the numbers $0, 1, \ldots, n$. The strategy behind the definition of the simplicial mapping spaces is the following. Given objects $i \leq j$ we encode a path from i to j by specifying the vertices of the corresponding path. Thus, let $P_{i,j}$ be the poset

$$P_{i,j} = \big\{ I \subseteq [i,j] \;\mid\; i,j \in I \big\}$$

ordered by inclusion where $[i,j]$ is short hand notation for $\{i, i+1, \ldots, j-1, j\}$. Considering these posets as categories, we can define the simplicial mapping spaces in $C[\Delta^n]$ by

$$\mathrm{Map}_{C[\Delta^n]}(i,j) = \begin{cases} NP_{i,j}\,, & i \leq j, \\ \emptyset\,, & i > j. \end{cases}$$

The composition is induced by the union of subsets, which fits fine with the strategy to encode a path by specifying the vertices one passes along. It is also immediate that identities are given by the singletons $\{i\}$. This concludes the definition of $C[\Delta^n] \in s\mathcal{C}at$.

One easily checks, that this definition specializes to the pictures we had in mind in low dimensions (14.2.20). For example in dimension $n = 2$, there is the following table of non-degenerate k-simplices in the mapping spaces $\mathrm{Map}_{C[\Delta^2]}(i,j)$:

k	$i=j=0$	$i=0, j=1$	$i=0, j=2$
0	$\{0\}$	$\{0,1\}$	$\{0,2\},\ \{0,1,2\}$
1			$\{0,2\} \subseteq \{0,1,2\}$

By definition $\{0,1,2\} = \{1,2\} \circ \{0,1\}$ is the composition and we see that the non-degenerate 1-simplex in $\mathrm{Map}_{C[\Delta^2]}(0,2)$ encodes the homotopy

$$\{0,2\} \to \{1,2\} \circ \{0,1\}$$

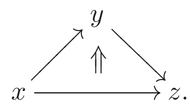
we were aiming for in (14.2.6).

It is straightforward to check that the assignment $[n] \mapsto C[\Delta^n]$ defines a cosimplicial object $C[\Delta^\bullet] \colon \Delta \to s\mathcal{C}at$. This allows us to give the following definition which appears to be due to Cordier [25].

Definition 14.2.21. The *coherent nerve* $N_\Delta(\mathcal{C})$ of a simplicial category $\mathcal{C}$ is the simplicial set

$$N_\Delta(\mathcal{C})_\bullet = \hom_{s\mathcal{C}at}(C[\Delta^\bullet], \mathcal{C}).$$

Thus we have a coherent nerve functor $N_\Delta \colon s\mathcal{C}at \to s\mathcal{S}et$. By the very definition, this coherent nerve construction takes into account the higher structure on a simplicial category given by the mapping spaces. For example a 2-simplex in such a coherent nerve is given by a homotopy as depicted in



Note that such a 2-simplex is, in general, not determined by its restriction to the horn $\Lambda^2_1 \subset \Delta^2$.

Using the observation that $s\mathcal{C}at$ is cocomplete (see Remark 14.2.22) we can extend the cosimplicial object $\Delta \to s\mathcal{C}at \colon [n] \mapsto C[\Delta^n]$ to a colimit-preserving functor $C[-] \colon s\mathcal{S}et \to s\mathcal{C}at$. More explicitly, for $X \in s\mathcal{S}et$ we make the definition

$$C[X] = \operatorname{colim}_{(\Delta/X)} C[-] \circ p \qquad (14.2.7)$$

where (Δ/X) is the category of simplices of X and $p \colon (\Delta/X) \to \Delta$ is the canonical functor. It turns out that this extension defines a left adjoint to the coherent nerve N_Δ,

$$(C[-], N_\Delta) \colon s\mathcal{S}et \rightleftarrows s\mathcal{C}at. \qquad (14.2.8)$$

(In fact, this can be considered as an example of *Yoneda extensions* in the sense of Digression 14.2.7.) We observe that the notation $C[-]$ is not in conflict with the notation $C[\Delta^n]$ for the simplicial thickening of $[n]$ since the colimit-preserving extension $C[-]$ applied to Δ^n is isomorphic to what we just defined. This follows because the category of simplices (Δ/Δ^n) has $([n], \mathrm{id} \colon \Delta^n \to \Delta^n)$ as terminal object so that the defining colimit in (14.2.7) simplifies accordingly.

Remark 14.2.22. The cocompleteness of $s\mathcal{C}at$ is a special instance of a more general result. Given a symmetric monoidal category $\mathcal{M}$ let us denote by $\mathcal{C}at_{\mathcal{M}}$ the category of (small) $\mathcal{M}$-enriched categories and $\mathcal{M}$-enriched functors. Thus in this notation we have $s\mathcal{C}at = \mathcal{C}at_{s\mathcal{S}et}$. It turns out that if $\mathcal{M}$ is complete and cocomplete then so is $\mathcal{C}at_{\mathcal{M}}$. The harder part is the cocompleteness and was established by Wolff in [116]. As important examples we deduce that the categories of categories, simplicial categories, topological categories, spectral categories, differential-graded categories, and 2-categories are complete and cocomplete.

As we explain next, the adjunction (14.2.8) is in fact a Quillen equivalence with respect to the *Joyal model structure* on $s\mathcal{S}et$ and the *Bergner model structure* on $s\mathcal{C}at$. Since we will not make an intensive use of the Bergner model structure, we do not go too much into detail and instead refer to [14]. Let us recall that every simplicial category $\mathcal{C}$ has an underlying *path component category* or *homotopy category* $\pi_0\mathcal{C}$. This is an ordinary category with the same objects while the sets of morphisms are obtained by applying π_0 to the simplicial mapping spaces. For example, if we endow $s\mathcal{S}et$ with the usual simplicial enrichment given by (14.2.4), then $\pi_0 s\mathcal{S}et$ is the naive homotopy category with all simplicial sets as objects and simplicial homotopy classes as morphisms.

We can now define the weak equivalences in the Bergner model structure.

Definition 14.2.23. A simplicial functor $F\colon \mathcal{C} \to \mathcal{D}$ is a *weak equivalence* if

(i) the induced functor $\pi_0 F\colon \pi_0\mathcal{C} \to \pi_0\mathcal{D}$ is essentially surjective and

(ii) for all objects $x, y \in \mathcal{C}$ the map $\mathrm{Map}_{\mathcal{C}}(x, y) \to \mathrm{Map}_{\mathcal{D}}(Fx, Fy)$ is a weak equivalence of simplicial sets (i.e., it induces a weak equivalence on geometric realizations).

Recall that a functor between ordinary categories is an equivalence if and only if it is essentially surjective and fully faithful. The definition of a weak equivalence between simplicial categories can be read as a higher categorical generalization of equivalences since it is asking that the simplicial functor is *homotopically essentially surjective* and *homotopically fully faithful.* Such a functor is also called a *Dwyer–Kan equivalence* , attributing credit to [37]. Obviously, such a weak equivalence $\mathcal{C} \to \mathcal{D}$ induces an equivalence $\pi_0\mathcal{C} \to \pi_0\mathcal{D}$ but having a weak equivalence is, in general, a much stronger statement.

Building on work of Dwyer and Kan, Bergner [14] established the following result.

Theorem 14.2.24. *The category $s\mathcal{C}at$ carries a left proper combinatorial model structure with the Dwyer–Kan equivalences as weak equivalences. With respect to this model structure, a simplicial category is fibrant if and only if it is* locally fibrant*, i.e., if all simplicial mapping spaces are Kan complexes.*

This model structure is referred to as the *Bergner model structure* and provides us with an example of a *homotopy theory of homotopy theories*. For more details about it we refer to [14].

Perspective 14.2.25. We have already seen that the functor $C[-]\colon s\mathcal{S}et \to s\mathcal{C}at$ is essentially determined by the cosimplicial object $C[\Delta^\bullet]\colon \Delta \to s\mathcal{C}at$ given by the 'simplicial thickenings' of the finite ordinals. These simplicial thickenings arise more conceptually as follows. If we consider the categories $[n]$ as discrete simplicial categories, then we obtain a cosimplicial object $[\bullet]\colon \Delta \to s\mathcal{C}at$. The Bergner model structure of Theorem 14.2.24 induces a Reedy model structure on the category $\mathrm{Fun}(\Delta, s\mathcal{C}at)$ of cosimplicial objects in $s\mathcal{C}at$. It turns out that $C[\Delta^\bullet]\colon \Delta \to s\mathcal{C}at$ gives us a Reedy cofibrant replacement of $[\bullet]\colon \Delta \to s\mathcal{C}at$. For an additional perspective on these simplicial thickenings we refer to Perspective 14.3.5.

With a view towards the Joyal model structure on $s\mathcal{S}et$, we make the following definition.

Definition 14.2.26. A map $f\colon X \to Y$ in $s\mathcal{S}et$ is a *categorical equivalence* if the induced simplicial functor $C[f]\colon C[X] \to C[Y]$ is a Dwyer–Kan equivalence.

This terminology is not the original one of Joyal. The maps in this definition are called *weak categorical equivalences* by Joyal [67], while he has a stronger notion of categorical equivalence. However, his notions of categorical equivalence and weak categorical equivalence coincide when only maps between ∞-categories are considered.

For simplicity, a categorical equivalence between ∞-categories is called an *equivalence of ∞-categories* and we say that the ∞-categories are *equivalent*. The fact that it suffices to consider direct equivalences as opposed to more complicated zig-zags is a consequence of the following important theorem of Joyal [67].

Theorem 14.2.27. *The category $s\mathcal{S}et$ carries a left proper combinatorial model structure with the monomorphisms as cofibrations and the categorical equivalences as weak equivalences. Moreover, a simplicial set is fibrant with respect to this model structure if and only if it is an ∞-category.*

The model structure of this theorem is referred to as the *Joyal model structure*. We add some comments on the fibrations in the Joyal model structure. For this we first recall that in the Kan–Quillen model structure on $s\mathcal{S}et$ the fibrations are the Kan fibrations. A Kan fibration $p\colon X \to S$ gives us a family of Kan complexes, i.e., ∞-groupoids, namely the fibers X_s defined by the pullbacks

$$\begin{array}{ccc} X_s & \longrightarrow & X \\ \downarrow & \lrcorner & \downarrow{\scriptstyle p} \\ \Delta^0 & \xrightarrow[s]{} & S. \end{array}$$

Similarly, there is the following class of maps giving families of ∞-categories.

Definition 14.2.28. A morphism of simplicial sets $p\colon X \to S$ is an *inner fibration* if it has the right lifting property with respect to $\Lambda^n_k \to \Delta^n$, $0 < k < n$.

Joyal uses the term *mid-fibration* instead of inner fibration. Since any class of morphisms defined by a right lifting property is closed under pullbacks this implies that the fibers of an inner fibration are ∞-categories. The dependence of the fiber on the base point is only functorial in a very weak sense, namely in the sense of *correspondences* (see [82, p. 97]), which are also known as *distributors, profunctors* or *bimodules* in the classical setting (see [18, §7.8]).

Now, the *categorical fibrations*, i.e., the fibrations in the Joyal model structure happen to be a bit difficult to describe. However, for a morphism $p\colon X \to S$ such that the target S is an ∞-category, Joyal gave the following characterization: such a map is a categorical fibration if and only if it is an inner fibration and an *isofibration*. More precisely, since p is an inner fibration and S is an ∞-category, also X is an ∞-category, and a map $X \to S$ between ∞-categories is an *isofibration* if and only if every equivalence $p(x) \to s$ in S can be lifted to an equivalence in X with domain x.

The nerve $N(F)$ of a functor $F\colon A \to B$ of ordinary categories is automatically an inner fibration; thus the notion of inner fibrations does not have a classical analogue. In particular, a functor can always be thought of as a family of categories parametrized by the objects of the target category. In the ∞-categorical world however in many definitions this condition has to be imposed. This is the case for the categorical fibrations but also for further classes of fibrations as we will see later.

As already mentioned, the original definition of categorical equivalences due to Joyal [67] is different. He gives a definition without reference to simplicial categories and his proof of the existence of the Joyal model structure is purely combinatorial. Lurie gives this alternative definition because he is heading for the following comparison result [82, Thm. 2.2.5.1].

Theorem 14.2.29. *The adjunction $(C[-], N_\Delta)\colon s\mathcal{S}et \rightleftarrows s\mathcal{C}at$ is a Quillen equivalence with respect to the Joyal model structure and the Bergner model structure,*

$$(C[-], N_\Delta)\colon s\mathcal{S}et \rightleftarrows s\mathcal{C}at.$$

A similar result can also be obtained by a combination of results due to Bergner, Joyal, Rezk, and Tierney. For more details we refer to the chapter by Bergner in this Handbook.

Theorem 14.2.29 makes precise in which sense the approaches to a theory of $(\infty, 1)$-categories given by ∞-categories and simplicial categories are equivalent. The proof of this theorem is actually hard work including a deep 'rigidification or straightening result' and can be found in [82, §2.2.5]. An alternative proof was given by Dugger and Spivak in [34, 35].

As a corollary, we have the following result which can also be obtained directly and without any mention of model structures (see for example the proof of [26, Theorem 2.1]). However, with Theorem 14.2.29 in mind the result is put into perspective. Recall that a simplicial category is called locally fibrant if all mapping spaces are Kan complexes.

Corollary 14.2.30. *The coherent nerve of a locally fibrant simplicial category is an ∞-category.*

With this corollary at our disposal we can now give some typical examples of ∞-categories that are neither nerves of categories nor singular complexes of spaces. So, these are somehow our first honest examples of ∞-categories. It turns out that the first class of examples is generic in a sense which is made precise by Theorem 14.4.15.

Examples 14.2.31. (i) Let $\mathcal{M}$ be a simplicial model category [92, §II.2] and let $\mathcal{M}_{\mathrm{cf}} \subseteq \mathcal{M}$ be the full simplicial subcategory spanned by the fibrant and cofibrant objects. It is an immediate consequence of Quillen's axiom (SM7) that $\mathcal{M}_{\mathrm{cf}}$ is a locally fibrant simplicial category. Thus, via the coherent nerve construction, we obtain the ∞-category $N_\Delta(\mathcal{M}_{\mathrm{cf}})$, the *underlying ∞-category* of the simplicial model category $\mathcal{M}$.

(ii) As a more specific example let us consider $s\mathcal{S}et$ endowed with the usual Kan–Quillen model structure. This is a simplicial model category and with respect to this model structure we have $s\mathcal{S}et_{\mathrm{cf}} = \mathcal{K}an$, where $\mathcal{K}an$ is the full simplicial subcategory spanned by the Kan complexes. The *∞-category $\mathcal{S}$ of spaces* is given by

$$\mathcal{S} = N_\Delta(\mathcal{K}an).$$

This is only one model for the ∞-category of spaces; there are others, for example the underlying ∞-category of topological spaces. For all purposes of ∞-category theory it turns out that any ∞-category equivalent to $\mathcal{S}$ is equally good and that the precise model is irrelevant. All these ∞-categories satisfy the same universal property of being the 'free cocomplete ∞-category on a single generator' (see Corollary 14.4.11).

(iii) Another class of examples is induced by additive categories. Given an additive category $\mathcal{A}$, the category $\mathrm{Ch}(\mathcal{A})$ of chain complexes in $\mathcal{A}$ can be enriched over the category $\mathrm{Ch}(\mathbb{Z})$ of chain complexes of abelian groups. The enrichment is set up in a way that if we take the 0-cycles of all the mapping complexes then we obtain the usual category of chain complexes, while cycles of positive dimensions give chain homotopies and higher chain homotopies. In what follows we restrict the enrichment to the category $\mathrm{Ch}_{\geq 0}(\mathbb{Z})$ of non-negative chain complexes.

The *Dold–Kan correspondence* gives us an equivalence of categories $\mathrm{DK}\colon \mathrm{Ch}_{\geq 0}(\mathbb{Z}) \to s\mathcal{A}b$, where $s\mathcal{A}b$ is the category of simplicial abelian groups. An inverse to DK is given by the *normalized chain complex functor* which can be shown to be lax comonoidal with respect to the levelwise tensor product on $s\mathcal{A}b$ and the usual one on $\mathrm{Ch}_{\geq 0}(\mathbb{Z})$ (in fact, this lax comonoidal structure is induced by the Alexander–Whitney maps). It follows by abstract nonsense that DK carries canonically a lax *monoidal* structure (see [74] for the abstract framework and [103] for precisely this context). Thus, we can apply DK to the morphism complexes in $\mathrm{Ch}(\mathcal{A})$ in order to obtain the category $\mathrm{DK}_*(\mathrm{Ch}(\mathcal{A}))$ enriched in simplicial abelian groups. Since simplicial abelian groups are Kan complexes this gives us a locally fibrant simplicial category, and we can define the *∞-category* $\mathrm{Ch}(\mathcal{A})$ *of chain complexes* in $\mathcal{A}$ by

$$\mathrm{Ch}(\mathcal{A}) = N_\Delta\big(\mathrm{DK}_*(\mathrm{Ch}(\mathcal{A}))\big).$$

A smaller model for these ∞-categories can be obtained by means of the *differential-graded nerve construction* (see [83, §1.3.1]).

We conclude this section by a short perspective on enriched ∞-categories.

Perspective 14.2.32. Recall that a 2-category is simply a category enriched over categories. Similarly, a strict n-category is just a category enriched over $(n-1)$-categories. This definition via iterated enrichments suggests that $(\infty, 1)$-categories should be categories enriched over $(\infty, 0)$-categories, i.e., Kan complexes by Corollary 14.2.18. And as we just saw, simplicial categories with the property that all mapping spaces are Kan complexes play a special role (see Theorem 14.2.24).

However, if we take the philosophy of ∞-categories seriously, then the good notion of enrichment in that context should be that of a 'weak enrichment'. We saw that for every pair of objects x, y in an ∞-category $\mathcal{C}$, there is a space of morphisms $\mathrm{Map}_\mathcal{C}(x, y)$. In contrast to the case of simplicial categories, for ∞-categories there is not a specified strictly associative and strictly unital composition law $\mathrm{Map}_\mathcal{C}(y, z) \times \mathrm{Map}_\mathcal{C}(x, y) \to \mathrm{Map}_\mathcal{C}(x, z)$. In fact, the idea of having a weak enrichment should be expressed by the existence of composition laws which are only *coherently associative*. Taking this perspective of $\mathbb{A}_\infty$-multiplications seriously, an abstract theory of enriched ∞-categories has been proposed by Gepner and Haugseng [45]. Their framework even allows one to study weak enrichments in not necessarily Cartesian contexts. For related rectification results see [53].

14.3 Categorical constructions with ∞-categories

The aim of this section is to extend some key constructions and notions from ordinary category theory to the world of ∞-categories (or more general simplicial sets) in a way that the following principles are satisfied.

(i) The concepts are extensions of the ordinary concepts in that everything is compatible with the fully faithful nerve functor $N\colon \mathcal{C}at \to s\mathcal{S}et$.

(ii) The notions are *coherent* variants of the classical notions, i.e., ∞-category theory realizes some kind of homotopy coherent category theory.

(iii) The extensions are often defined for arbitrary simplicial sets, and when applied to ∞-categories we want these extensions to again give rise to ∞-categories.

(iv) All concepts are *invariant concepts*, i.e., an application of these constructions to equivalent input ∞-categories yields equivalent output ∞-categories.

We are mostly interested in a robust theory of *limits* and *colimits* (see §14.3.5) giving us an ∞-categorical version of the more well-known homotopy (co)limits in model categories, but this needs some preparation. In §14.3.1 we discuss ∞-categories of functors. In §§14.3.2-14.3.3 we introduce join and slice constructions, which allow us to speak about ∞-categories of cones and cocones on diagrams. In §14.3.4 we introduce initial and final objects as special cases of limits and colimits, respectively. Finally, in §14.3.5 everything is put together and we introduce limits as final objects in slice categories (and dually for colimits).

The reader who is less inclined towards abstract categorical constructions is asked to consider the goal of a brief discussion of colimits as a justification for the discussion of the constructions in §14.3.2 and §14.3.3. For a more detailed discussion of the *theory* of limits we refer to [82].

14.3.1 Functors

Since ∞-categories are simply particular simplicial sets we can make the following definition.

Definition 14.3.1. Let K be a simplicial set and let $\mathcal{C}$ be an ∞-category. A *functor* $F\colon K \to \mathcal{C}$ is a map of simplicial sets. Similarly, a *natural transformation* is a map $\Delta^1 \times K \to \mathcal{C}$. More generally, the *space of functors* $\mathrm{Fun}(K, \mathcal{C})$ is

$$\mathrm{Fun}(K, \mathcal{C})_\bullet = \mathrm{Map}_{s\mathcal{S}et}(K, \mathcal{C})_\bullet = \hom_{s\mathcal{S}et}(\Delta^\bullet \times K, \mathcal{C}) \in s\mathcal{S}et.$$

This extends the classical concept of a functor because $N\colon \mathcal{C}at \to s\mathcal{S}et$ is fully faithful. More generally, there is the following refinement of this observation.

Lemma 14.3.2. *For categories A, B there is a natural isomorphism of simplicial sets*

$$N(\mathrm{Fun}(A, B)) \cong \mathrm{Fun}(NA, NB).$$

Proof. For $[n] \in \Delta$, there are the following natural bijections

$$\begin{aligned} N(\mathrm{Fun}(A, B))_n &= \hom_{\mathcal{C}at}([n], \mathrm{Fun}(A, B)) && (14.3.1)\\ &\cong \hom_{\mathcal{C}at}([n] \times A, B) && (14.3.2)\\ &\cong \hom_{s\mathcal{S}et}(N([n] \times A), NB) && (14.3.3)\\ &\cong \hom_{s\mathcal{S}et}(\Delta^n \times NA, NB) && (14.3.4)\\ &= \mathrm{Fun}(NA, NB)_n, && (14.3.5) \end{aligned}$$

given by the exponential laws, the fully faithfulness of N, the fact that N preserves products, and the isomorphism $N([n]) \cong \Delta^n$. □

This seemingly naive definition of a functor is actually the good one, as we want to indicate now (but see also Perspective 14.3.5).

Example 14.3.3. Let $A \in \mathcal{C}at$ be an ordinary category and let $\mathcal{M}$ be a locally fibrant simplicial category. By Corollary 14.2.30 we know that $N_\Delta(\mathcal{M})$ is an ∞-category and we want to unravel a bit what it means to have a functor $F\colon NA \to N_\Delta(\mathcal{M})$. The behavior on 0-simplices and 1-simplices amounts to saying that associated to each arrow $x \to y$ in A there is a morphism $Fx \to Fy$ in $\mathcal{M}$. Moreover, given a pair of composable arrows $x \xrightarrow{f} y \xrightarrow{g} z$ in A, i.e., a 2-simplex $\sigma\colon \Delta^2 \to NA$, we obtain a 2-simplex $F(\sigma)\colon \Delta^2 \to N_\Delta(\mathcal{M})$. By Definition 14.2.21 this means that we are given a simplicial functor $C[\Delta^2] \to \mathcal{M}$ which boils down to having a diagram in $\mathcal{M}$ of the form

$$\begin{array}{ccccc} & & Fy & & \\ & {\scriptstyle Ff}\nearrow & \Uparrow & \searrow{\scriptstyle Fg} & \\ Fx & & \xrightarrow[F(g\circ f)]{} & & Fz. \end{array}$$

Thus, the functor F preserves compositions up to specified homotopies.

However, there is still much more information encoded by F, namely all the higher simplices obtained from longer sequences of composable arrows in A. These encode the idea that F is not only a 'functor up to homotopy' but gives us a 'functor up to *coherent* homotopy', i.e., a *homotopy coherent diagram*. For a precise statement see [25] and also Perspective 14.3.5. In order to at least give a partial justification for this claim let us consider a 3-simplex $\tau\colon \Delta^3 \to NA$, i.e., a chain consisting of three composable arrows f, g, and h. By the above we know that the four faces of the 3-simplex $F(\tau)\colon \Delta^3 \to N_\Delta(\mathcal{M})$ altogether give us two different composite homotopies from $F(h\circ g\circ f)$ to $F(h)\circ F(g)\circ F(f)$ as in the boundary of

$$\begin{array}{ccc} F(h\circ g\circ f) & \longrightarrow & F(h)\circ F(g\circ f) \\ \downarrow & & \downarrow \\ F(h\circ g)\circ F(f) & \longrightarrow & F(h)\circ F(g)\circ F(f). \end{array} \tag{14.3.6}$$

The 3-simplex $F(\tau)$ specifies one more homotopy $F(h\circ g\circ f) \to F(h)\circ F(g)\circ F(f)$ together with two 2-homotopies in $\mathcal{M}$ relating this additional homotopy to the two compositions in (14.3.6). This follows from the fact that $\mathrm{Map}_{C[\Delta^3]}(0,3)$ is isomorphic to the product $\Delta^1\times\Delta^1$. (More generally, we have $\mathrm{Map}_{C[\Delta^n]}(i,j) \cong (\Delta^1)^{\times(j-i-1)}$ for $0 \le i < j \le n$.)

Given two ordinary categories A and B, it is easy to check that the functors from A to B together with the natural transformations assemble to a category $\mathrm{Fun}(A,B)$. Moreover, if we have equivalences $A \simeq A'$ and $B \simeq B'$, then there is a canonical equivalence $\mathrm{Fun}(A,B) \simeq \mathrm{Fun}(A',B')$. Similar results also hold true in the world of ∞-categories, but there — compared to the classical context — a proof requires some work. One proof uses certain stability properties of the class of categorical equivalences and the so-called *inner anodyne maps* (see for example [69]). Using these properties, one is able to deduce the following result [82, p. 94].

Proposition 14.3.4. *Let $\mathcal{C}$ and $\mathcal{D}$ be ∞-categories and let K and L be simplicial sets.*

(i) The simplicial set $\mathrm{Fun}(K,\mathcal{C})$ *is an* ∞*-category.*

(ii) If $\mathcal{C}\to\mathcal{D}$ *is an equivalence of* ∞*-categories, then also the induced map* $\mathrm{Fun}(K,\mathcal{C})\to \mathrm{Fun}(K,\mathcal{D})$ *is an equivalence of* ∞*-categories.*

(iii) If $K\to L$ *is a categorical equivalence of simplicial sets, then the induced map* $\mathrm{Fun}(L,\mathcal{C})\to\mathrm{Fun}(K,\mathcal{C})$ *is an equivalence of* ∞*-categories.*

Thus this proposition tells us that the formation of functor ∞-categories is an invariant notion — as it should be the case for all categorical constructions in the world of ∞-categories. On a more conceptual side, this proposition is a consequence of the fact that with respect to the Joyal model structures the categorical product $\times\colon s\mathcal{S}et\times s\mathcal{S}et\to s\mathcal{S}et$ is a left Quillen functor of two variables.

Perspective 14.3.5. The theory of ∞-categories should really be thought of as *homotopy coherent category theory*. In particular, the basic notion of a functor is to model the idea of having a *homotopy coherent diagram*. Homotopy coherent category theory has quite some history and references include [114, 25, 26, 27]. Here we content ourselves by including a short detour only.

To begin with, whenever we have an adjunction $(L,R)\colon\mathcal{C}\rightleftarrows\mathcal{D}$ we obtain a comonad C on $\mathcal{D}$ with $C=LR\colon\mathcal{D}\to\mathcal{D}$ as underlying functor. The structure morphisms of the comonad are induced by the unit and counit of the adjunction, and the necessary relations follow directly from the triangular identities of an adjunction. These comonads can be used to form simplicial resolutions of objects in $\mathcal{D}$ (which play, for example, some role in relative homological algebra; see e.g. [115, §8]). Slightly more precisely, given an object $d\in\mathcal{D}$, the associated simplicial object $C_*(d)$ is given by $C_n(d)=C^{n+1}(d)$. (Of course, there is also a dual part of the story leading to monads and cosimplicial resolutions.)

Now, the adjunction that is in the background of our context is the adjunction between graphs and categories given by the forgetful functor U sending a category to the underlying reflexive graph (a graph in which every vertex has a specified identity edge) and the free category functor F,

$$(F,U)\colon\mathcal{G}raph\rightleftarrows\mathcal{C}at. \tag{14.3.7}$$

Note that both the free category functor and the underlying graph functor preserve the set of objects. Thus, given a category A, the resulting simplicial resolution $C_*A\in\mathrm{Fun}(\Delta^{\mathrm{op}},\mathcal{C}at)$ also has a constant set of objects and hence defines an object $C_*A\in s\mathcal{C}at$, i.e., a simplicially enriched category as opposed to merely a simplicial object in categories.

This allows us to make the following definition which goes back to Vogt [114]. Given a small category A and a simplicial category $\mathcal{M}\in s\mathcal{C}at$, a *homotopy coherent diagram* in $\mathcal{M}$ of shape A is a simplicial functor $C_*A\to\mathcal{M}$. Thus, the set $\mathrm{coh}(A,\mathcal{M})$ of such homotopy coherent diagrams is defined by

$$\mathrm{coh}(A,\mathcal{M})=\hom_{s\mathcal{C}at}(C_*A,\mathcal{M}). \tag{14.3.8}$$

If as a special case we consider $A=[n]$, then the resulting simplicial category $C_*[n]$ is isomorphic to the 'simplicial thickening' $C[\Delta^n]$ introduced at the beginning of §14.2.2. Thus, the coherent nerve $N_\Delta(\mathcal{M})$ of $\mathcal{M}$ is precisely obtained by considering *homotopy coherent chains of composable arrows* in the given $\mathcal{M}$.

More generally, Riehl [96, Theorem 6.7] showed that for every $A \in \mathcal{C}at$ there is a natural isomorphism of simplicial categories $C[NA] \cong C_*(A)$. In particular, for $A \in \mathcal{C}at$ and $\mathcal{M} \in s\mathcal{C}at$ we have a bijection

$$\mathrm{coh}(A, \mathcal{M}) \cong \hom_{s\mathcal{C}at}(C[NA], \mathcal{M}) \cong \hom_{s\mathcal{S}et}(NA, N_\Delta \mathcal{M}).$$

This shows that the seemingly naive Definition 14.3.1 subsumes the concept of homotopy coherent diagrams, and justifies thinking more generally of functors in the sense of that definition as homotopy coherent diagrams.

From a conceptual perspective one might argue that it is not very nice that we defined homotopy coherent diagrams in simplicial categories by means of *specific simplicial resolutions* — namely the simplicial comonad resolutions associated to (14.3.7). In fact, the actual choice of resolution should not matter, and in some modern treatments like [46, §8] homotopy coherent diagrams are defined by means of *more general* simplicial resolutions.

Remark 14.3.6. (i) Proposition 14.3.4 reveals one of the technical advantages of ∞-categories over model categories: ∞-categories are stable under the formation of functor categories without any further assumption. In this respect, model categories are less well-behaved since one has to impose certain conditions on the model categories involved to obtain this stability property: for cofibrantly-generated model categories, associated diagram categories always admit the *projective* model structure [56, p.224], whereas in the case of combinatorial model categories the *projective* and the *injective* structure both always exist on the diagram categories [82, p.824]. Note however that Proposition 14.3.4 is significantly more general since these results only tell us something about model structures on $\mathcal{M}^A$, $A \in \mathcal{C}at$. For example, we never dared to ask for the existence of a 'canonical' model category of functors between two given model categories.

(ii) A further technical advantage of ∞-categories over model categories is the following one. The 'correct' notion of equivalence for model categories is the notion of *Quillen equivalence*. Since, in general, a Quillen equivalence can not be inverted, the equivalence relation generated by this notion is quite complicated: frequently model categories are only Quillen equivalent through a zig-zag of Quillen equivalences pointing in different directions. The appropriate notion of equivalence for ∞-categories is the notion of *categorical equivalence* (Definition 14.2.26). Since ∞-categories are precisely the fibrant and cofibrant objects with respect to the Joyal model structure where categorical equivalences are the weak equivalences, it follows that a zig-zag of categorical equivalences can always be replaced by a single categorical equivalence.

(iii) As we saw in Perspective 14.3.5, the notion of homotopy coherent diagrams is quite easily established in the world of ∞-categories as a map of simplicial sets with the domain given by the nerve of an ordinary category. There will be further advantages of this flavor, i.e., where 'higher coherences' are easily encoded in the setting of ∞-categories. For example the notions of $\mathbb{A}_\infty$- and $\mathbb{E}_\infty$-algebras in monoidal and symmetric monoidal ∞-categories are conveniently introduced in this setting as specific sections of certain

∞-categorical versions of Grothendieck opfibrations, called coCartesian fibrations by Lurie. We will come back to this in §14.5.

We conclude this subsection with a perspective on how to get an ∞-category of ∞-categories and also an underlying ∞-category for arbitrary model categories.

Perspective 14.3.7. We just introduced the notion of a functor between ∞-categories and hence obtain a category of ∞-categories. But to stick more seriously to the ∞-categorical framework, we would like to have an *∞-category $\mathcal{C}at_\infty$ of ∞-categories*. Since non-invertible natural transformations play an important role in classical category theory it would be even nicer to have an $(\infty, 2)$-category of ∞-categories but let us content ourselves with such an ∞-category. Recall that the Joyal model structure is not simplicial, so that we cannot apply Corollary 14.2.30 directly in order to get such a gadget. Luckily, there is the Quillen equivalent *simplicial* model category $s\mathcal{S}et^+$ of so-called *marked simplicial sets* which hence serves the purpose (among many others), and which we want to describe very briefly.

A *marked simplicial set* is a pair $(X, \mathcal{E}_X)$ consisting of a simplicial set X together with a subset $\mathcal{E}_X \subseteq X_1$ of edges containing the degenerate ones. A morphism of marked simplicial sets $(X, \mathcal{E}_X) \to (Y, \mathcal{E}_Y)$ is a morphism $f\colon X \to Y$ of simplicial sets such that $f(\mathcal{E}_X) \subseteq \mathcal{E}_Y$. This defines the category $s\mathcal{S}et^+$ of marked simplicial sets, a category playing a key role in the relative context [82, §§3.1-3.2]. (See also Perspective 14.5.10 for further motivational remarks.)

Degenerate edges, i.e., identity morphisms in an ∞-category are equivalences and it follows that every ∞-category $\mathcal{C}$ gives us a marked simplicial set $\mathcal{C}^\natural$ by marking the equivalences. There is a simplicial model structure on $s\mathcal{S}et^+$ such that the fibrant and cofibrant objects are precisely the marked simplicial sets of the form $\mathcal{C}^\natural$ for some ∞-category $\mathcal{C}$ [82, Prop. 3.1.4.1, Cor. 3.1.4.4]. Thus, the *∞-category of ∞-categories* $\mathcal{C}at_\infty$ can be defined as the underlying ∞-category of $s\mathcal{S}et^+$,

$$\mathcal{C}at_\infty = N_\Delta(s\mathcal{S}et^+_{\mathrm{cf}}).$$

14.3.2 Join construction

The main motivation for us to study the join construction in this context is that it allows us to define the slice construction (see §14.3.3) which in turn is fundamental to the theory of (co)limits (see §14.3.5). The join construction has its origins in classical topology, but here we immediately aim for the simplicial analogue. As a preparation, we recall the classical situation in category theory.

Given categories A and B, one can form a new category $A \star B$, the *join* of A and B, as follows. The class of objects $\mathrm{obj}(A \star B)$ is given by the disjoint union of $\mathrm{obj}(A)$ and $\mathrm{obj}(B)$. For the morphisms, there are the following four different cases

$$\hom_{A\star B}(x,y) = \begin{cases} \hom_A(x,y), & x, y \in A, \\ \hom_B(x,y), & x, y \in B, \\ \quad * \quad, & x \in A, y \in B, \\ \quad \emptyset \quad, & x \in B, y \in A, \end{cases}$$

and the composition is completely determined by requiring that A and B are full subcategories of $A \star B$ in the obvious way. Note that the construction is not symmetric in A and B. To illustrate the join construction, we mention a few basic examples.

Examples 14.3.8. (i) If $A \in \mathcal{C}at$ is arbitrary and if $B = \mathbb{1}$ is the terminal category, then $A^{\triangleright} = A \star \mathbb{1}$ is the *right cone* or *cocone* on A. It is obtained by adjoining a new terminal object ∞ to A, and plays a central role in the study of colimits.

(ii) Dually, if $A = \mathbb{1}$ is terminal and $B \in \mathcal{C}at$ is arbitrary, then $B^{\triangleleft} = \mathbb{1} \star B$ is the *left cone* or *cone* on B. This category is obtained from B by adjoining a new initial object $-\infty$ and is central to the theory of limits.

(iii) As a more specific example, let A be the category occurring in the study of pushout diagrams,

$$\begin{array}{ccc} (0,0) & \longrightarrow & (1,0) \\ \downarrow & & \\ (0,1). & & \end{array}$$

Then the cocone on A is given by the commutative square $A^{\triangleright} \cong [1] \times [1]$, which can of course be depicted as

$$\begin{array}{ccc} (0,0) & \longrightarrow & (1,0) \\ \downarrow & = & \downarrow \\ (0,1) & \longrightarrow & (1,1). \end{array}$$

In particular, a diagram of this shape consists of four morphisms in the target category such that the two compositions agree. Similarly, if B is the diagram occurring in the study of pullback diagrams,

$$\begin{array}{ccc} & & (1,0) \\ & & \downarrow \\ (0,1) & \longrightarrow & (1,1), \end{array}$$

then also the cone $B^{\triangleleft}$ is the square.

The join construction can be extended to simplicial sets. There is a very conceptual approach to this construction as described by Joyal in [69] (basically as a Day convolution construction [29] applied to the ordinal addition). In [69] one can also find many 'elementary relations' about this join construction. Since we only want to rush through the theory of this notion, we instead give the following more direct 'definition'.

Definition 14.3.9. Let K and L be simplicial sets. The *join* $K \star L$ of K and L is the simplicial set defined by

$$(K \star L)_n = K_n \cup L_n \cup \bigcup_{i+1+j=n} K_i \times L_j, \quad n \geq 0.$$

We leave it to the reader to define the structure maps of $K \star L$ and to check that this defines a functor $\star \colon s\mathcal{S}et \times s\mathcal{S}et \to s\mathcal{S}et$. It follows from those details that $K \star L$ comes with canonical inclusions $K \to K \star L$ and $L \to K \star L$. The join operation for simplicial sets is in fact characterized by the following two properties.

Proposition 14.3.10. *(i) The partial join functors $K \star (-) \colon s\mathcal{S}et \to s\mathcal{S}et_{K/}$ and $(-) \star L \colon s\mathcal{S}et \to s\mathcal{S}et_{L/}$ preserve colimits.*

(ii) For the standard simplices we find $\Delta^i \star \Delta^j \cong \Delta^{i+1+j}, i, j \geq 0$, and these isomorphisms are compatible with the obvious inclusions of Δ^i and Δ^j.

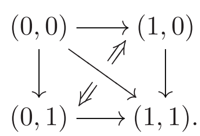
The following lemma is immediate and the proof is recommended as an exercise to those readers who just saw these notions for the first time. The solution to this exercise also suggests how to define the structure maps in Definition 14.3.9.

Lemma 14.3.11. *The nerve is compatible with the join constructions in that there is a natural isomorphism $N(A) \star N(B) \to N(A \star B), A, B \in \mathcal{C}at$.*

To give some examples, we consider the (co)cone constructions and again pushout and pullback diagrams.

Examples 14.3.12. (i) If $K \in s\mathcal{S}et$ is arbitrary and $L = \Delta^0$, then $K^{\triangleright} = K \star \Delta^0$ is the *right cone* or the *cocone* on K. Dually, if $L \in s\mathcal{S}et$ is arbitrary then $L^{\triangleleft} = \Delta^0 \star L$ is the *left cone* or *cone* on L.

(ii) Let $K = \Lambda^2_0$. Using the description of this horn as a pushout, it follows immediately from Proposition 14.3.10 that the cocone $(\Lambda^2_0)^{\triangleright}$ is isomorphic to the square $\square = \Delta^1 \times \Delta^1$, i.e., we have the picture



Note that if $\mathcal{M}$ is a simplicial category then a diagram of this shape in $N_\Delta(\mathcal{M})$ consists of five morphisms and two homotopies as depicted by the diagram. In particular, in general, we do not have a commutative square but only a coherent version thereof. This example can be dualized and we obtain a similar isomorphism $(\Lambda^2_2)^{\triangleleft} \cong \Delta^1 \times \Delta^1$.

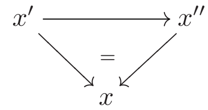
A careful analysis of the join construction allows one to establish the following important properties ([82, Prop. 1.2.8.3] and [82, Cor. 4.2.1.2]).

Proposition 14.3.13. *(i) If $\mathcal{C}$ and $\mathcal{D}$ are ∞-categories, then the join $\mathcal{C} \star \mathcal{D}$ is again an ∞-category.*

(ii) If $F \colon \mathcal{C} \to \mathcal{C}'$ and $G \colon \mathcal{D} \to \mathcal{D}'$ are equivalences of ∞-categories, then also the induced map $F \star G \colon \mathcal{C} \star \mathcal{D} \to \mathcal{C}' \star \mathcal{D}'$ is an equivalence.

14.3.3 Slice construction

We again begin by recalling the more classical situation of ordinary category theory. Given a category B and an object $x \in B$, one can form the *overcategory* $B_{/x}$ where objects are morphisms $x' \to x$ in B with target x. Given two such objects $x' \to x$, $x'' \to x$, a morphism $(x' \to x) \to (x'' \to x)$ in $B_{/x}$ is simply a morphism $x' \to x''$ in B making the obvious triangle



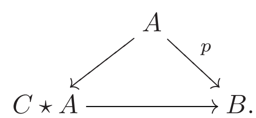
(14.3.9)

commute.

One generalization of this notion is obtained by replacing the object $x\colon \mathbb{1} \to B$ by a more general diagram in B. More precisely, if we are given a functor $p\colon A \to B$, then we can form the *slice category* $B_{/p}$ of *objects over* p or *cones on* p. Thus, an object in $B_{/p}$ is simply a cone on p, i.e., an object $b \in B$ together with a natural transformation from the constant functor with value b to the given p. Morphisms in this category are simply morphisms in B that are compatible with the natural transformations. Using the join construction one can see that the slice construction satisfies a universal property: for any category C, there is a natural bijection

$$\mathrm{Fun}(C, B_{/p}) \cong \mathrm{Fun}_p(C \star A, B),$$

where the right-hand side denotes all functors $C \star A \to B$ making the following triangle commute



To emphasize a bit more that the right-hand side in the above universal property takes certain structure maps into account, we rewrite this as

$$\hom_{\mathcal{C}at}(C, B_{/p}) \cong \hom_{\mathcal{C}at_{A/}}(A \to C \star A, A \xrightarrow{p} B), \quad C \in \mathcal{C}at.$$

Of course this sounds like an unnecessarily complicated reformulation of something very simple. But the point of this reformulation is that it gives us an idea on how to extend these notions to ∞-categories — as it was done by Joyal in [68].

Proposition 14.3.14. *Let $p\colon L \to \mathcal{C}$ be a map of simplicial sets with $\mathcal{C}$ an ∞-category. There is an* ∞-category $\mathcal{C}_{/p}$ *characterized by the following universal property: For every simplicial set K, there is a bijection*

$$\hom_{s\mathcal{S}et}(K, \mathcal{C}_{/p}) \cong \hom_{s\mathcal{S}et_{L/}}(L \to K \star L, L \to \mathcal{C})$$

which is natural in K. The ∞-category $\mathcal{C}_{/p}$ is the ∞-category of cones on p.

To check that there is such a *simplicial set* $\mathcal{C}_{/p}$, one can use the universal property as a definition. More precisely, the special cases of the standard simplices $K = \Delta^n$ give us by the Yoneda lemma a description of the n-simplices of $\mathcal{C}_{/p}$ as

$$(\mathcal{C}_{/p})_n \cong \hom_{s\mathcal{S}et_{L/}}(L \to \Delta^n \star L, L \to \mathcal{C}). \tag{14.3.10}$$

To show that one actually obtains an ∞-*category* requires more work and will not be done here [82, Cor. 2.1.2.2].

Remark 14.3.15. Let $p\colon L \to \mathcal{C}$ be a map of simplicial sets with $\mathcal{C}$ an ∞-category.

(i) In both the classical and the ∞-categorical situation the constructions can be dualized. For example, there is the *∞-category* $\mathcal{C}_{p/}$ *of cocones on* p. The ∞-categories $\mathcal{C}_{p/}$ and $\mathcal{C}_{/p}$ are also referred to as *slice ∞-categories*.

(ii) One can show that the slice construction is an *invariant* notion. First, for every equivalence of ∞-categories $q\colon \mathcal{C} \to \mathcal{D}$ the induced functor $\mathcal{C}_{p/} \to \mathcal{D}_{qp/}$ is an equivalence [82, §2.4.5]. Second, for every equivalence $v\colon L' \to L$ the induced functor $\mathcal{C}_{p/} \to \mathcal{C}_{pv/}$ is an equivalence (as a special case of [82, Prop. 4.1.1.7]).

There is the following result about the compatibility of the nerve $N\colon \mathcal{C}at \to s\mathcal{S}et$ with slice constructions.

Lemma 14.3.16. *For every functor* $p\colon A \to B$ *there is a natural isomorphism of simplicial sets*

$$N(B_{/p}) \cong N(B)_{/N(p)}.$$

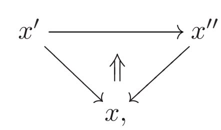
Proof. In simplicial dimension n we have the following chain of natural bijections

$$\begin{aligned} N(B_{/p})_n &= \hom_{\mathcal{C}at}([n], B_{/p}) && (14.3.11)\\ &\cong \hom_{\mathcal{C}at_{A/}}(A \to [n] \star A, A \to B) && (14.3.12)\\ &\cong \hom_{s\mathcal{S}et_{N(A)/}}(N(A) \to N([n] \star A), N(A) \to N(B)) && (14.3.13)\\ &\cong \hom_{s\mathcal{S}et_{N(A)/}}(N(A) \to \Delta^n \star N(A), N(A) \to N(B)) && (14.3.14)\\ &\cong \hom_{s\mathcal{S}et}(\Delta^n, N(B)_{/N(p)}), && (14.3.15) \end{aligned}$$

where in step three we used Lemma 14.3.11 and it is obvious which canonical isomorphisms were used in the remaining steps. □

Example 14.3.17. Let $\mathcal{C}$ be an ∞-category and let $x \in \mathcal{C}$ be an object, classified by the map $\kappa_x\colon \Delta^0 \to \mathcal{C}$. Then the ∞-category $\mathcal{C}_{/\kappa_x}$ is called the *∞-category of objects over* x, and is simply denoted by $\mathcal{C}_{/x}$. Dually, the ∞-category $\mathcal{C}_{\kappa_x/}$ is called the *∞-category of objects under* x, and is denoted by $\mathcal{C}_{x/}$.

Let us spell out some of the details about $\mathcal{C}_{/x}$ in order to convince ourselves that this is the expected coherent version of the corresponding classical construction. By (14.3.10), an object in $\mathcal{C}_{/x}$ is a map $f\colon \Delta^1 \cong \Delta^0 \star \Delta^0 \to \mathcal{C}$ such that $d_0(f) = x$, i.e., we are given a map in $\mathcal{C}$ with target x. Similarly, (14.3.10) tells us that a map in $\mathcal{C}_{/x}$ is given by a 2-simplex $\sigma\colon \Delta^2 \cong \Delta^1 \star \Delta^0 \to \mathcal{C}$ such that $\sigma(2) = x$. In the case that $\mathcal{C}$ is the coherent nerve of a (locally fibrant) simplicial category, the picture of σ is



indicating that we obtained the coherent version of (14.3.9) we were aiming for.

14.3.4 Final and initial objects

We now come to the ∞-categorical variant of final and initial objects. In classical category theory, final objects are characterized by the property that for all objects there is a unique morphism to the final object. In these notes, we take a slightly different approach to the ∞-categorical generalization (but see Proposition 14.3.19). The following definition is not the original one of Joyal [68], but it is an equivalent one as shown in [68].

Definition 14.3.18. An object $x \in \mathcal{C}$ of an ∞-category $\mathcal{C}$ is a *final object* if the canonical map $\mathcal{C}_{/x} \to \mathcal{C}$ is an acyclic fibration of simplicial sets.

Thus, an object is final if 'forgetting that we lived above it does not result in a loss of information'. This notion can be reformulated in the following more usual way.

Proposition 14.3.19. *The following are equivalent for an object x of an ∞-category $\mathcal{C}$.*

(i) The object x is final.

(ii) The mapping spaces $\mathrm{Map}_{\mathcal{C}}(x', x)$ are acyclic Kan complexes for all $x' \in \mathcal{C}$.

(iii) Every simplicial sphere $\alpha\colon \partial\Delta^n \to \mathcal{C}$ such that $\alpha(n) = x$ can be filled to an entire n-simplex $\Delta^n \to \mathcal{C}$.

We expect a replacement for the fact that in classical category theory any two terminal objects are canonically isomorphic, and for this purpose we first make precise the notion of a *full* subcategory of an ∞-category. There is a more general notion of subcategories of an ∞-category but since we will not need this additional generality we stick to full subcategories. Given objects $\mathcal{D}_0 \subseteq \mathcal{C}_0$ in an ∞-category $\mathcal{C}$, let $\mathcal{D} \subseteq \mathcal{C}$ be the simplicial subset consisting precisely of those simplices $\Delta^n \to \mathcal{C}$ that have the property that all vertices belong to $\mathcal{D}_0$. It is immediate that $\mathcal{D}$ is again an ∞-category, the *full subcategory of $\mathcal{C}$ spanned by $\mathcal{D}_0$*. Obviously, $\mathcal{D}$ comes with an inclusion $\mathcal{D} \to \mathcal{C}$.

Corollary 14.3.20. *Let $\mathcal{C}$ be an ∞-category and let $\mathcal{D} \subseteq \mathcal{C}$ be the full subcategory spanned by the final objects of $\mathcal{C}$. The ∞-category $\mathcal{D}$ is either empty or a contractible Kan complex.*

Proof. This is immediate from Proposition 14.3.19. □

Remark 14.3.21. The conclusion of Corollary 14.3.20 is typical for uniqueness statements in ∞-category theory: It states that a space 'parametrizing universal objects' is empty or a contractible Kan complex. In classical category theory, if universal objects exist then they are unique up to unique isomorphism. This can be reformulated by saying that if universal objects exist then the category of such is a *contractible groupoid*: the existence of comparison maps shows that it is a connected category, while the uniqueness of these maps tells us that it is a groupoid with no nontrivial endomorphisms. Thus, the possibly non-trivial π_0 and π_1 both vanish. Now, with Corollary 14.2.18 in mind we see that the two uniqueness statements are morally very similar.

One example of such a uniqueness statement was already obtained in the discussion of Theorem 14.2.14, namely the space of compositions of two composable arrows is a contractible Kan complex. With Corollary 14.3.20 we have a further example of such a statement. In §14.3.5 we will define (co)limits as universal objects of certain slice categories and will hence again obtain such uniqueness statements. Let us mention that uniqueness results of this kind are also ubiquitous in the theory of model categories. Compare for example to [56] where many categories of choices (for example, cofibrant replacements) are shown to have contractible nerves.

We conclude by a statement about the compatibility with the nerve construction, but leave the proof to the reader.

Lemma 14.3.22. *Let A be a category. An object $a \in A$ is final if and only if $a \in N(A)$ is final.*

14.3.5 Limits and colimits

Having the basic categorical notions at our disposal, we are ready to talk about (co)limits in the framework of ∞-categories. Recall that the colimit of an ordinary functor $p\colon A \to B$ consists of an object $\mathrm{colim}_A\, p$ in B together with a universal cocone. Said differently, such a pair is equivalently specified by an initial object in the category $B_{p/}$ of cocones on p. This definition can now readily be extended to ∞-categories (see [68]).

Definition 14.3.23. Let K be a simplicial set and let $\mathcal{C}$ be an ∞-category. A *colimit* of a diagram $p\colon K \to \mathcal{C}$ is an initial object in $\mathcal{C}_{p/}$. An ∞-category is *cocomplete* if it admits colimits of all small diagrams. Dually, we define *limits* of diagrams and *complete* ∞-categories.

By Corollary 14.3.20 for every $p\colon K \to \mathcal{C}$ the full subcategory $\mathcal{D} \subseteq \mathcal{C}_{p/}$ spanned by the colimits of p is either empty or a contractible Kan complex. Thus, if a colimit exists, then it is unique up to a contractible choice.

Remark 14.3.24. (i) The nerve functor $N\colon \mathcal{C}at \to s\mathcal{S}et$ is compatible with the notion of (co)limits.

(ii) The notions of limits and colimits are invariant notions.

One justification for Definition 14.3.23 is that it extends the classical theory of (co)limits. Although this is certainly a convenient aspect of the notion, there is an additional justification which is more relevant for our purposes (as discussed by Lurie in [82, Theorem 4.2.4.1]). Namely, there is a precise meaning in that these notions of (co)limits coincides with the notion of *homotopy (co)limits* in simplicial categories. Joyal was fully aware of this in [68]. References to various aspects of the theory of homotopy (co)limits in other contexts include [22], [114], [56], [36], [105], and [32]. We follow Joyal and Lurie and simply speak of *(co)limits* instead of *homotopy (co)limits*, leading to a lighter terminology.

As a special case we have the following definition; see again Examples 14.3.12.

Definition 14.3.25. Let $\mathcal{C}$ be an ∞-category and let $q\colon \square \to \mathcal{C}$ be a square.

(i) The square q is a *pushout* if $q\colon (\Lambda^2_0)^{\triangleright} \to \mathcal{C}$ is a colimiting cocone.

(ii) The square q is a *pullback* if $q\colon (\Lambda^2_2)^{\triangleleft} \to \mathcal{C}$ is a limiting cone.

Having a good definition of (co)limits available, one would now like to establish batteries of techniques to play with these notions. Many classical facts from the calculus of ordinary (co)limits can be extended to the context of ∞-categories, although the proofs of even fundamental statements are significantly harder in this framework. Nevertheless, such an extension was achieved to an impressive extent by Lurie in [82]. Instead of pursuing this any further, we conclude this section by mentioning the following important theorem [82, Cor. 4.2.4.8].

Theorem 14.3.26. *The underlying ∞-category of a simplicial, combinatorial model category is complete and cocomplete.*

14.4 Presentable ∞-categories and the relation to model categories

In this section we give a short introduction to *presentable ∞-categories*, a class of ∞-categories having very good formal properties. For example, the adjoint functor theorem of Freyd, in this context, takes the form that a functor is a left adjoint if and only if it preserves colimits, and there is also a result for right adjoint functors. Besides these good formal properties, it turns out that a good deal of mathematics is encoded by presentable ∞-categories. Many typical ∞-categories showing up in nature, e.g., in algebra, topology, or (derived) algebraic geometry are presentable.

The main reason for us to include a short discussion of presentable ∞-categories is that they play an essential role in Lurie's treatment of the *smash product* on the ∞-category of spectra (see §14.6). Moreover, they allow us to indicate a more precise relation between ∞-categories and model categories (see §14.4.2 and, in particular, Theorem 14.4.15). The reader can skip this section on a first reading since this will only affect the understanding of some of the statements in §14.6.

The theory of presentable ∞-categories has (at least) two precursors; *locally presentable categories* in the classical context as well as *combinatorial model categories* in the homotopical framework. We emphasize that, in all these three cases, there are two main ideas in the background, which help organize the material.

(i) The idea of passing from a small category to a cocomplete category in a universal way is realized by categories of presheaves in the classical case, and by model categories and ∞-categories of simplicial presheaves in the remaining two. One can think of this passage as some kind of *free generation.*

(ii) All locally presentable categories can be obtained as certain localizations of presheaf categories, and similarly in the homotopical contexts using suitable notions of (Bousfield) localizations. The passage to (Bousfield) localizations can be thought of as a way of *imposing relations.*

14.4.1 Locally presentable categories

In this subsection we give a short review of the theory of locally presentable categories. For more details we refer to [42, 87, 2] or [18, 19].

To begin with, let us again take up the idea that passage to a category of presheaves (with values in sets) can be regarded as a form of cocompletion. If A is a small category, then we denote by $y\colon A \to \mathrm{Fun}(A^{\mathrm{op}}, \mathcal{S}et)$ the Yoneda embedding which sends $a \in A$ to the represented presheaf $\hom_A(-, a)$. Associated to each presheaf $X\colon A^{\mathrm{op}} \to \mathcal{S}et$ there is the comma category (y/X). Objects of (y/X) are pairs (a, α) consisting of an object $a \in A$ and a natural transformation $\alpha\colon y(a) \to X$, and a morphism $f\colon (a, \alpha) \to (a', \alpha')$ is a morphism $f\colon a \to a'$ in A such that the diagram

$$\begin{array}{ccc} y(a) & \xrightarrow{y(f)} & y(a') \\ & {\scriptstyle\alpha}\searrow \quad \swarrow{\scriptstyle\alpha'} & \\ & X & \end{array}$$

commutes. (Note that the Yoneda lemma implies that this category is isomorphic to the *category of elements* of X.) The category (y/X) comes with a projection functor $p\colon (y/X) \to A$ sending a pair (a, α) to a, and we can consider the composition

$$(y/X) \xrightarrow{p} A \xrightarrow{y} \mathrm{Fun}(A^{\mathrm{op}}, \mathcal{S}et).$$

Using these diagrams one can make precise our idea that presheaves on small categories are canonically colimits of representable ones (see for example [85, p.76]).

Proposition 14.4.1. *Let A be a small category and let $X\colon A^{\mathrm{op}} \to \mathcal{S}et$ be a set-valued presheaf. There is a canonical isomorphism*

$$\mathrm{colim}_{(y/X)}\, y \circ p \cong X.$$

Thus, we think of the passage to presheaves as a *cocompletion* or, more informally, as a *free generation*. In order to make a more precise statement, let $\mathrm{Fun}^{\mathrm{L}}(-, -)$ denote the category of colimit-preserving functors. The following result seems to go back to Ulmer [110, Rmk. 2.29].

Theorem 14.4.2. *Let A be a small category and let $\mathcal{C}$ be a cocomplete category. The restriction along the Yoneda embedding $y\colon A \to \mathrm{Fun}(A^{\mathrm{op}}, \mathcal{S}et)$ induces an equivalence of categories*

$$y^*\colon\ \mathrm{Fun}^{\mathrm{L}}(\mathrm{Fun}(A^{\mathrm{op}}, \mathcal{S}et), \mathcal{C}) \xrightarrow{\sim} \mathrm{Fun}(A, \mathcal{C}).$$

To motivate the notion of a locally presentable category we recall Freyd's Adjoint Functor Theorem. Let $F\colon \mathcal{C} \to \mathcal{D}$ be a functor between cocomplete categories. It is obvious that if F is a left adjoint, then F preserves all colimits, but, in general, the converse is not true. The celebrated *Adjoint Functor Theorem* of Freyd (see [40, pp. 84-86] or [85, p. 121]) gives *necessary and sufficient* conditions for the existence of a right adjoint. Slightly more precisely, the following are equivalent for a functor $F\colon \mathcal{C} \to \mathcal{D}$ between cocomplete categories.

(i) The functor F is a left adjoint.

(ii) The functor F preserves colimits and the *solution-set-condition* is satisfied.

Without going into detail, let us only mention that the solution-set-condition states that a certain *class* of arrows turns out to be small enough to actually form a *set*. Hence, one can imagine that this condition is automatically satisfied if we impose some 'smallness conditions' on the categories. Recall that a *small, cocomplete* category is necessarily a poset ([85, p. 114]). Thus, in order to not rule out interesting examples, it is essential that these 'smallness conditions' are chosen in a smart way.

Definition 14.4.3. A category is *locally presentable* if it is cocomplete and accessible.

The accessibility assumption in this definition is the smallness assumption alluded to above. The idea is that a category $\mathcal{C}$ is accessible if it admits certain filtered colimits and if it is formally determined by some small subcategory $\mathcal{D}$ consisting of small objects. To make this more precise, let us consider a regular cardinal number κ. A category $\mathcal{C}$ is κ*-accessible* if $\mathcal{C}$ admits κ-filtered colimits and if there is a small subcategory $\mathcal{D} \subseteq \mathcal{C}$ such that the following two conditions are satisfied.

(i) Every object of $\mathcal{C}$ can be canonically written as a κ-filtered colimit of objects in $\mathcal{D}$.

(ii) The set-valued functors $\hom_{\mathcal{C}}(d, -)\colon \mathcal{C} \to \mathcal{S}et,\ d \in \mathcal{D}$, preserve κ-filtered colimits — expressing the idea that objects in $\mathcal{D}$ are small.

We say that a category is *accessible* if it is κ-accessible for some κ. Similarly, a functor $F\colon \mathcal{C} \to \mathcal{D}$ is κ*-accessible* if $\mathcal{C}$ and $\mathcal{D}$ admit κ-filtered colimits and if they are preserved by F. Finally, a functor is *accessible* if it is κ-accessible for some κ.

For more details on the rich theory of accessible and locally presentable categories we again refer to [42, 87, 2] or [18, 19]. But let us emphasize that every locally presentable category is also a *complete* category. To indicate the ubiquity of locally presentable categories, we include the following list of examples.

Examples 14.4.4. (i) The category $\mathcal{S}et$ of sets is locally presentable.

(ii) If A is a small category, then the category $\mathrm{Fun}(A^{\mathrm{op}}, \mathcal{S}et)$ of presheaves on A is locally presentable. In particular, the category $s\mathcal{S}et$ of simplicial sets is locally presentable.

(iii) For a ring R the categories $\mathrm{Mod}(R)$ of R-modules and $\mathrm{Ch}(R)$ of chain complexes over R are locally presentable.

(iv) Recall that an abelian category with exact filtered colimits is *Grothendieck abelian* (see [52] and [38, §14]) if it admits a generator. It can be shown that an abelian category with exact filtered colimits is Grothendieck if and only if it is locally presentable. Important examples are given by categories of quasi-coherent $\mathcal{O}_X$-modules on any scheme X.

(v) Categories of modules of (multi-sorted) algebraic theories are locally presentable [3, §6].

(vi) If $T\colon \mathcal{C} \to \mathcal{C}$ is an accessible monad on a locally presentable category, then the category of T-algebras is locally presentable [2].

(vii) Every Grothendieck topos [6, 20, 86] is locally presentable.

(viii) The category $\mathcal{C}at$ of small categories is locally presentable. More generally, if $\mathcal{M}$ is a locally presentable, symmetric monoidal category, then the category $\mathcal{C}at_{\mathcal{M}}$ of $\mathcal{M}$-enriched categories is locally presentable (see [75]).

(ix) The category $\mathcal{T}op$ of topological spaces is *not* locally presentable, but this can be fixed by passing to the Quillen equivalent category of Δ-generated spaces. It is shown in [39] that this category is locally presentable.

Here is the simplified form of the adjoint functor theorem for locally presentable categories.

Theorem 14.4.5. *(i) A functor between locally presentable categories is a left adjoint if and only if it preserves colimits.*

(ii) A functor between locally presentable categories is a right adjoint if and only if it preserves limits and is accessible.

Other forms of representability results have appeared in various settings and these include the classical Brown representability theorem in stable homotopy theory [24], the Brown representability results for triangulated categories [90, §8], Watt's theorems in homological algebra [100, §5.3], representability theorems for Grothendieck categories [72, p.186], as well as versions of Watt's theorem in homotopical algebra [62].

It turns out that up to equivalence *all* locally presentable categories can be obtained as certain localizations of presheaf categories. To state this more precisely, let us begin by recalling that a *reflective localization* is an adjunction $(L, R)\colon \mathcal{C} \rightleftarrows \mathcal{D}$ such that the right adjoint R is fully faithful (see for example [18, §3.5 and §5.3]). In this situation it follows that $\mathcal{D}$ is equivalent to the localization $\mathcal{C}[S^{-1}]$ where S is the class of morphisms in $\mathcal{C}$ that are sent to isomorphisms by L. This is nicely illustrated by the following example.

Example 14.4.6. Let $\delta\colon \mathcal{S}et \to s\mathcal{S}et$ be the discrete simplicial set functor. The adjunction $(\pi_0, \delta)\colon s\mathcal{S}et \rightleftarrows \mathcal{S}et$ is a reflective localization. Thus, if in $s\mathcal{S}et$ we invert all maps inducing isomorphisms on π_0 then we simply get (discrete simplicial) sets.

We want to emphasize that the typical terminology from *Bousfield localization theory* [21] makes already perfectly well sense in this classical context, and that reflective localizations can be nicely described using that terminology (see Proposition 14.4.8).

Definition 14.4.7. Let $\mathcal{C}$ be a category and let S be a class of morphisms in $\mathcal{C}$.

(i) An object c in $\mathcal{C}$ is *S-local* if $f^*\colon \hom_{\mathcal{C}}(c_2, c) \to \hom_{\mathcal{C}}(c_1, c)$ is a bijection for all $f\colon c_1 \to c_2$ in S.

(ii) A morphism $f\colon c_1 \to c_2$ in $\mathcal{C}$ is an *S-local equivalence* if for all S-local objects $c \in \mathcal{C}$ the map $f^*\colon \hom_{\mathcal{C}}(c_2, c) \to \hom_{\mathcal{C}}(c_1, c)$ is a bijection.

The following is straightforward.

Proposition 14.4.8. *Let $(L, R)\colon \mathcal{C} \rightleftarrows \mathcal{D}$ be a reflective localization and let S be the morphisms in $\mathcal{C}$ that are inverted by L.*

(i) The essential image of R consists precisely of the S-local objects.

(ii) The S-local equivalences are precisely the maps in S.

With this preparation, we now state the 'classification result of locally presentable categories' (see for example [2]). Let us recall that an *accessible, reflective localization* is a reflective localization (L, R) such that the right adjoint R is accessible, i.e., preserves κ-filtered colimits for a sufficiently large regular cardinal κ.

Theorem 14.4.9. *A category is locally presentable if and only if it is equivalent to an accessible, reflective localization of* $\mathrm{Fun}(A^{\mathrm{op}}, \mathcal{S}et)$ *for some small category A.*

Thus, a category is locally presentable if and only if it can be obtained from a small category by a free generation (cocompletion) followed by imposing relations in a suitable way (accessible reflective localization). In §14.4.2 we will see that there are variants of Theorem 14.4.2 and Theorem 14.4.9 valid in the context of ∞-categories (and also in the context of model categories).

14.4.2 Presentable ∞-categories

One reason why *set-valued presheaves* play such a central role in classical category theory is that questions about the existence of universal constructions can be reformulated as representability questions for certain set-valued presheaves. In higher category theory, the representable functors take values in the category of simplicial sets. So one might expect that the central role is now taken by *simplicial presheaf categories*. In the world of model categories, these were intensively studied by Jardine (see for example [65]).

We now discuss simplicial presheaves in the framework of ∞-categories. Recall that we denote the ∞-category of spaces by $\mathcal{S} = N_\Delta(\mathcal{K}an)$ (see Examples 14.2.31). Given a simplicial set K, the ∞-category $\mathcal{P}(K)$ of *(simplicial) presheaves* on K is defined by

$$\mathcal{P}(K) = \mathrm{Fun}(K^{\mathrm{op}}, \mathcal{S}).$$

It follows from Proposition 14.3.4 that $\mathcal{P}(K)$ is an ∞-category.

In order to define the Yoneda embedding we recall that we have the adjunction $(C[-], N_\Delta)\colon s\mathcal{S}et \rightleftarrows s\mathcal{C}at$; see (14.2.8). The simplicial category $C[K]$ of an arbitrary simplicial set K is not locally fibrant. This can be fixed by choosing a product-preserving fibrant replacement functor for $s\mathcal{S}et$, like the one induced by the Quillen equivalence $s\mathcal{S}et \rightleftarrows \mathcal{T}op$ or Kan's Ex^∞-functor; see [71]. Composing the mapping space functor of $C[K]$ with such a replacement functor, we obtain a simplicial functor $C[K]^{\mathrm{op}} \times C[K] \to \mathcal{K}an$. Combining this with the canonical map $C[K^{\mathrm{op}} \times K] \to C[K]^{\mathrm{op}} \times C[K]$ and passing to adjoints, this yields a map $K^{\mathrm{op}} \times K \to N_\Delta(\mathcal{K}an) = \mathcal{S}$. The exponential law finally gives us the *Yoneda embedding*

$$y\colon K \to \mathrm{Fun}(K^{\mathrm{op}}, \mathcal{S}) = \mathcal{P}(K), \tag{14.4.1}$$

which can be shown to be fully faithful ([82, Prop. 5.1.3.1]).

The Yoneda embedding provides a model for the cocompletion. In order to make this precise let us introduce the following notation. Given ∞-categories $\mathcal{C}$ and $\mathcal{D}$, we denote by

$$\mathrm{Fun}^{\mathrm{L}}(\mathcal{C}, \mathcal{D}) \subseteq \mathrm{Fun}(\mathcal{C}, \mathcal{D})$$

the full subcategory spanned by the *colimit-preserving* functors. Lurie establishes the following result ([82, Thm. 5.1.5.6]) which is an ∞-categorical version of Theorem 14.4.2. (See also [31] and [99] for a variant in the language of model categories.)

Theorem 14.4.10. *Let K be a small simplicial set and let $\mathcal{C}$ be a cocomplete ∞-category. Restriction along the Yoneda embedding* (14.4.1) *induces an equivalence of ∞-categories*

$$\mathrm{Fun}^{\mathrm{L}}(\mathcal{P}(K), \mathcal{C}) \xrightarrow{\sim} \mathrm{Fun}(K, \mathcal{C}).$$

In particular, the *∞-category $\mathcal{S}$ of spaces is freely generated by* $\Delta^0 \in \mathcal{S}$ in the sense of the following corollary.

Corollary 14.4.11. *For any cocomplete ∞-category $\mathcal{C}$ the evaluation on the 0-simplex $\Delta^0 \in \mathcal{S}$ induces an equivalence of ∞-categories*

$$\mathrm{Fun}^{\mathrm{L}}(\mathcal{S}, \mathcal{C}) \xrightarrow{\sim} \mathcal{C}.$$

In [82, §4] Lurie establishes batteries of techniques which allow us to manipulate (co)limits in the context of ∞-categories. In [82, §5] more advanced notions are introduced, including filtered colimits and small objects. Once the basic notions are in place, more advanced concepts from classical category theory can be formally extended to the ∞-categorical framework. We want to emphasize once more that although impressively many *statements* are still true in this more general framework, at least at present the *proofs*, in general, are more involved.

Definition 14.4.12. An ∞-category is *presentable* if it is cocomplete and accessible.

As in ordinary category theory, it can be shown that a presentable ∞-category is automatically also complete ([82, Cor. 5.5.2.4]). There are also ∞-categorical variants of Freyd's special adjoint functor theorem (see Theorem 14.4.5). A first step towards this of course consists of making precise the notion of an adjunction between two ∞-categories. We will not get into this here and instead refer the reader to [82, p.337] and [80, §3]. With this concept at hand, there is the following result due to Lurie ([82, Cor. 5.5.2.9]).

Theorem 14.4.13. *(i) A functor between presentable ∞-categories is a left adjoint if and only if it preserves colimits.*

(ii) A functor between presentable ∞-categories is a right adjoint if and only if it preserves limits and is accessible.

Lurie also establishes a classification result for presentable ∞-categories similar to the one given by Theorem 14.4.9. Using the concept of an adjunction of ∞-categories, one can extend additional concepts from ordinary category theory to the context of ∞-categories. A functor between two ∞-categories is a *reflective localization* if it admits a fully faithful right adjoint, and there is also the notion of an *accessible, reflective localization.* With these definitions at hand, the classification result for presentable ∞-categories takes the following form as proved by Lurie as [82, Thm. 5.5.1.1]. In that section, Lurie attributes the result to Simpson [106].

Theorem 14.4.14. *For an ∞-category $\mathcal{C}$ the following are equivalent.*

(i) The ∞-category $\mathcal{C}$ is presentable.

(ii) There is a small ∞-category $\mathcal{D}$ such that $\mathcal{C}$ is an accessible, reflective localization of $\mathcal{P}(\mathcal{D})$.

In light of Theorem 14.4.14, if one wants to understand presentable ∞-categories then it seems to be important to have good control over accessible localizations of ∞-categories of presheaves or, more generally, of presentable ∞-categories. Given a reflective localization $(L, R)\colon \mathcal{C} \rightleftarrows \mathcal{D}$ let us also write $L\colon \mathcal{C} \to \mathcal{C}$ for the *localization functor* $\mathcal{C} \to \mathcal{D} \to \mathcal{C}$, and let S_L be those maps in $\mathcal{C}$ that are sent to equivalences by L. An object $x \in \mathcal{C}$ is S_L*-local* if all maps $f^*\colon \operatorname{Map}_{\mathcal{C}}(z, x) \to \operatorname{Map}_{\mathcal{C}}(y, x)$ induced by $f \in S_L$ are weak equivalences. One can then show that the essential image of L consists precisely of the S_L-local objects (see [82, Prop. 5.5.4.2]).

In particular, an accessible localization L of a presentable ∞-category $\mathcal{C}$ is thus completely determined by the class S_L. In that case, the class S_L is closed under the formation of colimits in S_L as a subcategory of $\mathcal{C}^{[1]}$, is stable under the formation of retracts, contains the equivalences, satisfies the 2-out-of-3 property (with respect to 2-simplices), and is stable under cobase change. Lurie calls a class of morphisms with these closure properties *strongly saturated*. Intersections of strongly saturated classes are again strongly saturated as is the class of all morphisms. Thus, for each class T of morphisms in $\mathcal{C}$ there is a smallest strongly saturated class $\bar{T}$ that contains T. We call a strongly saturated class S *of small generation* if there is a *subset* $T \subseteq S$ such that $S = \bar{T}$. Lurie establishes the wonderful fact that a strongly saturated class S in a presentable ∞-category $\mathcal{C}$ is of small generation if and only if there is an accessible localization $L\colon \mathcal{C} \to \mathcal{C}$ such that $S = S_L$ (see [82, §5.5.4]).

Lurie then shows that in the case of simplicial presheaves the localization theory of ∞-categories interacts nicely with the localizaton theory of certain associated model categories (see [82, Appendix 3.7]). Having established all this theory, he is then able to build on Dugger's work [30] in order to deduce the following result (see [82, Appendix 3]). Recall that a model category is combinatorial if and only if it is cofibrantly generated and the underlying category is locally presentable (see for instance [12, 30, 26, 56, 98]).

Theorem 14.4.15. *An ∞-category $\mathcal{C}$ is presentable if and only if there is a combinatorial, simplicial model category $\mathcal{M}$ such that $\mathcal{C}$ is equivalent to the underlying ∞-category $N_\Delta(\mathcal{M}_{\mathrm{cf}})$.*

More generally, the underlying ∞-category of a combinatorial model category is presentable by [83, Prop. 1.3.4.22].

In this section we considered the 'theme of locally presentable categories' in three different frameworks, namely in ordinary category theory, in ∞-category theory, and briefly in model category theory. We conclude this section with a short perspective in which we give an outlook on a similar picture for *Grothendieck topoi* (and we refer to [82, §5.5.8] and [28, 44] for a similar picture on *algebraic categories*).

Perspective 14.4.16. Let us recall that a *Grothendieck topos* can be defined as a category equivalent to a category of set-valued sheaves on a Grothendieck site; references for this vast subject include the original [6, 7, 8] and the monographs [20, 86]. It turns out that Grothendieck topoi admit a different characterization as suitable localizations of presheaf categories. The 'theme of sheaves and topoi' was taken up again in homotopical frameworks, both using the language of model categories (see for example [65, 66, 33, 109, 95]) as well as in the ∞-categorical picture [82, §§6-7].

14.5 Monoidal and symmetric monoidal ∞-categories

In this section we give an introduction to the theory of monoidal ∞-categories. Let us recall that a monoidal structure on a category $\mathcal{M}$ consists of a monoidal pairing $\otimes\colon \mathcal{M}\times\mathcal{M}\to\mathcal{M}$ and a monoidal unit $\mathbb{S}\in\mathcal{M}$ together with three natural isomorphisms, namely the associativity and left and right unitality constraints. Moreover, to obtain the notion we have in mind we have to impose certain compatibility assumptions. In particular, we have to ask axiomatically that the two ways of comparing four-fold products $\big((X\otimes Y)\otimes Z\big)\otimes W$ and $X\otimes\big(Y\otimes(Z\otimes W)\big)$ as described by the boundary in

$$\begin{array}{ccccc} \big((X\otimes Y)\otimes Z\big)\otimes W & & \longrightarrow & & (X\otimes Y)\otimes(Z\otimes W) \\ \downarrow & & & & \downarrow \\ \big(X\otimes(Y\otimes Z)\big)\otimes W & & & & X\otimes\big(Y\otimes(Z\otimes W)\big) \\ & \searrow & & \nearrow & \\ & & X\otimes\big((Y\otimes Z)\otimes W\big) & & \end{array} \tag{14.5.1}$$

coincide. This leads to the *classical presentation* of monoidal categories ([84, 73]).

In the ∞-categorical setting this presentation will not model the good notion anymore. Instead one expects a monoidal structure on an ∞-category to be some kind of a monoidal pairing that is coherently associative and unital in the sense of $\mathbb{A}_\infty$-multiplications. In particular, it is therefore insufficient to consider Mac Lane's pentagon (14.5.1) and instead one expects that all Stasheff associahedra with their complicated combinatorics play a key role (see [108] or [89, §I.1.6 and §II.1.6]).

Luckily, all this structure does not have to be made explicit if one chooses a *different presentation* of ordinary monoidal categories, namely, as suitable Grothendieck opfibrations over Δ^{op} (the same observation but in the context of $\mathbb{A}_\infty$-spaces motivated Adams to refer to the category Δ as a 'storehouse of formulas' [5]). This different presentation extends more readily to ∞-categories and is obtained by combining two main ideas, namely

(i) the Segal perspective on $\mathbb{A}_\infty$-monoids as 'special simplicial objects' and

(ii) the Grothendieck construction applied to category-valued functors.

The passage to *symmetric* monoidal ∞-categories then essentially amounts to a change of combinatorics, i.e., one replaces the category Δ^{op} by a skeleton $\mathcal{F}\mathrm{in}$ of the category of finite pointed sets.

14.5.1 Monoidal categories via Grothendieck opfibrations

Let us consider the following classical situation. Let $p\colon \mathcal{C} \to \mathcal{D}$ be a functor between ordinary categories and let $d \in \mathcal{D}$ be an object. We denote by $\mathcal{C}_d$, the *fiber* of p over d, defined by the following pullback diagram

$$\begin{array}{ccc} \mathcal{C}_d & \longrightarrow & \mathcal{C} \\ \downarrow & \lrcorner & \downarrow{\scriptstyle p} \\ [0] & \xrightarrow[d]{} & \mathcal{D}. \end{array}$$

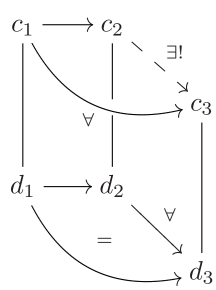
Thus, $\mathcal{C}_d \subseteq \mathcal{C}$ is the (in general not full) subcategory given by the objects $c \in \mathcal{C}$ such that $p(c) = d$ and those morphisms in $\mathcal{C}$ that are sent to id_d . A functor $p\colon \mathcal{C} \to \mathcal{D}$ can always be thought of as a collection of categories $\mathcal{C}_d$ parametrized by the objects in $\mathcal{D}$.

Our first aim is to find conditions which ensure that the fiber $\mathcal{C}_d$ *depends covariantly* on the object $d \in \mathcal{D}$.

Definition 14.5.1. Let $p\colon \mathcal{C} \to \mathcal{D}$ be a functor and let $f\colon c_1 \to c_2$ be a morphism in $\mathcal{C}$ with image $p(f) = \alpha\colon d_1 \to d_2$. The morphism f is *p-coCartesian* or a *p-coCartesian lift* of α if it has the following property: For every $h\colon c_1 \to c_3$ in $\mathcal{C}$ with image $\gamma = p(h)\colon d_1 \to d_3$ and every $\beta\colon d_2 \to d_3$ such that $\gamma = \beta \circ \alpha$ there is a unique $g\colon c_2 \to c_3$ in $\mathcal{C}$ such that

$$\beta = p(g) \qquad \text{and} \qquad h = g \circ f.$$

Thus, the defining property of a p-coCartesian arrow can be described by the following diagram



in which the vertical lines indicate the effect of an application of $p\colon \mathcal{C} \to \mathcal{D}$. Note that a morphism $f\colon c_1 \to c_2$ is p-coCartesian if and only if the diagram

$$\begin{array}{ccc} \hom_{\mathcal{C}}(c_2, c_3) & \xrightarrow{f^*} & \hom_{\mathcal{C}}(c_1, c_3) \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle p} \\ \hom_{\mathcal{D}}(p(c_2), p(c_3)) & \xrightarrow[p(f)^*]{} & \hom_{\mathcal{D}}(p(c_1), p(c_3)) \end{array} \tag{14.5.2}$$

is a pullback diagram for all objects $c_3 \in \mathcal{C}$. This slightly cryptic reformulation of Definition 14.5.1 will have its uses when it comes to extending these notions to the setting of ∞-categories (see Lemma 14.5.5 and Definition 14.5.6). To develop some feeling for the notion we recommend the reader to give the easy proof of the following lemma.

Lemma 14.5.2. *Let $f'\colon c \to c'$ and $f''\colon c \to c''$ be p-coCartesian arrows with the same image $\alpha = p(f') = p(f'')$. Then there is a unique isomorphism $\phi\colon c' \to c''$ in the fiber $\mathcal{C}_{p(c')} = \mathcal{C}_{p(c'')}$ such that $\phi \circ f' = f''$.*

This lemma tells us that p-coCartesian lifts with a fixed domain are essentially unique if they exist. In particular, for $\alpha\colon d_1 \to d_2$ the targets of p-coCartesian lifts are (uniquely compatibly) isomorphic as objects in the fiber $\mathcal{C}_{d_2}$. Thus, in order to obtain a covariant dependence of the fiber it seems to be a good strategy to ask for a sufficient supply of p-coCartesian arrows.

Definition 14.5.3. A functor $p\colon \mathcal{C} \to \mathcal{D}$ is a *Grothendieck opfibration* if for all $c_1 \in \mathcal{C}$ and for all morphisms α in $\mathcal{D}$ with domain $p(c_1)$ there is a p-coCartesian lift $f\colon c_1 \to c_2$ of α.

Let $p\colon \mathcal{C} \to \mathcal{D}$ be a Grothendieck opfibration. Then we can *choose* for each $c \in \mathcal{C}$ and for each morphism $\alpha\colon p(c) \to d$ a p-coCartesian lift. We now fix a morphism $\alpha\colon d_1 \to d_2$ in $\mathcal{D}$ and define

$$\alpha_!\colon \mathcal{C}_{d_1} \to \mathcal{C}_{d_2}\colon c_1 \mapsto c_2,$$

where c_2 is the codomain of the *chosen* p-coCartesian lift $f\colon c_1 \to c_2$ of α. This defines $\alpha_!$ on objects, and we recommend the reader to check that this can be extended to define a functor $\alpha_!\colon \mathcal{C}_{d_1} \to \mathcal{C}_{d_2}$.

If we now consider an additional morphism $\beta\colon d_2 \to d_3$ in $\mathcal{D}$, then we obtain associated functors

$$\mathcal{C}_{d_1} \xrightarrow{\alpha_!} \mathcal{C}_{d_2} \xrightarrow{\beta_!} \mathcal{C}_{d_3}, \qquad \mathcal{C}_{d_1} \xrightarrow{(\beta\circ\alpha)_!} \mathcal{C}_{d_3}.$$

In general, these two functors are not equal since their definitions depend on certain choices of p-coCartesian lifts: they send an object $c_1 \in \mathcal{C}_{d_1}$ to the respective targets of two possibly different lifts of $\beta\circ\alpha$ to p-coCartesian morphisms with domain c_1 (Exercise: The composition of two p-coCartesian morphisms is again p-coCartesian.). But one can deduce from Lemma 14.5.2 that there is a unique natural isomorphism

$$\beta_! \circ \alpha_! \cong (\beta \circ \alpha)_! \tag{14.5.3}$$

of functors $\mathcal{C}_{d_1} \to \mathcal{C}_{d_3}$, i.e., all components of the natural isomorphism are sent to the identity of d_3 via p.

As an upshot, we essentially succeeded in obtaining a covariant dependence of the fiber by considering Grothendieck opfibrations. To put this in a slightly technical language, we have observed that for such functors the fibers depend *pseudo-functorially* on $d \in \mathcal{D}$, i.e., that the assignment $d \mapsto \mathcal{C}_d$ defines a pseudo-functor $\mathcal{D} \to \mathcal{C}AT$. One might be disappointed about this lack of strict functoriality, but for our purposes this is very convenient: it allows us to *encode* or, better, *hide a lot of structure* in the natural isomorphisms (14.5.3) associated to a Grothendieck opfibration. We illustrate this by the following example.

Example 14.5.4. Let $\mathcal{M}$ be a monoidal category with monoidal pairing $\otimes\colon \mathcal{M} \times \mathcal{M} \to \mathcal{M}$ and monoidal unit $\mathbb{S} \in \mathcal{M}$. We form a new category $\mathcal{M}^{\otimes}$ in the following way. The objects of $\mathcal{M}^{\otimes}$ are (possibly empty) finite sequences of objects in $\mathcal{M}$,

$$(M_1, \ldots, M_n), \quad n \geq 0,\ M_i \in \mathcal{M}.$$

Given two such sequences $(M_1, \ldots, M_n)$ and $(L_1, \ldots, L_k)$, a morphism

$$(\alpha, \{f_i\}_i)\colon (M_1, \ldots, M_n) \to (L_1, \ldots, L_k)$$

consists of a morphism $\alpha\colon [k] \to [n]$ in Δ together with morphisms

$$f_i\colon M_{\alpha(i-1)+1} \otimes \ldots \otimes M_{\alpha(i)} \to L_i, \quad i = 1, \ldots, k. \tag{14.5.4}$$

Thus given such a morphism, α encodes the domains of the f_i. In particular, if there is an $i \in [k]$ such that $\alpha(i-1) = \alpha(i)$, then by convention the corresponding map (14.5.4) is to be read as a map $f_i\colon \mathbb{S} \to L_i$. The composition of morphisms in $\mathcal{M}^{\otimes}$ is defined using the compositions in Δ and $\mathcal{M}$ together with the associativity constraints of the monoidal structure on $\mathcal{M}$. The identity of an object $(M_1, \ldots, M_n)$ is easily seen to be given by $(\mathrm{id}_{[n]}, \{\mathrm{id}_{M_i}\}_i)$.

There is an obvious projection functor $p\colon \mathcal{M}^{\otimes} \to \Delta^{\mathrm{op}}$ which sends a string $(M_1, \ldots, M_n)$ to $[n]$ and a morphism $(\alpha, \{f_i\}_i)$ to its first component α. One easily checks that $p\colon \mathcal{M}^{\otimes} \to \Delta^{\mathrm{op}}$ is a Grothendieck opfibration. Indeed, let us consider a lifting problem given by an object $(M_1, \ldots, M_n)$ in $\mathcal{M}^{\otimes}_{[n]}$ together with a morphism $\alpha^{\mathrm{op}}\colon [n] \to [k]$ in Δ^{op},

$$\begin{array}{ccc} (M_1, \ldots, M_n) & & \\ \Big| & & \\ [n] & \xrightarrow[\alpha^{\mathrm{op}}]{} & [k], \end{array}$$

$$[n] \xleftarrow[\alpha]{} [k].$$

Then a p-coCartesian lift of α with domain the given string is obtained from any family of isomorphisms

$$f_i\colon M_{\alpha(i-1)+1} \otimes \cdots \otimes M_{\alpha(i)} \xrightarrow{\cong} L_i, \quad i = 1, \ldots, k.$$

More precisely, these L_i specify an object $(L_1, \ldots, L_k) \in \mathcal{M}^{\otimes}_{[k]}$, and the morphism

$$(\alpha, \{f_i\}_i)\colon (M_1, \ldots, M_n) \to (L_1, \ldots, L_k) \tag{14.5.5}$$

is the desired p-coCartesian lift.

Now, this Grothendieck opfibration $p\colon \mathcal{M}^{\otimes} \to \Delta^{\mathrm{op}}$ has the property that the fiber $\mathcal{M}_{[n]}$ is canonically equivalent to the n-fold product of $\mathcal{M}^{\otimes}_{[1]} \simeq \mathcal{M}$ in the following sense. Let $\iota_{\{i-1,i\}}\colon [1] \to [n]$ be the inclusion of the i-th length one interval, i.e., the unique monomorphism in Δ with image $\{i-1, i\}$, and let us write $\iota_i = \iota^{\mathrm{op}}_{\{i-1,i\}}\colon [n] \to [1]$ for the opposite morphism in Δ^{op}. Since $p\colon \mathcal{M}^{\otimes} \to \Delta^{\mathrm{op}}$ is a Grothendieck opfibration, we obtain induced functors

$$(\iota_i)_!\colon \mathcal{M}^{\otimes}_{[n]} \to \mathcal{M}^{\otimes}_{[1]} = \mathcal{M}, \qquad i = 1, \ldots, n, \tag{14.5.6}$$

which, taken together, induce the *Segal maps*

$$\sigma = ((\iota_1)_!, \ldots, (\iota_n)_!)\colon \mathcal{M}^{\otimes}_{[n]} \xrightarrow{\simeq} \underbrace{\mathcal{M} \times \ldots \times \mathcal{M}}_{n \text{ times}}. \tag{14.5.7}$$

Note that the explicit construction of p-coCartesian lifts in (14.5.5) implies that these Segal maps are equivalences. We refer to this observation by saying that the Grothendieck opfibration $p\colon \mathcal{M}^{\otimes} \to \Delta^{\mathrm{op}}$ satisfies the *Segal condition*. Let us emphasize that the Segal condition in simplicial degree zero amounts to saying that $\mathcal{M}^{\otimes}_{[0]}$ is equivalent to the terminal category $\mathbb{1}$.

It turns out that the monoidal product $\otimes\colon \mathcal{M} \times \mathcal{M} \to \mathcal{M}$ can be recovered up to equivalence from $p\colon \mathcal{M}^{\otimes} \to \Delta^{\mathrm{op}}$. But something seemingly more general works: Any Grothendieck opfibration $p\colon \mathcal{C} \to \Delta^{\mathrm{op}}$ satisfying the Segal condition defines a monoidal structure on the fiber $\mathcal{M} = \mathcal{C}_{[1]}$. We content ourselves by sketching a proof of this result. As usual, let $d_1\colon [2] \to [1]$ denote the face map in Δ^{op} that is opposite to the coface map $d^1\colon [1] \to [2]$. *Choosing* an inverse of the equivalence $\sigma\colon \mathcal{C}_{[2]} \to \mathcal{M} \times \mathcal{M}$ given by one of the Segal maps we can define a functor

$$\otimes\colon \mathcal{M} \times \mathcal{M} \overset{\simeq}{\leftarrow} \mathcal{C}_{[2]} \overset{(d_1)_!}{\to} \mathcal{C}_{[1]} = \mathcal{M}.$$

In order to construct an associativity constraint for $\otimes$ we will invoke the cosimplicial identity $d^2 \circ d^1 = d^1 \circ d^1\colon [1] \to [3]$. As a special instance of (14.5.3) we thus obtain a natural isomorphism

$$(d_1)_! \circ (d_2)_! \cong (d_1)_! \circ (d_1)_!\colon \mathcal{C}_{[3]} \to \mathcal{C}_{[1]} = \mathcal{M}.$$

It is an instructive exercise to use the Segal condition to translate this into a natural isomorphism

$$\alpha\colon M_1 \otimes (M_2 \otimes M_3) \cong (M_1 \otimes M_2) \otimes M_3, \qquad M_1, M_2, M_3 \in \mathcal{M}.$$

With slightly more effort, one can use the different factorizations of the map $\{0, 4\}\colon [1] \to [4]$ to deduce that this associativity constraint satisfies Mac Lane's pentagon axiom; see (14.5.1). And finally, in a similar way one checks that a monoidal unit $\mathbb{S} \in \mathcal{M}$ and the remaining coherence isomorphisms are also encoded by the Grothendieck opfibration $p\colon \mathcal{C} \to \Delta^{\mathrm{op}}$ satisfying the Segal condition.

The point of this lengthy example was to show that there is an equivalent way of encoding monoidal structures. Instead of making a specific choice of a monoidal pairing, a monoidal unit, and coherence isomorphisms one can consider one 'global object' which nicely hides all this structure, namely a Grothendieck opfibration $p\colon \mathcal{C} \to \Delta^{\mathrm{op}}$ satisfying the Segal condition. In particular, one does not have to make precise the coherence axioms of a monoidal category since they will follow from the (co)simplicial identities. (In §14.5.3 we will briefly mention how monoidal functors and monoid objects fit into this opfibration picture.)

If one only cares about ordinary monoidal categories it might be arguable how large the benefit is by changing the classical perspective on monoidal categories to the Grothendieck opfibration picture. However, as mentioned in the introduction, in the ∞-categorical setting it allows us to avoid the complicated combinatorics of the Stasheff associahedra.

14.5.2 Monoidal ∞-categories via coCartesian fibrations

We now turn to ∞-categorical variants of the above concepts. The main reference for the remainder of §14.5 is Lurie's second book [83]. For the convenience of the reader we also include references to the former volumes [80, 81] which are now subsumed in [83] but which might be a bit more accessible as a first reference.

To begin with, we extend the notion of p-coCartesian morphisms to the context of ∞-categories. For this purpose it is handy to observe that the following is true (which is a refined version of the statement that (14.5.2) is a pullback square).

Lemma 14.5.5. *Let $p\colon \mathcal{C} \to \mathcal{D}$ be a functor between ordinary categories. A morphism $f\colon c_1 \to c_2$ in $\mathcal{C}$ is p-coCartesian if and only if the following functor is an isomorphism*

$$\mathcal{C}_{f/} \to \mathcal{C}_{c_1/} \underset{\mathcal{D}_{p(c_1)/}}{\times} \mathcal{D}_{p(f)/}.$$

In §14.3.3 we already introduced ∞-categorical versions of slice categories. With this lemma at hand, there is the following formal generalization of p-coCartesian arrows to the ∞-categorical setting [82, Definition 2.4.1.1].

Definition 14.5.6. Let $p\colon \mathcal{C} \to \mathcal{D}$ be a functor between ∞-categories. A morphism $f\colon c_1 \to c_2$ in $\mathcal{C}$ is *p-coCartesian* or a *p-coCartesian lift* of $\alpha = p(f)$ if the following map is an acyclic Kan fibration

$$\mathcal{C}_{f/} \to \mathcal{C}_{c_1/} \underset{\mathcal{D}_{p(c_1)/}}{\times} \mathcal{D}_{p(f)/}.$$

The ∞-categorical concept corresponding to Grothendieck opfibrations is that of a *coCartesian fibration* (Joyal [67] uses the terminology *Grothendieck opfibrations* instead). The main idea is again to axiomatically ask for a sufficient supply of coCartesian morphisms (see [82, Definition 2.4.2.1]).

Definition 14.5.7. A functor $p\colon \mathcal{C} \to \mathcal{D}$ between ∞-categories is a *coCartesian fibration* if the following two properties are satisfied.

(i) The functor p is an inner fibration (Definition 14.2.28).

(ii) For every object $c_1 \in \mathcal{C}$ and every morphism $\alpha\colon p(c_1) = d_1 \to d_2$ in $\mathcal{D}$, there is a p-coCartesian lift $f\colon c_1 \to c_2$ of α.

As in the case of categories, in this ∞-categorical context the fiber $\mathcal{C}_d$ of a functor $\mathcal{C} \to \mathcal{D}$ is defined via a pullback square,

$$\begin{array}{ccc} \mathcal{C}_d & \longrightarrow & \mathcal{C} \\ \downarrow & \lrcorner & \downarrow{\scriptstyle p} \\ \Delta^0 & \xrightarrow[d]{} & \mathcal{D}. \end{array}$$

Lurie proves that a coCartesian fibration gives rise to a covariantly depending family of ∞-categories. The fact that the fibers are ∞-*categories* is immediate since a coCartesian fibration is an inner fibration and inner fibrations are stable under pullbacks. The hard part is to show that they assemble to a functor; see Perspective 14.5.10. In particular, any map $\alpha\colon d_1 \to d_2$ in $\mathcal{D}$ induces an essentially unique functor $\alpha_!\colon \mathcal{C}_{d_1} \to \mathcal{C}_{d_2}$ defined by means of coCartesian lifts.

We saw in §14.5.1 that monoidal categories can be alternatively encoded by Grothendieck opfibrations satisfying the Segal condition. Having introduced the corresponding notion of coCartesian fibrations, we now turn this into a definition of monoidal ∞-categories [80, Definition 1.1.2].

Definition 14.5.8. A *monoidal ∞-category* is a coCartesian fibration $p\colon \mathcal{M}^{\otimes} \to N(\Delta^{\mathrm{op}})$ such that the Segal maps are equivalences,

$$\mathcal{M}^{\otimes}_{[n]} \xrightarrow{\sim} (\mathcal{M}^{\otimes}_{[1]})^{\times n}, \qquad n \geq 0.$$

For simplicity, we refer to the ∞-category $\mathcal{M} = \mathcal{M}^{\otimes}_{[1]}$ as a monoidal ∞-category. If one wants to be very precise, then one should call the coCartesian fibration $p\colon \mathcal{M}^{\otimes} \to N(\Delta^{\mathrm{op}})$ a *monoidal structure* on $\mathcal{M}$.

The interpretation of such a coCartesian fibration $p\colon \mathcal{M}^{\otimes} \to N(\Delta^{\mathrm{op}})$ is now similar to the one in classical category theory. To give an example we just make the following remark. One immediate consequence of the axioms is that the fiber $\mathcal{M}^{\otimes}_{[0]}$ is a contractible space. The unique map $s^0\colon [1] \to [0]$ in Δ induces a functor

$$\eta = (s_0)_!\colon \mathcal{M}^{\otimes}_{[0]} \to \mathcal{M}^{\otimes}_{[1]},$$

and we call any object in its image a *monoidal unit* of $\mathcal{M}$. We leave it to the reader to justify this terminology in the classical case.

It can be shown that given a monoidal ∞-category $\mathcal{C}$ the homotopy category $\mathrm{Ho}(\mathcal{C})$ inherits a monoidal structure (see [80, Rmk. 1.1.6]). This underlines the idea that the monoidal structure on $\mathcal{C}$ is associative and unital up to homotopy. Let us however emphasize that Definition 14.5.8 really encodes much more structure, namely that of a monoidal product that is associative and unital up to *coherent homotopy* (see Perspective 14.5.21 for a short discussion in the symmetric monoidal context).

We now turn to important examples of monoidal ∞-categories (but see also §14.6).

Examples 14.5.9. (i) Let $\mathcal{M}$ be a monoidal category and $p\colon \mathcal{M}^{\otimes} \to \Delta^{\mathrm{op}}$ be the associated Grothendieck opfibration. An application of the nerve functor yields a monoidal ∞-category

$$N(p)\colon N(\mathcal{M}^{\otimes}) \to N(\Delta^{\mathrm{op}}).$$

(ii) A special case of a monoidal category is given by the terminal category $\mathbb{1}$. The associated Grothendieck construction can be identified with the identity functor $\Delta^{\mathrm{op}} \to \Delta^{\mathrm{op}}$, and we obtain a monoidal ∞-category $N(\Delta^{\mathrm{op}}) \to N(\Delta^{\mathrm{op}})$. This examples is of some interest in the study of algebra objects (see §14.5.3).

(iii) Again, first 'honest examples' are obtained from suitable model categorical input, more precisely from suitably compatibly closed monoidal and simplicial model category structures, by passing to coherent nerves. Such a category comes with three additional structures, namely a simplicial enrichment, a monoidal structure, and a model structure, which have to be suitably compatible. The assumptions made by Lurie in [80, Prop. 1.6.5] are the following ones.

(a) The closed monoidal structure is compatible with the enrichment in that $\otimes$ and the adjunction expressing the closedness are simplicial.

(b) The monoidal pairing $\otimes\colon \mathcal{M}\times\mathcal{M}\to\mathcal{M}$ is a *left Quillen bifunctor.*

(c) The monoidal unit $\mathbb{S}\in\mathcal{M}$ is cofibrant.

Under these assumptions, one can form a simplicial version of the category $\mathcal{M}^{\otimes}$ from §14.5.1. More precisely, given two finite strings $(M_1,\ldots,M_n)$ and $(L_1,\ldots,L_k)$ of objects in $\mathcal{M}$, the corresponding simplicial mapping space in $\mathcal{M}^{\otimes}$ is given by

$$\coprod_{\alpha\colon [k]\to[n]}\prod_{i=1}^{k}\mathrm{Map}_{\mathcal{M}}(M_{\alpha(i-1)+1}\otimes\ldots\otimes M_{\alpha(i)},L_i).$$

This simplicial category comes with an obvious simplicial functor $\mathcal{M}^{\otimes}\to\Delta^{\mathrm{op}}$ (regarding Δ^{op} as a discrete simplicial category). In order to obtain an ∞-category we consider the full simplicial subcategory

$$\mathcal{M}^{\otimes}_{\mathrm{cf}}\subseteq\mathcal{M}^{\otimes}$$

spanned by the finite strings of fibrant and cofibrant objects. It is then a consequence of the above compatibility assumptions that $\mathcal{M}^{\otimes}_{\mathrm{cf}}$ is a locally fibrant simplicial category so that $N_\Delta(\mathcal{M}^{\otimes}_{\mathrm{cf}})$ is an ∞-category (Corollary 14.2.30). In this situation, Lurie establishes as [80, Prop. 1.6.5] that

$$N_\Delta(\mathcal{M}^{\otimes}_{\mathrm{cf}})\to N_\Delta(\Delta^{\mathrm{op}})=N(\Delta^{\mathrm{op}})$$

endows the ∞-category $N_\Delta(\mathcal{M}_{\mathrm{cf}})$ with a monoidal structure.

This result and variants of it (such as [80, Thm. 1.6.16]) yield important special cases of monoidal ∞-categories. For example, the projective model structure on unbounded chain complexes over a commutative ring gives rise to the monoidal ∞-category of chain complexes [80, Ex. 1.6.17]. Similarly, the monoidal category of symmetric spectra can be endowed with a compatible model structure, and this leads to an extrinsic construction of the monoidal ∞-category of spectra [80, Ex. 1.6.18].

(iv) In ordinary category theory, important monoidal structures are given by categorical products and coproducts. These monoidal structures are referred to as the *Cartesian* and *coCartesian* monoidal structures, respectively. Given an ∞-category $\mathcal{C}$ with finite (co)products, then one can construct (co)Cartesian monoidal structures on $\mathcal{C}$, given by coCartesian fibrations

$$\mathcal{C}^{\times}\to N(\Delta^{\mathrm{op}})\qquad\text{and}\qquad\mathcal{C}^{\sqcup}\to N(\Delta^{\mathrm{op}}),$$

respectively. See [83, §2.4] for more details.

Perspective 14.5.10. CoCartesian fibrations are an ∞-categorical analogue of Grothendieck opfibrations. We saw in §14.5.1 that such an opfibration $p\colon\mathcal{C}\to\mathcal{D}$ encodes the idea of having a family of categories $\mathcal{C}_d, d\in\mathcal{D}$, that depends covariantly on the object $d\in\mathcal{D}$. More precisely, by choosing certain p-coCartesian lifts we obtain a pseudo-functor

$$F_p\colon\mathcal{D}\to\mathcal{C}AT\colon d\mapsto\mathcal{C}_d.$$

There is also a construction in the converse direction called the *Grothendieck construction* . Given a pseudo-functor $F\colon \mathcal{D} \to \mathcal{C}AT$, one can form a new category $\mathcal{E}(F)$ defined as follows.

(i) An object in $\mathcal{E}(F)$ is a pair (d, x) consisting of an object $d \in \mathcal{D}$ and an object $x \in F(d)$.

(ii) A morphism $(d, x) \to (d', x')$ in $\mathcal{E}(F)$ is a pair (α, f) consisting of a morphism $\alpha\colon d \to d'$ in $\mathcal{D}$ and a morphism $f\colon F(\alpha)(x) \to x'$ in $F(d')$.

(iii) Compositions and identities are defined in the obvious way.

The category $\mathcal{E}(F)$ comes with a forgetful functor

$$p_F\colon \mathcal{E}(F) \to \mathcal{D}$$

which projects objects and morphisms onto their respective first components, and one checks that p_F is a Grothendieck opfibration. In fact, given an object (d, x) in $\mathcal{E}(F)$ and a morphism $\alpha\colon p_F(d, x) = d \to d'$ in $\mathcal{D}$, then a p_F-coCartesian lift is given by

$$(\alpha, \mathrm{id}_{F(\alpha)(x)})\colon (d, x) \to (d', F(\alpha)(x)).$$

It turns out that if we fix a category $\mathcal{D}$, then these two constructions tell us that category-valued pseudo-functors defined on $\mathcal{D}$ and Grothendieck opfibrations with target $\mathcal{D}$ are essentially the same. References for this theory include [19, §8] and [112].

In [82, §3], Lurie has generalized these constructions to the ∞-categorical setting. Roughly speaking, he has established a result saying that giving a coCartesian fibration $p\colon \mathcal{C} \to \mathcal{D}$ is equivalent to giving a functor $\mathcal{D} \to \mathcal{C}at_\infty$, where $\mathcal{C}at_\infty$ is the ∞-category of ∞-categories introduced in Perspective 14.3.7. In fact, there is a Quillen equivalence between certain model structures making this idea precise. Let us only mention that the homotopy theory of coCartesian fibrations above $\mathcal{D}$ is encoded by the *coCartesian model structure* on $s\mathcal{S}et^+_{/\mathcal{D}}$, the category of marked simplicial sets above $\mathcal{D} = (\mathcal{D}, \mathcal{D}_1)$. An object $p\colon \mathcal{C} \to \mathcal{D}$ in this model structure is fibrant if and only if p is a coCartesian fibration and the marked edges in $\mathcal{C}$ are precisely the p-coCartesian arrows.

In ordinary category theory, there are variants to these notions for contravariant category-valued pseudo-functors. The ∞-categorical analogue is that of a *Cartesian fibration.* Moreover, in the classical context — in particular, in the theory of stacks in algebraic geometry or algebraic topology — one frequently considers *groupoid-valued* pseudo-functors (of either variance). These notions are sometimes also referred to as categories (co)fibered in groupoids. The ∞-categorical analogues of these are *left* and *right fibrations* and again there are suitable associated model categories in the background. For more details we refer to [82, §3].

14.5.3 Algebra objects and monoidal functors

Let $\mathcal{M}$ be an ordinary monoidal category with monoidal pairing $\otimes$ and unit object $\mathbb{S} \in \mathcal{M}$. An *algebra* or *monoid* in $\mathcal{M}$ is an object $A \in \mathcal{M}$ together with a multiplication map

$\mu\colon A \otimes A \to A$ and a unit map $\eta\colon \mathbb{S} \to A$ that satisfy obvious associativity and unitality conditions. We also say that (μ, η) specifies an *algebra structure* on A.

In §14.5.1 we saw how to encode monoidal structures by means of Grothendieck opfibrations $p\colon \mathcal{M}^{\otimes} \to \Delta^{\mathrm{op}}$ satisfying the Segal condition. We now want to describe algebra structures in that picture. As a first guess for a definition of algebra objects we might consider sections of $p\colon \mathcal{M}^{\otimes} \to \Delta^{\mathrm{op}}$. Given an arbitrary such section

$$A\colon \Delta^{\mathrm{op}} \to \mathcal{M}^{\otimes},$$

the equivalences given by the Segal maps (14.5.6) imply that the value $A_{[n]} \in \mathcal{M}^{\otimes}_{[n]}$ corresponds to n objects of $\mathcal{M} = \mathcal{M}^{\otimes}_{[1]}$,

$$\mathcal{M}_{[n]} \ni A_{[n]} \quad \longleftrightarrow \quad A^1_{[n]}, \ldots, A^n_{[n]} \in \mathcal{M}. \tag{14.5.8}$$

Let us again consider the face map $d_1\colon [2] \to [1]$ which we saw in §14.5.1 encodes the monoidal pairing $\otimes\colon \mathcal{M} \times \mathcal{M} \to \mathcal{M}$. The section A evaluated on d_1 gives us by means of the identification (14.5.8) a map

$$(A^1_{[2]}, A^2_{[2]}) \to A^1_{[1]} \tag{14.5.9}$$

defined on the pair of objects $(A^1_{[2]}, A^2_{[2]})$. Moreover, by the discussion in §14.5.1, we have a description of the p-coCartesian arrows for the Grothendieck opfibration $p\colon \mathcal{M}^{\otimes} \to \Delta^{\mathrm{op}}$. Applied to our situation, a p-coCartesian lift of d_1 with domain $A_{[2]}$ corresponds under (14.5.8) to a morphism

$$(A^1_{[2]}, A^2_{[2]}) \to A^1_{[2]} \otimes A^2_{[2]}.$$

The universal property of this p-coCartesian lift (as expressed by (14.5.2) being a pullback square) implies that (14.5.9) factors uniquely over this lift and we hence obtain an induced map

$$A^1_{[2]} \otimes A^2_{[2]} \to A^1_{[1]}. \tag{14.5.10}$$

If we now want to get a classical algebra object, we would like (14.5.10) to be a map of the form $M \otimes M \to M$ for some fixed $M \in \mathcal{M}$. Thus we should ensure that, among other things, the objects $A^1_{[2]}, A^2_{[2]}$, and $A^1_{[1]}$ are isomorphic.

Definition 14.5.11. A morphism $\alpha\colon [n] \to [k]$ in Δ is *convex* if it is injective and the image $\mathrm{im}(\alpha) \subseteq [k]$ is convex, i.e., the image is given by the interval $[\alpha(0), \alpha(n)]$.

It follows from the construction of the p-coCartesian arrows of $p\colon \mathcal{M}^{\otimes} \to \Delta^{\mathrm{op}}$ (compare to the discussion around (14.5.5)) that the p-coCartesian lifts of convex maps $\alpha\colon [n] \to [k]$ in Δ induce projection functors $\mathcal{M}^{\times k} \to \mathcal{M}^{\times n}$ onto some of the factors. In particular, p-coCartesian lifts defining the functors $(\iota_i)_!, i = 1, 2$, as in (14.5.6) can be identified with the maps

$$(A^1_{[2]}, A^2_{[2]}) \to A^1_{[2]} \qquad \text{and} \qquad (A^1_{[2]}, A^2_{[2]}) \to A^2_{[2]},$$

respectively. If the images of $\iota_i\colon [2] \to [1], i = 1, 2$, under A are p-coCartesian arrows, then by Lemma 14.5.2 we obtain the desired isomorphisms

$$A^1_{[2]} \cong A^1_{[1]} \qquad \text{and} \qquad A^2_{[2]} \cong A^1_{[1]}.$$

With this preparation it is straightforward to establishes the following result.

Proposition 14.5.12. *Let $p\colon \mathcal{M}^{\otimes} \to \Delta^{\mathrm{op}}$ be a monoidal structure on $\mathcal{M} = \mathcal{M}^{\otimes}_{[1]}$. Then a section $A\colon \Delta^{\mathrm{op}} \to \mathcal{M}^{\otimes}$ of p that sends convex arrows to p-coCartesian arrows encodes an algebra structure on $A_{[1]} \in \mathcal{M}$. Conversely, any algebra object in $\mathcal{M}$ determines such a section of $p\colon \mathcal{M}^{\otimes} \to \Delta^{\mathrm{op}}$.*

In the world of ∞-categories, we turn this observation into a definition ([80, Definition 1.1.14]).

Definition 14.5.13. Let $p\colon \mathcal{M}^{\otimes} \to N(\Delta^{\mathrm{op}})$ be a monoidal ∞-category. A section $A\colon N(\Delta^{\mathrm{op}}) \to \mathcal{M}^{\otimes}$ of p is an *(associative) algebra object* in $\mathcal{M}^{\otimes}$ if A sends convex morphisms to p-coCartesian arrows in $\mathcal{M}^{\otimes}$.

Note that it is already a certain abuse of language to speak of algebra objects of $\mathcal{M}^{\otimes}$ since the notion of algebra object obviously also depends on the coCartesian fibration. A further comfortable abuse of language is to simply speak of algebra objects in $\mathcal{M}$.

As in the case of monoidal structures on ∞-categories, an algebra object encodes quite a lot of structure: namely, given an algebra object A in $\mathcal{M}$, the underlying object $A_{[1]}$ is endowed with a multiplication map which is associative and unital up to *coherent homotopy* (see Perspective 14.5.21 for an explanation of this similarity in the commutative case). In particular, an algebra object in a monoidal ∞-category defines an ordinary algebra object in the underlying homotopy category, but not conversely.

Algebra objects in monoidal ∞-categories are special cases of lax monoidal functors between monoidal ∞-categories. We include the following definition and leave it to the reader to check that in the case of ordinary categories this reduces to the usual concepts.

Definition 14.5.14. Let $p\colon \mathcal{M}^{\otimes} \to N(\Delta^{\mathrm{op}})$ and $q\colon \mathcal{N}^{\otimes} \to N(\Delta^{\mathrm{op}})$ be monoidal ∞-categories. A *lax monoidal functor* $F\colon \mathcal{M}^{\otimes} \to \mathcal{N}^{\otimes}$ is a functor over $N(\Delta^{\mathrm{op}})$,

$$\begin{array}{ccc} \mathcal{M}^{\otimes} & \xrightarrow{\quad F \quad} & \mathcal{N}^{\otimes} \\ & {\scriptstyle p}\searrow \quad = \quad \swarrow{\scriptstyle q} & \\ & N(\Delta^{\mathrm{op}}), & \end{array}$$

that sends p-coCartesian lifts of convex morphisms in $N(\Delta^{\mathrm{op}})$ to q-coCartesian arrows. A *monoidal functor* $F\colon \mathcal{M}^{\otimes} \to \mathcal{N}^{\otimes}$ is a functor over $N(\Delta^{\mathrm{op}})$ that sends arbitrary p-coCartesian arrows to q-coCartesian ones.

Of course we also want to consider monoidal transformations between monoidal functors. More generally, (lax) monoidal functors between monoidal ∞-categories $\mathcal{M}^{\otimes}$ and $\mathcal{N}^{\otimes}$ are organized into ∞-categories

$$\mathrm{Fun}^{\otimes,\mathrm{lax}}(\mathcal{M}^{\otimes}, \mathcal{N}^{\otimes}) \qquad \text{and} \qquad \mathrm{Fun}^{\otimes}(\mathcal{M}^{\otimes}, \mathcal{N}^{\otimes}),$$

respectively. The ∞-category $\mathrm{Fun}^{\otimes,\mathrm{lax}}(\mathcal{M}^{\otimes}, \mathcal{N}^{\otimes})$ is the full subcategory of the ∞-category $\mathrm{Map}_{N(\Delta^{\mathrm{op}})}(\mathcal{M}^{\otimes}, \mathcal{N}^{\otimes})$ of functors over $N(\Delta^{\mathrm{op}})$ spanned by the lax monoidal functors. Here,

$\mathrm{Map}_{N(\Delta^{\mathrm{op}})}(\mathcal{M}^{\otimes},\mathcal{N}^{\otimes})$ is of course defined as the pullback

$$\begin{array}{ccc} \mathrm{Map}_{N(\Delta^{\mathrm{op}})}(\mathcal{M}^{\otimes},\mathcal{N}^{\otimes}) & \longrightarrow & \mathrm{Map}(\mathcal{M}^{\otimes},\mathcal{N}^{\otimes}) \\ \downarrow & & \downarrow q_* \\ \Delta^0 & \xrightarrow[p]{} & \mathrm{Map}(\mathcal{M}^{\otimes}, N(\Delta^{\mathrm{op}})), \end{array} \tag{14.5.11}$$

which is an ∞-category because q and so q_* is an inner fibration [82, Corollary 2.3.2.5]. Similarly, $\mathrm{Fun}^{\otimes}(\mathcal{M}^{\otimes},\mathcal{N}^{\otimes}) \subseteq \mathrm{Fun}^{\otimes,\mathrm{lax}}(\mathcal{M}^{\otimes},\mathcal{N}^{\otimes})$ is the full subcategory spanned by the monoidal functors.

As a special case, we see that algebra objects in a monoidal ∞-category $\mathcal{M}^{\otimes}$ are themselves organized in an ∞-category. In fact, the *∞-category* $\mathrm{Alg}_{\mathbb{A}_\infty}(\mathcal{M}^{\otimes})$ *of algebra objects* in $\mathcal{M}^{\otimes}$ can be defined as

$$\mathrm{Alg}_{\mathbb{A}_\infty}(\mathcal{M}^{\otimes}) = \mathrm{Fun}^{\otimes,\mathrm{lax}}(N(\Delta^{\mathrm{op}}),\mathcal{M}^{\otimes}), \tag{14.5.12}$$

where $N(\Delta^{\mathrm{op}}) \to N(\Delta^{\mathrm{op}})$ is the trivial monoidal structure as in Examples 14.5.9. We will say a bit more about similar ∞-categories in the context of symmetric monoidal structures in §14.5.4.

14.5.4 Symmetric monoidal ∞-categories

In order to obtain a theory of *symmetric* monoidal ∞-categories one combines the Segal perspective on $\mathbb{E}_\infty$-monoids and the ∞-categorical Grothendieck construction, and this section is hence inspired by the theory of 'special Γ-spaces' (see [104, 102]). A good part of this section simply amounts to translating parts of §§14.5.1-14.5.3 to the context of symmetric monoidal ∞-categories, and therefore we are rather sketchy.

We again begin with the classical situation in ordinary category theory. In §14.5.1, we described monoidal categories in terms of Grothendieck opfibrations $\mathcal{M}^{\otimes} \to \Delta^{\mathrm{op}}$. In that picture, the monoidal product was encoded by the induced functor

$$(d_1)_!\colon \mathcal{M}^{\otimes}_{[2]} \to \mathcal{M}^{\otimes}_{[1]} = \mathcal{M}.$$

If we want to have a similar description of *symmetric* monoidal categories, we must be able to encode symmetry isomorphisms, thus we have to encode the flip map

$$t\colon \mathcal{M}^{\times 2} \to \mathcal{M}^{\times 2}\colon (X,Y) \mapsto (Y,X)$$

and, more generally, any permutation of n objects in $\mathcal{M}$. It is hence plausible that the role of Δ is taken by 'a category of finite sets with all maps between them'. The details are as follows.

For a natural number $n \geq 0$, let $\langle n\rangle$ be the finite pointed set

$$\langle n\rangle = \{0,1,\dots,n\}$$

with $0 \in \langle n \rangle$ as base point. The category $\mathcal{F}\mathrm{in}$ is the full subcategory of the category of pointed sets spanned by the objects $\langle n \rangle, n \geq 0$. Note that the natural ordering on $\langle n \rangle$ does not play a role in the definition of $\mathcal{F}\mathrm{in}$ but it will have its uses in the formation of higher monoidal products. We denote by

$$\rho^j\colon \langle n \rangle \to \langle 1 \rangle, \quad n \geq 1, \quad j = 1, \ldots, n, \tag{14.5.13}$$

the unique pointed map $\langle n \rangle \to \langle 1 \rangle$ with $(\rho^j)^{-1}(1) = \{j\}$.

Let now $\mathcal{M}$ be a *symmetric* monoidal category with monoidal product $\otimes$ and monoidal unit $\mathbb{S} \in \mathcal{M}$. Following a pattern similar to §14.5.1, we construct a new category $\mathcal{M}^{\otimes}$ as follows. An object in $\mathcal{M}^{\otimes}$ is a finite (possibly empty) sequence of objects in $\mathcal{M}$,

$$(M_1, \ldots, M_n), \quad M_i \in \mathcal{M}, \quad n \geq 0.$$

A morphism $(M_1, \ldots, M_n) \to (L_1, \ldots, L_k)$ between two such sequences is a pair $(\alpha, \{f_i\}_i)$ consisting of a morphism $\alpha\colon \langle n \rangle \to \langle k \rangle$ in $\mathcal{F}\mathrm{in}$ together with morphisms

$$f_i\colon \bigotimes_{j \in \alpha^{-1}(i)} M_j \to L_i, \quad i = 1, \ldots, k,$$

where the tensor product is formed according to the ordering on $\alpha^{-1}(i)$. Again, if the set $\alpha^{-1}(i)$ is empty, then this is to be read as a map $f_i\colon \mathbb{S} \to L_i$. The composition is obtained from the compositions in $\mathcal{M}$ and $\mathcal{F}\mathrm{in}$ together with the coherence constraints of $\mathcal{M}$. There is an obvious projection functor $p\colon \mathcal{M}^{\otimes} \to \mathcal{F}\mathrm{in}$ given by $(M_1, \ldots, M_n) \mapsto \langle n \rangle$ and $(\alpha, \{f_i\}_i) \mapsto \alpha$.

Proposition 14.5.15. *For any symmetric monoidal category $\mathcal{M}$ the functor $p\colon \mathcal{M}^{\otimes} \to \mathcal{F}\mathrm{in}$ is a Grothendieck opfibration. Moreover, this functor satisfies the* Segal condition*, i.e., the Segal maps*

$$(\rho^1_!, \ldots, \rho^n_!)\colon \mathcal{M}^{\otimes}_{\langle n \rangle} \to \mathcal{M}^{\times n}, \quad n \geq 0,$$

are equivalences. Conversely, any Grothendieck opfibration $p\colon \mathcal{C} \to \mathcal{F}\mathrm{in}$ satisfying the Segal condition encodes a symmetric monoidal structure on $\mathcal{M} = \mathcal{C}_{\langle 1 \rangle}$.

Here is a sketch of a proof. Given $(M_1, \ldots, M_n) \in \mathcal{M}^{\otimes}_{\langle n \rangle}$ and $\alpha\colon \langle n \rangle \to \langle k \rangle$, an associated p-coCartesian is obtained by choosing isomorphisms

$$f_i\colon \bigotimes_{j \in \alpha^{-1}(i)} M_j \to L_i, \quad i = 1, \ldots, k.$$

In the special case of ρ^j as in (14.5.13), it follows that such lifts are of the form

$$(\rho^j, \mathrm{id}_{M_j})\colon (M_1, \ldots, M_n) \to M_j,$$

and the associated functor $\rho^j_!\colon \mathcal{M}^{\otimes}_{\langle n \rangle} \to \mathcal{M}^{\otimes}_{\langle 1 \rangle}$ can hence be identified with a projection functor, implying the Segal condition.

Conversely, let us consider a Grothendieck opfibration $p\colon \mathcal{C} \to \mathcal{F}\mathrm{in}$ satisfying the Segal condition. We content ourselves by mentioning the following three steps towards the construction of a symmetric monoidal structure on $\mathcal{M} = \mathcal{C}_{\langle 1 \rangle}$.

(i) Let $m\colon \langle 2\rangle \to \langle 1\rangle$ be the map in $\mathcal{F}\text{in}$ determined by $m(1) = m(2) = 1$. By means of the Segal condition, we can then define a functor

$$\otimes\colon \mathcal{M} \times \mathcal{M} \xleftarrow{\simeq} \mathcal{C}_{\langle 2\rangle} \xrightarrow{m_!} \mathcal{C}_{\langle 1\rangle} = \mathcal{M},$$

which will be the monoidal pairing.

(ii) As a special case of the Segal condition we obtain an equivalence $\mathcal{C}_{\langle 0\rangle} \simeq \mathbb{1}$. Thus, for the unique map $n\colon \langle 0\rangle \to \langle 1\rangle$ in $\mathcal{F}\text{in}$, the induced functor

$$n_!\colon \mathcal{C}_{\langle 0\rangle} \to \mathcal{C}_{\langle 1\rangle} = \mathcal{M}$$

essentially classifies an object in $\mathcal{M}$, which will be the monoidal unit $\mathbb{S}$.

(iii) The *twist map* $t\colon \langle 2\rangle \to \langle 2\rangle$ in $\mathcal{F}\text{in}$ is the automorphism that interchanges 1 and 2. The equality $m = m \circ t\colon \langle 2\rangle \to \langle 1\rangle$ together with Lemma 14.5.2 yields a unique natural isomorphism

$$\sigma\colon m_! \cong m_! \circ t_!\colon \mathcal{C}_{\langle 2\rangle} \to \mathcal{C}_{\langle 1\rangle} = \mathcal{M}$$

over $\langle 1\rangle$, which can be shown to induce a symmetry constraint for $\otimes$.

By similar arguments, one obtains associativity and unitality constraints and establishes the coherence axioms. As in the non-symmetric case, we now turn this observation into a definition [78].

Definition 14.5.16. A *symmetric monoidal ∞-category* is a coCartesian fibration $p\colon \mathcal{M}^{\otimes} \to N(\mathcal{F}\text{in})$ such that the Segal maps are equivalences,

$$(\rho_!^1, \ldots, \rho_!^n)\colon \mathcal{M}^{\otimes}_{\langle n\rangle} \xrightarrow{\simeq} (\mathcal{M}^{\otimes}_{\langle 1\rangle})^{\times n}, \qquad n \geq 0.$$

We include a few remarks which are parallel to corresponding statements in §14.5.2.

Remark 14.5.17. (i) A symmetric monoidal ∞-category $p\colon \mathcal{M}^{\otimes} \to N(\mathcal{F}\text{in})$ endows the underlying ∞-category $\mathcal{M} = \mathcal{M}^{\otimes}_{\langle 1\rangle}$ with a monoidal pairing which is associative and commutative *up to coherent homotopies*; see Perspective 14.5.21. In particular, the homotopy category of a symmetric monoidal ∞-category is canonically a symmetric monoidal category.

(ii) Applying the nerve construction to Grothendieck opfibrations associated to ordinary symmetric monoidal categories we obtain symmetric monoidal ∞-categories. In particular, the identity functor $N(\mathcal{F}\text{in}) \to N(\mathcal{F}\text{in})$ is a symmetric monoidal ∞-category. As a variant, given a closed, symmetric monoidal, simplicial model category satisfying suitable compatibility assumptions, by means of the coherent nerve construction we obtain 'honest examples' of symmetric monoidal ∞-categories ([78, §8]).

Before we turn to algebra objects in the symmetric monoidal context, we expand a bit on the relation between monoidal and symmetric monoidal ∞-categories. For this purpose, we consider the functor

$$\phi\colon \Delta^{op} \to \mathcal{F}\text{in} \tag{14.5.14}$$

that on objects is given by $[n] \mapsto \langle n\rangle$. Given a morphism $\alpha\colon [k] \to [n]$ in Δ, the induced pointed map $\phi(\alpha)\colon \langle n\rangle \to \langle k\rangle$ is defined by

$$\phi(\alpha)(j) = \begin{cases} i & \text{if there is an } i \text{ such that } j \in [\alpha(i-1)+1, \alpha(i)], \\ * & \text{otherwise.} \end{cases}$$

Since α is monotone, such an i is unique if it exists, and we leave it to the reader to check that this defines a functor. In fact, up to a restriction of the codomain, the functor ϕ is simply the simplicial circle $S^1 \in s\mathcal{S}et_*$ defined as the coequalizer

$$\Delta^0 \rightrightarrows \Delta^1 \to S^1$$

and considered as a pointed simplicial set. As a special case the image of the opposite ι_j of $\iota_{\{j-1,j\}}\colon [1] \to [n]$ under ϕ is the map ρ^j as in (14.5.13).

The role of convex maps in the theory of monoidal ∞-categories is taken by *inert* maps in the theory of symmetric monoidal ∞-categories.

Definition 14.5.18. A morphism $\alpha\colon \langle n\rangle \to \langle k\rangle$ in $\mathcal{F}$in is *inert* if $\alpha^{-1}(i)$ is a singleton for every $1 \le i \le k$.

We note that $\alpha\colon [k] \to [n]$ in Δ is convex if and only if $\phi(\alpha)\colon \langle n\rangle \to \langle k\rangle$ in $\mathcal{F}$in is inert, and that the maps (14.5.13) are inert. Given a symmetric monoidal ∞-category $p\colon \mathcal{M}^{\otimes} \to N(\mathcal{F}\text{in})$ we already observed that the induced functors $\rho^j_!$ are projection functors. Similarly, general inert morphisms induce projection and permutation functors. This suggests the following definition.

Definition 14.5.19. Let $p\colon \mathcal{M}^{\otimes} \to N(\mathcal{F}\text{in})$, $q\colon \mathcal{N}^{\otimes} \to N(\mathcal{F}\text{in})$ be symmetric monoidal ∞-categories and let $F\colon \mathcal{M}^{\otimes} \to \mathcal{N}^{\otimes}$ be a functor over $N(\mathcal{F}\text{in})$.

(i) The functor F is *symmetric monoidal* if it sends p-coCartesian arrows to q-coCartesian arrows.

(ii) The functor F is *lax symmetric monoidal* if it sends p-coCartesian lifts of inert morphisms to q-coCartesian arrows.

Symmetric monoidal and lax symmetric monoidal functors respectively are organized in ∞-categories, namely the corresponding full subcategories

$$\mathrm{Fun}^{\otimes}(\mathcal{M}^{\otimes}, \mathcal{N}^{\otimes}) \subseteq \mathrm{Fun}^{\otimes,\mathrm{lax}}(\mathcal{M}^{\otimes}, \mathcal{N}^{\otimes}) \subseteq \mathrm{Map}_{N(\mathcal{F}\mathrm{in})}(\mathcal{M}^{\otimes}, \mathcal{N}^{\otimes})$$

where $\mathrm{Map}_{N(\mathcal{F}\mathrm{in})}(\mathcal{M}^{\otimes}, \mathcal{N}^{\otimes})$ is defined in analogy to (14.5.11). As a special case we obtain *∞-categories of commutative algebra objects* $\mathrm{Alg}_{\mathbb{E}_\infty}(\mathcal{M}^{\otimes})$ defined by

$$\mathrm{Alg}_{\mathbb{E}_\infty}(\mathcal{M}^{\otimes}) = \mathrm{Fun}^{\otimes,\mathrm{lax}}(N(\mathcal{F}\mathrm{in}), \mathcal{M}^{\otimes}). \tag{14.5.15}$$

More explicitly, a commutative algebra object is a section $E\colon N(\mathcal{F}\text{in}) \to \mathcal{M}^{\otimes}$ of $p\colon \mathcal{M}^{\otimes} \to N(\mathcal{F}\text{in})$ sending inert morphisms to p-coCartesian ones. Such a section endows the underlying object $E_{\langle 1\rangle} \in \mathcal{M}$ with a coherently associative and commutative pairing $E_{\langle 1\rangle} \otimes E_{\langle 1\rangle} \to E_{\langle 1\rangle}$.

Remark 14.5.20. A symmetric monoidal ∞-category has an underlying monoidal ∞-category. In fact, given a symmetric monoidal ∞-category $\mathcal{M}^{\otimes} \to N(\mathcal{F}\mathrm{in})$, then the underlying monoidal ∞-category $U(\mathcal{M}^{\otimes})$ is defined as the pullback

$$\begin{array}{ccc} U(\mathcal{M}^{\otimes}) & \longrightarrow & \mathcal{M}^{\otimes} \\ \downarrow & & \downarrow p \\ N(\Delta^{\mathrm{op}}) & \xrightarrow[N(\phi)]{} & N(\mathcal{F}\mathrm{in}), \end{array}$$

where ϕ is again the simplicial circle (14.5.14). Similarly, one shows that commutative algebra objects have underlying associative algebra objects.

Perspective 14.5.21. In this section we saw that symmetric monoidal ∞-categories are endowed with a coherently associative and commutative monoidal structure and we just claimed that something similar is true for commutative algebra objects in the sense of (14.5.15). In this perspective we make the similarity between these two situations precise and refer the reader to [78, §2] for details.

Given an ∞-category $\mathcal{C}$ with finite products, there is the Cartesian monoidal structure $\mathcal{C}^{\times} \to N(\mathcal{F}\mathrm{in})$ and there are two associated ∞-categories.

(i) As a special case of (14.5.15) we have the ∞-category $\mathrm{Alg}_{\mathbb{E}_\infty}(\mathcal{C}^{\times})$ of commutative algebra objects in $\mathcal{C}^{\times}$.

(ii) On the other hand, we can consider *commutative monoids* in $\mathcal{C}$, i.e., functors $M\colon N(\mathcal{F}\mathrm{in}) \to \mathcal{C}$ such that the Segal maps

$$M_{\langle n\rangle} \xrightarrow{\sim} M_{\langle 1\rangle} \times \cdots \times M_{\langle 1\rangle}$$

are equivalences. The ∞-category $\mathrm{Mon}_{\mathbb{E}_\infty}(\mathcal{C})$ of commutative monoids in $\mathcal{C}$ is the full subcategory of $\mathrm{Fun}(N(\mathcal{F}\mathrm{in}), \mathcal{C})$ spanned by the commutative monoids.

And it turns out that there is an equivalence $\mathrm{Alg}_{\mathbb{E}_\infty}(\mathcal{C}^{\times}) \simeq \mathrm{Mon}_{\mathbb{E}_\infty}(\mathcal{C})$ over $\mathcal{C}$.

We now apply this to the ∞-category $\mathcal{C}at_\infty$ of ∞-categories (Perspective 14.3.7), which is an example of an ∞-category admitting finite products. The Grothendieck construction (Perspective 14.5.10) implies that $\mathrm{Fun}(N(\mathcal{F}\mathrm{in}), \mathcal{C}at_\infty)$ is equivalent to the ∞-category of coCartesian fibrations $p\colon \mathcal{M} \to N(\mathcal{F}\mathrm{in})$. And under this equivalence commutative monoids in $\mathcal{C}at_\infty$ and symmetric monoidal ∞-categories correspond to each other since both are defined by similar Segal conditions. Thus, as an upshot, we obtain an equivalence of ∞-categories

$$\mathrm{Alg}_{\mathbb{E}_\infty}(\mathcal{C}at_\infty^{\times}) \simeq \mathcal{C}at_\infty^{\mathrm{sMon}},$$

where $\mathcal{C}at_\infty^{\mathrm{sMon}}$ denotes the ∞-category of symmetric monoidal ∞-categories and symmetric monoidal functors, explaining why in both cases we obtain similar coherence data. There is a similar equivalence in the case of monoidal ∞-categories and monoidal functors, namely

$$\mathrm{Alg}_{\mathbb{A}_\infty}(\mathcal{C}at_\infty^{\times}) \simeq \mathcal{C}at_\infty^{\mathrm{Mon}}.$$

Obviously, having introduced commutative algebra objects, one would now like to study modules over such algebra objects as well as the existence of limits and colimits in related ∞-categories. Such a theory exists and for more details we refer to [80, 81] and [83], as well as the chapter in this Handbook by David Gepner.

We content ourselves by concluding this section with the following result concerning *initial* objects in ∞-categories of commutative algebra objects. This result comes up again in the final section §14.6.

Proposition 14.5.22. *For every symmetric monoidal ∞-category* $\mathcal{M}^{\otimes} \to N(\mathcal{F}\mathrm{in})$ *the ∞-category* $\mathrm{Alg}_{\mathbb{E}_\infty}(\mathcal{M})$ *has an initial object. Moreover, a commutative algebra object E is initial if and only if the unit map* $\mathbb{S} \to E_{\langle 1 \rangle}$ *is an equivalence in* $\mathcal{M}$.

14.6 Stable ∞-categories and the universal property of spectra

In this final section we give an introduction to stable ∞-categories. By definition a finitely complete and finitely cocomplete ∞-categories is stable if it admits a zero object and if a square in it is a pullback if and only if it is a pushout. Typical examples of stable ∞-categories arise in homological algebra (∞-categories of chain complexes) and stable homotopy theory (the ∞-category of spectra). In fact, it turns out the ∞-category of spectra is the universal example of a stable ∞-category in a certain precise sense.

Stable ∞-categories are an enhancement of triangulated categories. In §14.6.1 we sketch some of ingredients involved in a proof that homotopy categories of stable ∞-categories can be turned into triangulated categories. In §14.6.2 we briefly discuss the *stabilization* of nice ∞-categories which is obtained by passing to internal spectrum objects. We conclude this subsection by a precise universal property of this stabilization process. Preparing the ground for the construction of the smash product, in §14.6.3 we discuss the tensor product of presentable ∞-categories. Following Lurie, this allows us in §14.6.4 to give a very conceptual construction of the smash product monoidal structure on spectra and hence to define associative and commutative ring spectra.

14.6.1 Stable ∞-categories

General references for the first two subsections are [79] and [83, §1]. As a first step we collect a few basics concerning *pointed* ∞-categories.

Definition 14.6.1. An ∞-category is *pointed* if it admits a zero object, i.e., an object that is initial and final.

Thus, an ∞-category $\mathcal{C}$ is pointed if there is an object $0 \in \mathcal{C}$ such that for all $x \in \mathcal{C}$ the mapping spaces $\mathrm{Map}_{\mathcal{C}}(x, 0)$ and $\mathrm{Map}_{\mathcal{C}}(0, x)$ are contractible. It follows, that for any two objects $x, y \in \mathcal{C}$ there is a zero map

$$0 = 0_{x,y} \colon x \to y,$$

well-defined up to a contractible space of choices. Again by Proposition 14.3.19, if an ∞-category $\mathcal{C}$ is pointed, then the full subcategory spanned by the zero objects is a contractible Kan complex.

Examples 14.6.2. (i) Let $\mathcal{C}$ be an ∞-category with a terminal object $* \in \mathcal{C}$. The undercategory $\mathcal{C}_* = \mathcal{C}_{*/}$ (see 14.3.17) is a pointed ∞-category, called the ∞-category of *pointed objects* in $\mathcal{C}$. Adding a disjoint base point defines a functor $_+ \colon \mathcal{C} \to \mathcal{C}_*$ which is left adjoint to the forgetful functor $_- \colon \mathcal{C}_* \to \mathcal{C}$,

$$(_+, _-) \colon \mathcal{C} \rightleftarrows \mathcal{C}_*.$$

(For the notion of an adjunction between ∞-categories we again refer the reader to [82, p. 337] and [80, §3].) As a special case we obtain the *∞-category $\mathcal{S}_*$ of pointed spaces* and the corresponding adjunction

$$(_+, _-) \colon \mathcal{S} \rightleftarrows \mathcal{S}_*.$$

(ii) The underlying ∞-category of a pointed simplicial model category is pointed; see Examples 14.2.31. An ordinary category is pointed if and only if the nerve is a pointed ∞-category.

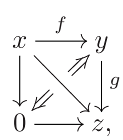
The ∞-category $\mathcal{S}_*$ of pointed spaces together with the 0-sphere $S^0 \in \mathcal{S}_*$ enjoys the following universal property (which is a pointed variant of Corollary 14.4.11).

Proposition 14.6.3. *Let $\mathcal{D}$ be a pointed, presentable ∞-category. Evaluation at the 0-sphere $S^0 \in \mathcal{S}_*$ induces an equivalence of ∞-categories*

$$\mathrm{Fun}^{\mathrm{L}}(\mathcal{S}_*, \mathcal{D}) \xrightarrow{\sim} \mathcal{D}.$$

This result makes precise that $\mathcal{S}_*$ is the *free pointed, presentable ∞-category generated by S^0*.

A *triangle* τ in a pointed ∞-category $\mathcal{C}$ is a diagram $\tau \colon \square \to \mathcal{C}$,



(14.6.1)

that vanishes at the lower left corner. Thus, a triangle in $\mathcal{C}$ encodes a pair of composable arrows $f \colon x \to y$ and $g \colon y \to z$, a further arrow $h \colon x \to z$ together with a homotopy $h \simeq g \circ f$ and a *null-homotopy* $h \simeq 0$. Recall the definition of pullback and pushout squares in Definition 14.3.25.

Definition 14.6.4. A triangle in a pointed ∞-category is *exact* if it is a pullback square. Dually, a triangle is *coexact* if it is a pushout square.

For every finitely complete, finitely cocomplete, and pointed ∞-category $\mathcal{C}$ we denote by

$$\mathcal{C}^{\Sigma} \subseteq \mathrm{Fun}(\square, \mathcal{C})$$

the full subcategory spanned by the coexact triangles that also vanish on the upper right corner,

$$\begin{array}{ccc} x & \longrightarrow & 0' \\ \downarrow & & \downarrow \\ 0 & \longrightarrow & y. \end{array}$$

There is a dually defined ∞-category $\mathcal{C}^{\Omega} \subseteq \mathrm{Fun}(\square, \mathcal{C})$ of exact triangles vanishing on the upper right corner. Morally, such diagrams should be determined by the value in the upper left corner in the first and by the value in the lower right corner in the second case. This is made precise by the following result (see [83, pp. 23-24]).

Proposition 14.6.5. *Let $\mathcal{C}$ be a finitely complete, finitely cocomplete, and pointed ∞-category. The evaluation maps*

$$\mathrm{ev}_{(0,0)}\colon \mathcal{C}^{\Sigma} \to \mathcal{C} \qquad \textit{and} \qquad \mathrm{ev}_{(1,1)}\colon \mathcal{C}^{\Omega} \to \mathcal{C}$$

are acyclic Kan fibrations.

This proposition and many similar results in this section are consequences of an ∞-categorical version of the calculus of Kan extensions. One of the key facts of constant use is [82, Prop. 4.3.2.15]. In this chapter we will not pursue this calculus any further, but we only consider results from this calculus which are 'similarly plausible' as Proposition 14.6.5. For more details on the calculus of homotopy Kan extensions we also refer to [48] and references therein.

We briefly recall that given an acyclic Kan fibration $p\colon X \to Y$, then the space of sections $\Gamma(p) \in s\mathcal{S}et$ is a contractible Kan complex. In fact, for every simplicial set K the induced map $p_*\colon \mathrm{Map}(K, X) \to \mathrm{Map}(K, Y)$ between simplicial mapping spaces as defined by (14.2.4) is again an acyclic Kan fibration. Since acyclic Kan fibrations are stable under pullbacks, we conclude that $\Gamma(p)$ is a contractible Kan complex by considering the defining pullback diagram

$$\begin{array}{ccc} \Gamma(p) & \longrightarrow & \mathrm{Map}(Y,X) \\ \downarrow & & \downarrow{\scriptstyle p_*} \\ \Delta^0 & \xrightarrow[\mathrm{id}_Y]{} & \mathrm{Map}(Y,Y). \end{array} \qquad (14.6.2)$$

Thus, under the assumption of Proposition 14.6.5, we can choose sections

$$s_{\Sigma}\colon \mathcal{C} \to \mathcal{C}^{\Sigma} \qquad \text{and} \qquad s_{\Omega}\colon \mathcal{C} \to \mathcal{C}^{\Omega}$$

of the evaluation maps $\mathrm{ev}_{(0,0)}$ and $\mathrm{ev}_{(1,1)}$, respectively, and these sections are unique up to contractible spaces of choices. Consequently, the following is well-defined.

Definition 14.6.6. Let $\mathcal{C}$ be a finitely complete, finitely cocomplete, and pointed ∞-category. The *suspension functor* $\Sigma = \Sigma_{\mathcal{C}}\colon \mathcal{C} \to \mathcal{C}$ and the *loop functor* $\Omega = \Omega_{\mathcal{C}}\colon \mathcal{C} \to \mathcal{C}$ are respectively defined as

$$\Sigma\colon \mathcal{C} \xrightarrow{s_{\Sigma}} \mathcal{C}^{\Sigma} \xrightarrow{\mathrm{ev}_{(1,1)}} \mathcal{C} \qquad \text{and} \qquad \Omega\colon \mathcal{C} \xrightarrow{s_{\Omega}} \mathcal{C}^{\Omega} \xrightarrow{\mathrm{ev}_{(0,0)}} \mathcal{C}.$$

Proposition 14.6.7. *Let $\mathcal{C}$ be a finitely complete, finitely cocomplete, and pointed ∞-category. The suspension functor $\Sigma\colon \mathcal{C} \to \mathcal{C}$ is left adjoint to the loop functor $\Omega\colon \mathcal{C} \to \mathcal{C}$,*

$$(\Sigma, \Omega)\colon \mathcal{C} \rightleftarrows \mathcal{C}.$$

This is [83, Rmk. 1.1.2.8]. In a similar way one defines *cofibers* and *fibers* in pointed ∞-categories. In fact, in the case of cofibers, starting with a morphism $f\colon x \to y$, suitable combinations of Kan extensions yield a coexact triangle as in (14.6.1). Combining these Kan extensions with a restriction of such triangles to the vertical morphism on the right yields a *cofiber* functor $\mathsf{cof}\colon \mathrm{Fun}(\Delta^1, \mathcal{C}) \to \mathrm{Fun}(\Delta^1, \mathcal{C})$. Dualizing this, we obtain a *fiber* functor $\mathsf{fib}\colon \mathrm{Fun}(\Delta^1, \mathcal{C}) \to \mathrm{Fun}(\Delta^1, \mathcal{C})$.

Proposition 14.6.8. *Let $\mathcal{C}$ be a finitely complete, finitely cocomplete, and pointed ∞-category. The cofiber functor $\mathsf{cof}\colon \mathrm{Fun}(\Delta^1, \mathcal{C}) \to \mathrm{Fun}(\Delta^1, \mathcal{C})$ is left adjoint to the fiber functor $\mathsf{fib}\colon \mathrm{Fun}(\Delta^1, \mathcal{C}) \to \mathrm{Fun}(\Delta^1, \mathcal{C})$,*

$$(\mathsf{cof}, \mathsf{fib})\colon \mathrm{Fun}(\Delta^1, \mathcal{C}) \rightleftarrows \mathrm{Fun}(\Delta^1, \mathcal{C}).$$

One way of defining *stable* ∞-categories is as follows.

Definition 14.6.9. A finitely complete, finitely cocomplete, and pointed ∞-category is *stable* if a triangle in it is exact if and only if it is coexact.

This is one way of imposing the usual *linearity condition* axiomatizing stability. It turns out that this definition admits the following reformulations (see [83, Prop. 1.1.3.4] and [83, Cor. 1.4.2.27]).

Theorem 14.6.10. *The following are equivalent for a finitely complete, finitely cocomplete, and pointed ∞-category $\mathcal{C}$.*

(i) The adjunction $(\Sigma, \Omega)\colon \mathcal{C} \rightleftarrows \mathcal{C}$ is an equivalence.

(ii) The adjunction $(\mathsf{cof}, \mathsf{fib})\colon \mathrm{Fun}(\Delta^1, \mathcal{C}) \rightleftarrows \mathrm{Fun}(\Delta^1, \mathcal{C})$ is an equivalence.

(iii) The ∞-category $\mathcal{C}$ is stable.

(iv) A square in $\mathcal{C}$ is a pullback square if and only if it is a pushout square.

The last characterization can be reformulated in terms of cubical diagrams [51, 10], and we refer to [49] for additional characterizations of stability. The relation of these characterizations to abstract representation theory is discussed in [48] and references therein.

Examples 14.6.11. (i) The underlying ∞-category of a stable, combinatorial model category (Examples 14.2.31) is stable.

(ii) The ∞-category of spectra can be realized as the underlying ∞-category of the stable model category of simplicial symmetric spectra [63]. The homotopy category of this ∞-category Sp is *the* stable homotopy category $\mathcal{SHC}$ of Boardman (see [113] and [4, Part III]). We will see in §14.6.2 that there also is an intrinsic construction of this ∞-category.

(iii) In the context of homological algebra, there are stable *derived ∞-categories*; see [82, §1].

Like stable model categories also stable ∞-categories provide an enhancement of triangulated categories. In particular, the homotopy category of a stable ∞-category can be endowed with a triangulation. To this end, we define a *cofiber sequence* in a pointed ∞-category $\mathcal{C}$ to be a diagram $\Delta^2 \times \Delta^1 \to \mathcal{C}$ looking like

$$\begin{array}{ccccc} x & \xrightarrow{f} & y & \longrightarrow & 0' \\ \downarrow & & \downarrow g & & \downarrow \\ 0 & \longrightarrow & z & \xrightarrow[h]{} & w \end{array} \qquad (14.6.3)$$

and such that both squares are pushout squares. (A cofiber sequence is essentially obtained by two iterations of the passage to the cofiber of a morphism.) As in the case of ordinary category theory, it follows that also the composite square is a pushout square, and, by definition of the suspension functor (Definition 14.6.6), we obtain a canonical equivalence $\phi\colon w \simeq \Sigma x$. Thus, if we pass to homotopy classes of morphisms, then associated to (14.6.3) we obtain by means of this equivalence an *incoherent* cofiber sequence or *triangle*

$$T_f\colon \qquad x \xrightarrow{f} y \xrightarrow{g} z \xrightarrow{\phi\circ h} \Sigma x,$$

which is an ordinary diagram in the homotopy category $\mathrm{Ho}(\mathcal{C})$. If $\mathcal{C}$ is a stable ∞-category, then we say that a triangle in $\mathrm{Ho}(\mathcal{C})$ is *distinguished* if it is isomorphic to T_f for some $f\colon \Delta^1 \to \mathcal{C}$.

In the following result ([83, Thm. 1.1.2.14]) we also denote by $\Sigma\colon \mathrm{Ho}(\mathcal{C}) \to \mathrm{Ho}(\mathcal{C})$ the functor induced by $\Sigma\colon \mathcal{C} \to \mathcal{C}$. We assume that the reader is familiar with the notion of a triangulated category; see the original references of Puppe [91, Satz 3.5 and §4.1] or Verdier [111] (a reprint of his 1967 thesis) as well as the monographs [90, 59].

Theorem 14.6.12. *Let $\mathcal{C}$ be a stable ∞-category. The functor $\Sigma\colon \mathrm{Ho}(\mathcal{C}) \to \mathrm{Ho}(\mathcal{C})$ and the above class of distinguished triangles endow the homotopy category $\mathrm{Ho}(\mathcal{C})$ with the structure of a triangulated category.*

For closely related discussion of these triangulations we refer to the introduction to [48] and references there. A natural class of functors between stable ∞-categories is the class of exact functors in the sense of the following definition.

Definition 14.6.13. Let $\mathcal{C}$ and $\mathcal{D}$ be finitely complete and finitely cocomplete ∞-categories.

(i) A functor $\mathcal{C} \to \mathcal{D}$ is *left exact* if it preserves pullbacks and terminal objects.

(ii) A functor $\mathcal{C} \to \mathcal{D}$ is *right exact* if it preserves pushouts and initial objects.

(iii) A functor $\mathcal{C} \to \mathcal{D}$ is *exact* if it is left exact and right exact.

Clearly, limit-preserving functors (and hence, in particular, right adjoint functors) are left exact, and dually. If $\mathcal{C}$ and $\mathcal{D}$ are stable ∞-categories, then the three classes in Definition 14.6.13 agree. It turns out that the triangulations of Theorem 14.6.12 are natural with respect to exact functors in the following sense. Given an exact functor $F\colon \mathcal{C} \to \mathcal{D}$ of stable

∞-categories, then the functor $F\colon \mathrm{Ho}(\mathcal{C}) \to \mathrm{Ho}(\mathcal{D})$ can be endowed with the structure of an exact functor.

Remark 14.6.14. Note that, by the very definition, a stable ∞-category is obtained from the general notion of an ∞-category by imposing certain (easily motivated) exactness *properties*; namely, we ask that finite limits and finite colimits exist and that certain limit type constructions are colimit type constructions and conversely. Similarly, the good notion of morphisms of stable ∞-categories, namely, exact functors, are defined by asking for the property that certain (co)limits are preserved.

This is in contrast to the more classical notion of a triangulated category which addresses the bad categorical properties of derived categories of abelian categories or of homotopy categories of stable model categories or stable ∞-categories by imposing additional *structure*. Given an additive category, the axioms of a triangulated category ask for the existence of some non-canonical additional structure (the suspension functor and the class of distinguished triangles) satisfying certain properties. Similarly, given two triangulated categories $\mathcal{T}$ and $\mathcal{T}'$, a morphism should be an additive functor $F\colon \mathcal{T} \to \mathcal{T}'$ which sends distinguished triangles to distinguished triangles. In order to make this precise, one has to ask for the existence of an exact structure, i.e., a natural isomorphism $F\Sigma \cong \Sigma F$.

Despite their great successes in many areas of pure mathematics, it was obvious from the very beginning on (see for example already the introduction to [54]) that the axioms of a triangulated category suffer certain defects (non-functoriality of the cone construction, no good theory of homotopy limits and homotopy colimits, diagram categories of triangulated categories do not admit canonical triangulations).

There are more traditional attempts to improve the axioms of a triangulated category and the basic idea goes back at least to [11]. The idea is to ask for more structure, leading to *higher triangulations*. This use of the word ‘higher’ is meant to indicate that one asks for higher octahedron axioms, i.e., that one also encodes iterated (co)fibers associated to longer strings of composable morphisms (see [88] for a precise definition). It can be shown that homotopy categories of stable ∞-categories or stable model categories can be naturally endowed with higher triangulations [50, §13], illustrating the slogan that ‘these enhancements encode all the triangulated structure’.

14.6.2 Stabilization and the universality of spectra

In this section we discuss the stabilization process which can be realized by passing to ∞-categories of internal spectrum objects. Similar constructions were also carried out in the language of model categories (for example by Schwede [101] and Hovey [61]) as well as in the framework of derivators by Heller [55].

The basic notion to begin with is the following one [79, Def. 8.1].

Definition 14.6.15. Let $\mathcal{C}$ be a finitely complete, finitely cocomplete, and pointed ∞-category. A *prespectrum object* in $\mathcal{C}$ is a functor

$$X\colon N(\mathbb{Z} \times \mathbb{Z}) \to \mathcal{C}$$

such that for all $i \neq j$ the value $X(i,j)$ is a zero object. The full subcategory of $\mathrm{Fun}(N(\mathbb{Z} \times \mathbb{Z}), \mathcal{C})$ spanned by the prespectrum objects is denoted by $\mathrm{PSp}(\mathcal{C})$.

Here, we consider the poset $\mathbb{Z}$ with the natural ordering as a category. Since only the diagonal entries are possibly non-trivial, we use the shorthand notation $X_m = X(m,m)$. Thus, a part of a prespectrum object $X \in \mathrm{PSp}(\mathcal{C})$ looks like

$$\begin{array}{ccccc} & & 0 & \longrightarrow & X_{m+1} \\ & & \uparrow & & \uparrow \\ 0'' & \longrightarrow & X_m & \longrightarrow & 0' \\ \uparrow & & \uparrow & & \\ X_{m-1} & \longrightarrow & 0'''. & & \end{array}$$

By definition of the suspension and loop functors $(\Sigma, \Omega) \colon \mathcal{C} \rightleftarrows \mathcal{C}$, given $X \in \mathrm{PSp}(\mathcal{C})$ we obtain induced morphisms

$$\alpha_{m-1} \colon \Sigma X_{m-1} \to X_m \qquad \text{and} \qquad \beta_m \colon X_m \to \Omega X_{m+1}. \tag{14.6.4}$$

Definition 14.6.16. Let $\mathcal{C}$ be a finitely complete, finitely cocomplete, and pointed ∞-category and let $X \in \mathrm{PSp}(\mathcal{C})$.

(i) The prespectrum X is a *spectrum below n* if $\beta_m \colon X_m \xrightarrow{\sim} \Omega X_{m+1}$ is an equivalence for all $m < n$. The full subcategory of $\mathrm{PSp}(\mathcal{C})$ spanned by the spectra below n is denoted by $\mathrm{Sp}_n(\mathcal{C})$.

(ii) The prespectrum X is a *spectrum object* if $\beta_m \colon X_m \xrightarrow{\sim} \Omega X_{m+1}$ is an equivalence for all $m \in \mathbb{Z}$. The full subcategory of $\mathrm{PSp}(\mathcal{C})$ spanned by the spectrum objects is denoted by $\mathrm{Sp}(\mathcal{C})$.

Example 14.6.17. An important special case is obtained if we start with the pointed ∞-category $\mathcal{S}_*$ of pointed spaces. In that case, we simplify notation and write $\mathrm{Sp} = \mathrm{Sp}(\mathcal{S}_*)$ for the ∞-*category of spectra*.

Theorem 14.6.18. *The ∞-category* Sp *of spectra is stable and presentable.*

We discuss further below the fact that Sp is the *stabilization* of the ∞-category $\mathcal{S}$ of spaces. More generally, given an ∞-category $\mathcal{C}$, we refer to

$$\mathrm{Stab}(\mathcal{C}) = \mathrm{Sp}(\mathcal{C}_*)$$

as the *stabilization* of $\mathcal{C}$.

Under certain assumptions on a pointed ∞-category $\mathcal{C}$ we will now construct a *spectrification functor*, i.e., a left adjoint $L\colon \mathrm{PSp}(\mathcal{C}) \to \mathrm{Sp}(\mathcal{C})$ to the fully faithful inclusion functor $\iota\colon \mathrm{Sp}(\mathcal{C}) \to \mathrm{PSp}(\mathcal{C})$, exhibiting $\mathrm{Sp}(\mathcal{C})$ as a localization of $\mathrm{PSp}(\mathcal{C})$. To begin with there is the following result.

Proposition 14.6.19. *Let $\mathcal{C}$ be a finitely complete, finitely cocomplete, and pointed ∞-category. The fully faithful inclusion $\iota_n\colon \mathrm{Sp}_n(\mathcal{C}) \to \mathrm{PSp}(\mathcal{C})$ admits a left adjoint $L_n\colon \mathrm{PSp}(\mathcal{C}) \to \mathrm{Sp}_n(\mathcal{C})$,*

$$(L_n, \iota_n)\colon \mathrm{PSp}(\mathcal{C}) \rightleftarrows \mathrm{Sp}_n(\mathcal{C}).$$

The idea is of course that spectra below a certain level are somehow determined by the higher levels. And in fact, the left adjoint L_n can be constructed as follows. Given a prespectrum $X \in \mathrm{PSp}(\mathcal{C})$ we restrict it to the full subcategory of $N(\mathbb{Z}\times\mathbb{Z})$ spanned by

$$Q_n = \{(i,j) \in \mathbb{Z}\times\mathbb{Z} \mid i \neq j \text{ or } i = j \geq n\}$$

and then set

$$L_n(X) = \mathrm{RKan}_{Q_n \hookrightarrow N(\mathbb{Z}\times\mathbb{Z})}(X|_{Q_n}).$$

Here RKan stands for an ∞-categorical variant of the usual right Kan extension [82, §4.3]. Under suitable completeness assumptions on the ∞-categories involved, right Kan extensions can again be calculated pointwise, i.e., are given by limits over certain slice categories. In our situation, the corresponding slice categories are cofinally finite and the above right Kan extensions hence exist. The essential image of L_n consists of the spectra below n.

With a bit more care, one can show that there is a sequence of functors

$$\mathrm{id} \to L_0 \to L_1 \to L_2 \to \ldots\colon \mathrm{PSp}(\mathcal{C}) \to \mathrm{PSp}(\mathcal{C}),$$

and it is hence tempting to simply set $L := \mathrm{colim}_n L_n$. This in fact works if one imposes the following conditions on the ∞-category $\mathcal{C}$ ([79, Cor. 8.17]).

Proposition 14.6.20. *Let $\mathcal{C}$ be a finitely complete, countably cocomplete, and pointed ∞-category. If the loop functor $\Omega_{\mathcal{C}}\colon \mathcal{C} \to \mathcal{C}$ commutes with sequential colimits, then*

$$L := \mathrm{colim}_n L_n\colon \mathrm{PSp}(\mathcal{C}) \to \mathrm{PSp}(\mathcal{C})$$

is a localization with essential image $\mathrm{Sp}(\mathcal{C})$*. We refer to L as the* spectrification functor*.*

An important example of an ∞-category satisfying these assumptions is the ∞-category of pointed spaces. In this case, let $\mathcal{D}_n \subseteq \mathrm{Sp}_n$ be the full subcategory spanned by those X such that $\alpha_m\colon \Sigma X_m \to X_{m+1}$ defined in (14.6.4) is an equivalence for $m \geq n$. Thus, morally, such a prespectrum X is essentially determined by its value X_n. And in fact, the evaluation map $\mathrm{ev}_n\colon \mathcal{D}_n \to \mathcal{S}_*$ is an acyclic Kan fibration. Let us choose a section $s_{\tilde{\Sigma}^{\infty-n}}\colon \mathcal{S}_* \to \mathcal{D}_n$ of ev_n and set

$$\tilde{\Sigma}^{\infty-n}\colon \mathcal{S}_* \xrightarrow{s_{\tilde{\Sigma}^{\infty-n}}} \mathcal{D}_n \to \mathrm{PSp}\,.$$

(We note again that this is well-defined since the space of sections is a contractible Kan complex; see the discussion around (14.6.2).) Denoting the n-th evaluation functor $\mathrm{PSp} \to \mathcal{S}_*$ by $\tilde{\Omega}^{\infty-n}$ we obtain an adjunction

$$(\tilde{\Sigma}^{\infty-n}, \tilde{\Omega}^{\infty-n})\colon \mathcal{S}_* \rightleftarrows \mathrm{PSp}\,.$$

Combining this with Proposition 14.6.19 and Proposition 14.6.20 we deduce the following result.

Proposition 14.6.21. *There is the following sequence of adjunctions*

$$(\Sigma_+^{\infty-n}, \Omega_-^{\infty-n}) \colon \mathcal{S} \underset{-}{\overset{+}{\rightleftarrows}} \mathcal{S}_* \underset{\tilde{\Omega}^{\infty-n}}{\overset{\tilde{\Sigma}^{\infty-n}}{\rightleftarrows}} \mathrm{PSp} \underset{\iota}{\overset{L}{\rightleftarrows}} \mathrm{Sp}\,.$$

Dropping the first adjunction in the proposition we obtain the adjunction

$$(\Sigma^{\infty-n}, \Omega^{\infty-n}) \colon \mathcal{S}_* \rightleftarrows \mathrm{Sp},$$

and it turns out that a similar adjunction exists for arbitrary pointed, presentable ∞-categories. In fact, the evaluation functor $\Omega^{\infty-n} \colon \mathrm{Sp}(\mathcal{C}) \to \mathcal{C}$ clearly makes sense for every finitely complete, finitely cocomplete, and pointed ∞-category $\mathcal{C}$. If $\mathcal{C}$ is moreover presentable, then it can be shown that $\Omega^{\infty-n}$ satisfies the assumptions of the special adjoint functor theorem (Theorem 14.4.13). Thus, there is a left adjoint $\Sigma^{\infty-n} \colon \mathcal{C} \to \mathrm{Sp}(\mathcal{C})$, the *suspension spectrum functor* or the *n-th free spectrum functor*,

$$(\Sigma^{\infty-n}, \Omega^{\infty-n}) \colon \mathcal{C} \rightleftarrows \mathrm{Sp}(\mathcal{C}),$$

although in this generality the functor $\Sigma^{\infty-n}$ does not admit such a nice explicit description as in the case of $\mathcal{S}_*$. In the following important result [79, Cor. 15.5] we denote by $\mathrm{Fun}^{\mathrm{L}}(\mathcal{C}, \mathcal{D})$ the ∞-category of colimit-preserving functors between presentable ∞-categories $\mathcal{C}$ and $\mathcal{D}$.

Theorem 14.6.22. *Let $\mathcal{C}, \mathcal{D}$ be pointed, presentable ∞-categories and suppose that $\mathcal{D}$ is stable. Restriction along the suspension spectrum functor $\Sigma^\infty \colon \mathcal{C} \to \mathrm{Sp}(\mathcal{C})$ induces an equivalence of ∞-categories*

$$\mathrm{Fun}^{\mathrm{L}}(\mathrm{Sp}(\mathcal{C}), \mathcal{D}) \xrightarrow{\sim} \mathrm{Fun}^{\mathrm{L}}(\mathcal{C}, \mathcal{D}).$$

As a central special case [79, Cor. 15.6], let us again consider the ∞-category $\mathcal{S}_*$. We refer to the image of the zero sphere $S^0 = \Delta^0_+$ under $\Sigma^\infty \colon \mathcal{S}_* \to \mathrm{Sp}$ as the *sphere spectrum*.

Corollary 14.6.23. *Let $\mathcal{D}$ be a stable, presentable ∞-category. Evaluation at the sphere spectrum induces an equivalence of ∞-categories*

$$\mathrm{Fun}^{\mathrm{L}}(\mathrm{Sp}, \mathcal{D}) \xrightarrow{\sim} \mathcal{D}.$$

In fact, this follows from Theorem 14.6.22 by observing that the evaluation map factors as

$$\mathrm{Fun}^{\mathrm{L}}(\mathrm{Sp}, \mathcal{D}) \xrightarrow{\sim} \mathrm{Fun}^{\mathrm{L}}(\mathcal{S}_*, \mathcal{D}) \xrightarrow{\sim} \mathcal{D}$$

where the second equivalence is given by evaluation on S^0; see Proposition 14.6.3. This corollary makes precise the statement that the ∞-category Sp of spectra is the *free stable ∞-category on one generator*, namely on the sphere spectrum.

14.6.3 Tensor products of presentable ∞-categories

We now turn to the tensor product of presentable ∞-categories which plays a key role in the construction of the smash product on the ∞-category of spectra; see §14.6.4. We begin with the corresponding construction in ordinary category theory.

Let $\mathcal{C}_1$ and $\mathcal{C}_2$ be locally presentable categories. The *tensor product* of $\mathcal{C}_1$ and $\mathcal{C}_2$ is a locally presentable category $\mathcal{C}_1 \otimes \mathcal{C}_2$ together with a universal bilinear map, i.e., a functor

$$\mathcal{C}_1 \times \mathcal{C}_2 \to \mathcal{C}_1 \otimes \mathcal{C}_2 \tag{14.6.5}$$

that preserves colimits in both variables separately and that is universal in the following sense: For any locally presentable category $\mathcal{D}$ restricting along (14.6.5) induces an equivalence of categories

$$\mathrm{Fun}^{\mathrm{L}}(\mathcal{C}_1 \otimes \mathcal{C}_2, \mathcal{D}) \xrightarrow{\sim} \mathrm{Fun}^{\mathrm{L,L}}(\mathcal{C}_1 \times \mathcal{C}_2, \mathcal{D})$$

where $\mathrm{Fun}^{\mathrm{L,L}}(\mathcal{C}_1 \times \mathcal{C}_2, \mathcal{D}) \subseteq \mathrm{Fun}(\mathcal{C}_1 \times \mathcal{C}_2, \mathcal{D})$ is the full subcategory spanned by the functors that preserve colimits in both variables separately.

It can be shown that the tensor product always exists [17, Chapter 5] and that it admits the explicit description

$$\mathcal{C}_1 \otimes \mathcal{C}_2 = \mathrm{Fun}^{\mathrm{R}}(\mathcal{C}_1^{\mathrm{op}}, \mathcal{C}_2), \tag{14.6.6}$$

where $\mathrm{Fun}^{\mathrm{R}}(-,-) \subseteq \mathrm{Fun}(-,-)$ denotes the full subcategory spanned by the *limit-preserving* functors.

In [83] Lurie establishes a variant of this monoidal structure for presentable ∞-categories. Let $\mathcal{P}\mathrm{r}^{\mathrm{L}}$ be the (very large) *∞-category of presentable ∞-categories* and colimit-preserving functors.

Theorem 14.6.24. *The ∞-category $\mathcal{P}\mathrm{r}^{\mathrm{L}}$ admits a closed symmetric monoidal structure $\mathcal{P}\mathrm{r}^{\mathrm{L},\otimes} \to N(\mathcal{F}\mathrm{in})$ such that the following properties are satisfied.*

(i) The tensor product $\mathcal{C}_1 \otimes \mathcal{C}_2$ corepresents the functor that sends $\mathcal{D}$ to the ∞-category $\mathrm{Fun}^{\mathrm{L,L}}(\mathcal{C}_1 \times \mathcal{C}_2, \mathcal{D})$ of functors that preserve colimits separately in both variables.

(ii) The ∞-category $\mathcal{C}_1 \otimes \mathcal{C}_2$ is equivalent to $\mathrm{Fun}^{\mathrm{R}}(\mathcal{C}_1^{\mathrm{op}}, \mathcal{C}_2)$.

(iii) The ∞-category $\mathcal{S}$ of spaces is the monoidal unit.

(iv) The internal hom is given by $\mathrm{Fun}^{\mathrm{L}}(\mathcal{C}_1, \mathcal{C}_2)$.

We denote by $\mathcal{P}\mathrm{r}^{\mathrm{L,smon}}$ the ∞-category of presentable, symmetric monoidal closed ∞-categories and symmetric monoidal, colimit-preserving functors. The following is a variant of Perspective 14.5.21 and it follows from the results in [83, §4.8] (see [83, Rmk. 4.8.1.8]).

Proposition 14.6.25. *There is an equivalence of ∞-categories*

$$\mathrm{Alg}_{\mathbb{E}_\infty}(\mathcal{P}\mathrm{r}^{\mathrm{L},\otimes}) \simeq \mathcal{P}\mathrm{r}^{\mathrm{L,smon}}.$$

Thus, Proposition 14.5.22 applied to the monoidal structure of Theorem 14.6.24 implies that the ∞-category $\mathcal{S}$ of spaces endowed with a certain symmetric monoidal structure is an initial object in $\mathcal{P}\mathrm{r}^{\mathrm{L,smon}}$. As a special case of [83, Cor. 4.8.1.12], it turns out that the monoidal structure is the usual Cartesian monoidal structure and that it can be characterized by the properties that

(i) the pairing $\times\colon \mathcal{S} \times \mathcal{S} \to \mathcal{S}$ preserves colimits separately in both variables and

(ii) the point $\Delta^0 \in \mathcal{S}$ is a monoidal unit.

14.6.4 The smash product

In this subsection we briefly discuss an ∞-categorical version of the smash product monoidal structure on spectra. The stabilization of presentable ∞-categories can be summarized by the following theorem. We denote by $\mathcal{P}\mathrm{r}^{\mathrm{L}}_{\mathrm{St}} \subseteq \mathcal{P}\mathrm{r}^{\mathrm{L}}_{\mathrm{Pt}} \subseteq \mathcal{P}\mathrm{r}^{\mathrm{L}}$ the full subcategories spanned by stable, presentable and pointed and presentable ∞-categories, respectively. The following result is a consequence of the techniques in [83, §4.8] (see, in particular, [83, Prop. 4.8.2.11] and [83, Prop. 4.8.2.18]).

Theorem 14.6.26. *The stabilization* $\mathrm{Stab}\colon \mathcal{P}\mathrm{r}^{\mathrm{L}} \to \mathcal{P}\mathrm{r}^{\mathrm{L}}_{\mathrm{St}}$ *of presentable ∞-categories factors as a composition of adjunctions*

$$\mathcal{P}\mathrm{r}^{\mathrm{L}} \rightleftarrows \mathcal{P}\mathrm{r}^{\mathrm{L}}_{\mathrm{Pt}} \rightleftarrows \mathcal{P}\mathrm{r}^{\mathrm{L}}_{\mathrm{St}}.$$

It turns out that the symmetric monoidal structure $\mathcal{P}\mathrm{r}^{\mathrm{L},\otimes} \to N(\mathcal{F}\mathrm{in})$ has similar variants in the pointed and in the stable context. These results are instances of a more general construction of canonical monoidal structures on smashing localizations associated to idempotent objects (see again [83, §4.8.2]). Let us collect the variant of Theorem 14.6.24 for stable, presentable ∞-categories.

Theorem 14.6.27. *The ∞-category $\mathcal{P}\mathrm{r}^{\mathrm{L}}_{\mathrm{St}}$ admits a closed symmetric monoidal structure $\mathcal{P}\mathrm{r}^{\mathrm{L},\otimes}_{\mathrm{St}} \to N(\mathcal{F}\mathrm{in})$ such that the following properties are satisfied.*

(i) The tensor product $\mathcal{C}_1 \otimes \mathcal{C}_2$ corepresents the functor that sends $\mathcal{D}$ to the ∞-category $\mathrm{Fun}^{\mathrm{L,L}}(\mathcal{C}_1 \times \mathcal{C}_2, \mathcal{D})$ of functors that preserve colimits separately in both variables.

(ii) The ∞-category $\mathcal{C}_1 \otimes \mathcal{C}_2$ is equivalent to $\mathrm{Fun}^{\mathrm{R}}(\mathcal{C}_1^{\mathrm{op}}, \mathcal{C}_2)$.

(iii) The ∞-category Sp of spectra is the monoidal unit.

(iv) The internal hom is given by $\mathrm{Fun}^{\mathrm{L}}(\mathcal{C}_1, \mathcal{C}_2)$.

And there is a similar variant for $\mathcal{P}\mathrm{r}^{\mathrm{L}}_{\mathrm{Pt}}$ the only difference being that the ∞-category $\mathcal{S}_*$ of pointed spaces is the monoidal unit.

Let $\mathcal{P}\mathrm{r}^{\mathrm{L,smon}}_{\mathrm{St}}$ denote the ∞-category of stable, presentable, symmetric monoidal closed ∞-categories and symmetric monoidal, colimit-preserving functors. As a consequence of Perspective 14.5.21 there is the following result.

Proposition 14.6.28. *There is an equivalence of ∞-categories*

$$\mathrm{Alg}_{\mathbb{E}_\infty}(\mathcal{P}\mathrm{r}^{\mathrm{L},\otimes}_{\mathrm{St}}) \simeq \mathcal{P}\mathrm{r}^{\mathrm{L,smon}}_{\mathrm{St}}.$$

An application of Proposition 14.5.22 to $\mathcal{P}\mathrm{r}^{\mathrm{L},\otimes}_{\mathrm{St}}$ implies that the ∞-category Sp of spectra can be endowed with a certain symmetric monoidal structure, the *smash product*, such that the resulting symmetric monoidal ∞-category $\mathrm{Sp}^{\otimes}$ is an initial object in $\mathcal{P}\mathrm{r}^{\mathrm{L,smon}}_{\mathrm{St}}$. It turns out that the monoidal structure can be characterized by the properties that

(i) the monoidal structure $\otimes\colon \mathrm{Sp} \times \mathrm{Sp} \to \mathrm{Sp}$ preserves colimits separately in both variables and

(ii) the sphere spectrum is the monoidal unit.

Having the smash product at our disposal we can finally make the following definition.

Definition 14.6.29. (i) The ∞-*category of* $\mathbb{E}_\infty$-*ring spectra* is the ∞-category of commutative algebra objects $\mathrm{Alg}_{\mathbb{E}_\infty}(\mathrm{Sp}^\otimes)$.

(ii) The ∞-*category of* $\mathbb{A}_\infty$-*ring spectra* is the ∞-category of (associative) algebra objects $\mathrm{Alg}_{\mathbb{A}_\infty}(\mathrm{Sp}^\otimes)$.

These are ∞-categorical versions of the more classical model categories of commutative or associative ring spectra. In fact, for a precise statement along these lines using *symmetric spectra* see [81, Example 8.21]. Having these key notions in place one could now begin with an ∞-categorical study of stable homotopy theory [83] which together with the theory of ∞-topoi [82, §§6-7] provides the foundations for Lurie's ∞-categorical approach to derived algebraic geometry. For this we refer the reader to the literature and to David Gepner's chapter in this Handbook.

We conclude this section by a short discussion of monoidal aspects of the stabilization of presentable ∞-categories. References for the remainder of this sections are [83, §8.4.2] and [44, §3].

Theorem 14.6.30. *Let $\mathcal{C}^\otimes$ be a closed symmetric monoidal structure on a presentable ∞-category. The ∞-categories $\mathcal{C}_*$ and $\mathrm{Sp}(\mathcal{C})$ admit closed symmetric monoidal structures, that are uniquely determined by the requirement that the respective free functors from $\mathcal{C}$ are symmetric monoidal. Moreover, also the remaining left adjoint in*

$$\mathcal{C} \to \mathcal{C}_* \to \mathrm{Sp}(\mathcal{C})$$

is uniquely symmetric monoidal.

The monoidal structures in this theorem enjoy the following universal properties. Given closed symmetric monoidal, presentable ∞-categories $\mathcal{C}, \mathcal{D}$ we denote by $\mathrm{Fun}^{\mathrm{L},\otimes}(\mathcal{C}, \mathcal{D})$ the ∞-category of symmetric monoidal, colimit-preserving functors from $\mathcal{C}$ to $\mathcal{D}$.

Theorem 14.6.31. *Let $\mathcal{C}, \mathcal{D}$ be closed symmetric monoidal, presentable ∞-categories.*

(i) If $\mathcal{D}$ is pointed, then the symmetric monoidal functor $\mathcal{C} \to \mathcal{C}_*$ induces an equivalence of ∞-categories

$$\mathrm{Fun}^{\mathrm{L},\otimes}(\mathcal{C}_*, \mathcal{D}) \xrightarrow{\sim} \mathrm{Fun}^{\mathrm{L},\otimes}(\mathcal{C}, \mathcal{D}).$$

(ii) If $\mathcal{D}$ is stable, then the symmetric monoidal functor $\mathcal{C} \to \mathrm{Sp}(\mathcal{C})$ induces an equivalence of ∞-categories

$$\mathrm{Fun}^{\mathrm{L},\otimes}(\mathrm{Sp}(\mathcal{C}), \mathcal{D}) \xrightarrow{\sim} \mathrm{Fun}^{\mathrm{L},\otimes}(\mathcal{C}, \mathcal{D}).$$

One way to summarize some of these results is as follows.

Theorem 14.6.32. *The stabilization $\mathcal{P}\mathrm{r}^{\mathrm{L}} \to \mathcal{P}\mathrm{r}^{\mathrm{L}}_{\mathrm{St}}$ of presentable ∞-categories admits a monoidal refinement which factors as a composition of adjunctions*

$$\mathcal{P}\mathrm{r}^{\mathrm{L},\otimes} \rightleftarrows \mathcal{P}\mathrm{r}^{\mathrm{L},\otimes}_{\mathrm{Pt}} \rightleftarrows \mathcal{P}\mathrm{r}^{\mathrm{L},\otimes}_{\mathrm{St}}.$$

We conclude this course by the following perspective on a refined picture of the stabilization process [44].

Perspective 14.6.33. Let us recall that stable ∞-categories are obtained from pointed ∞-categories by imposing an additional exactness property, asking that pushouts and pullbacks agree. There are two more intermediate steps given by preadditive and additive ∞-categories, respectively, both of which are obtained by adding certain exactness properties to pointed ∞-categories. A *preadditive* ∞-category is a pointed ∞-category with finite biproducts. It follows from these axioms that every object can be canonically turned into an $\mathbb{E}_\infty$-monoid object and this monoid structure is given by the fold map. We say that a preadditive ∞-category is *additive* if these canonical $\mathbb{E}_\infty$-monoid structures actually are $\mathbb{E}_\infty$-group structures.

Focusing again on the context of presentable ∞-categories it can be shown that there are universal examples of such ∞-categories. In fact, the ∞-category $\mathrm{Mon}_{\mathbb{E}_\infty}(\mathcal{S})$ of $\mathbb{E}_\infty$-spaces is a preadditive, presentable ∞-category and the left adjoint $\mathcal{S} \to \mathrm{Mon}_{\mathbb{E}_\infty}(\mathcal{S})$ to the forgetful functor $\mathrm{Mon}_{\mathbb{E}_\infty}(\mathcal{S}) \to \mathcal{S}$ exhibits $\mathrm{Mon}_{\mathbb{E}_\infty}(\mathcal{S})$ as the free preadditive ∞-category on one generator, namely the free $\mathbb{E}_\infty$-space generated by Δ^0. More specifically, for every preadditive, presentable ∞-category $\mathcal{D}$, there are equivalences of ∞-categories

$$\mathrm{Fun}^{\mathrm{L}}(\mathrm{Mon}_{\mathbb{E}_\infty}(\mathcal{S}), \mathcal{D}) \xrightarrow{\sim} \mathrm{Fun}^{\mathrm{L}}(\mathcal{S}, \mathcal{D}) \xrightarrow{\sim} \mathcal{D}.$$

A similar universal property is enjoyed by $\mathcal{C} \to \mathrm{Mon}_{\mathbb{E}_\infty}(\mathcal{C})$ where $\mathcal{C}$ is a general presentable ∞-categories. And if we pass to the context of additive presentable ∞-categories instead, then the universal example is given by the ∞-category $\mathrm{Grp}_{\mathbb{E}_\infty}(\mathcal{S})$ of grouplike $\mathbb{E}_\infty$-spaces.

It turns out that the stabilization of presentable ∞-categories factors through the ∞-category of preadditive, presentable ∞-categories and also through the ∞-category of additive, presentable ∞-categories. More precisely, the obvious forgetful functors admit left adjoints and the stabilization hence factors as

$$\mathcal{P}\mathrm{r}^{\mathrm{L}} \rightleftarrows \mathcal{P}\mathrm{r}^{\mathrm{L}}_{\mathrm{Pt}} \rightleftarrows \mathcal{P}\mathrm{r}^{\mathrm{L}}_{\mathrm{Pre}} \rightleftarrows \mathcal{P}\mathrm{r}^{\mathrm{L}}_{\mathrm{Add}} \rightleftarrows \mathcal{P}\mathrm{r}^{\mathrm{L}}_{\mathrm{St}}.$$

Finally, let us also mention that this factorization admits a monoidal refinement parallel to the one in Theorem 14.6.32. For details and applications we refer the reader to [44].

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