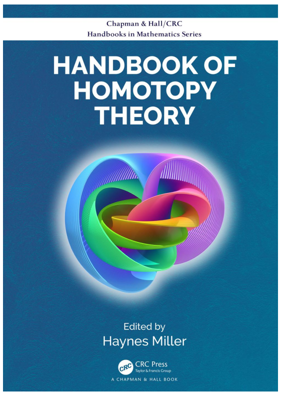

Handbook of Homotopy Theory

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Floer homotopy theory, revisited

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10

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Introduction

In three seminal papers in 1988 and 1989 A. Floer introduced Morse theoretic homological invariants that transformed the study of low dimensional topology and symplectic geometry. In [17] Floer defined an "instanton homology" theory for 3-manifolds that, when paired with Donaldson's polynomial invariants of 4-manifolds defined a gauge theoretic 4-dimensional topological field theory that revolutionized the study of low dimensional topology and geometry. In [18], Floer defined an infinite dimensional Morse theoretic homological invariant for symplectic manifolds , now referred to as "Symplectic" or "Hamiltonian" Floer homology, that allowed him to prove a well-known conjecture of Arnold on the number of fixed points of a diffeomorphism $\phi_1 : M \xrightarrow{\cong} M$ arising from a time-dependent Hamiltonian flow $\{\phi_t\}_{0 \leq t \leq 1}$. In [16] Floer introduced "Lagrangian intersection Floer theory" for the study of interesections of Lagrangian submanifolds of a symplectic manifold.

Since that time there have been many other versions of Floer theory that have been introduced in geometric topology, including a Seiberg-Witten Floer homology [31]. This is similar in spirit to Floer's "instanton homology", but it is based on the Seiberg-Witten equations rather than the Yang-Mills equations. There were many difficult, technical analytic issues in developing Seiberg-Witten Floer theory, and Kronheimer and Mrowka's book [31] deals with them masterfully and elegantly. Another important geometric theory is Heegard Floer theory introduced by Oszvath and Szabo [44]. This is an invariant of a closed 3-manifold equipped with a $spin^c$ structure. It is computed using a Heegaard diagram of the manifold. It allowed for a related "knot Floer homology" introduced by Oszvath and Szabo [45] and by Rasmussen [47]. Khovanov's important homology theory that gave a "categorification" of the Jones polynomial [27] was eventually shown to be related to Floer theory by Seidel and Smith [55] and Abouzaid and Smith [3]. Lipshitz and Sarkar [35] showed that there is an associated "Khovanov stable homotopy". There have been many other variations of Floer theories as well.

The rough idea in all of these theories is to associate a Morse-like chain complex generated by the critical points of a functional defined typically on an infinite dimensional space. Recall that in classical Morse theory, given a Morse function $f : M \to \mathbb{R}$ on a closed

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Riemannian manifold, the "Morse complex" is the chain complex

$$\cdots \to C_p(f) \xrightarrow{\partial_p} C_{p-1}(f) \to \cdots$$

where $C_p(f)$ is the free abelian group generated by the critical points of f of index p, and the boundary homomorphisms can be computed by "counting" the gradient flow-lines connecting critical points of relative index one. More specfically, if $Crit_q(f)$ is the set of critical points of f of index q, then if $a \in Crit_p(f)$, then

$$\partial_p([a]) = \sum_{b \in Crit_{p-1}(f)} \#\mathcal{M}(a,b)\,[b] \tag{10.0.1}$$

where $\mathcal{M}(a,b)$ is the moduli space of gradient flow lines connecting a to b, which, since a and b have relative index one is a closed, zero dimensional oriented manifold. $\#\mathcal{M}(a,b)$ reflects the "oriented count" of this finite set. More carefully $\#\mathcal{M}(a,b)$ is the integer in the zero dimensional oriented cobordism group, $\pi_0 MSO \cong \mathbb{Z}$ represented by $\mathcal{M}(a,b)$.

In Floer's original examples, the functionals he studied were in fact $\mathbb{R}/\mathbb{Z}$-valued. In the case of Floer's instanton theory, the relevant functional is the Chern-Simons map defined on the space of connections on a principal $SU(2)$-bundle over the 3-manifold. Its critical points are flat connections and its flow lines are "instantons", i.e. anti-self-dual connections on the three-manifold Y crossed with the real line. Modeling classical Morse theory, the "Floer complex" is generated by the critical points of this functional, suitably perturbed to make them nondegenerate, and the boundary homomorphisms are computed by taking oriented counts of the gradient flow lines, i.e. anti-self-dual connections on $Y \times \mathbb{R}$ that connect critical points of relative index one.

When Floer introduced what is now called "Symplectic" or "Hamiltonian" Floer homology ("SFH") to use in his proof of Arnold's conjecture, he studied the symplectic action defined on the free loop space of the underlying symplectic manifold M,

$$\mathcal{A} : LM \to \mathbb{R}/\mathbb{Z}.$$

He perturbed the action functional by a time dependent Hamiltonian function $H : \mathbb{R}/\mathbb{Z} \times M \to \mathbb{R}$. Call the resulting functional $\mathcal{A}_H$. The critical points of $\mathcal{A}_H$ are the 1-periodic orbits of the Hamiltonian vector field. That is, they are smooth loops $\alpha : \mathbb{R}/\mathbb{Z} \to M$ satisfying the differential equaiton

$$\frac{d\alpha}{dt} = X_H(t, \alpha(t))$$

where X_H is the Hamiltonian vector field. One way to think of X_H is that the symplectic 2-form ω on M defines, since it is nondegenerate, a bundle isomorphism $\omega : TM \xrightarrow{\cong} T^*M$. This induces an identification of the periodic one-forms, that is sections of the cotangent bundle pulled back over $\mathbb{R}/\mathbb{Z} \times M$, with periodic vector fields. The Hamiltonian vector field X_H corresponds to the differential dH under this identification. Floer showed that with respect to a generic Hamiltonian, the critical points of $\mathcal{A}_H$ are nondegenerate.

Of course to understand the gradient flow lines connecting critical points, one must have a metric. This is defined using the symplectic form and a compatible choice of almost complex structure J on TM. With respect to this structure the gradient flow lines are curves

$\gamma : \mathbb{R} \to LM$, or equivalently, maps of cylinders

$$\gamma : \mathbb{R} \times S^1 \to M$$

that satisfy (perturbed) Cauchy-Riemann equations. If $\mathcal{M}(\alpha_1, \alpha_2, H, J)$ is the moduli space of such "pseudo-holomorphic" cylinders that connect the periodic orbits α_1 and α_2, then the boundary homomorphisms Morse-Floer chain complex is defined by giving an oriented count of the zero dimensional moduli spaces, in analogy with the situation in classical Morse theory described above.

The other, newer examples of Floer homology tend to be similar. There is typically a functional that can be perturbed in such a way that its critical points and zero dimensional moduli spaces of gradient flow lines define a chain complex whose homology is invariant of the choices made.

In the case of a classical Morse function $f : M \to \mathbb{R}$ closed manifold, the Morse complex can be viewed as the cellular chain complex of a CW-complex X_f of the homotopy type of M. X_f has one cell of dimension λ, for every critical point of f of index λ. The attaching maps were studied by Franks in [19], and one may view the work of the author and his collaborators in [11], [12] as a continuation of that study. This led us to ask the question:

Q1: Does the Floer chain complex arise as the cellular chain complex of a CW-complex or a $C.W$-spectrum?

More specifically we asked the the following question:

Q2: What properties of the data of a Floer functional, i.e. its critical points and moduli spaces of gradient flow lines connecting them, are needed to define a CW-spectrum realizing the Floer chain complex?

More generally one might ask the following question:

Q3: Given a finite chain complex, is there a reasonable way to classify the CW-spectra that realize this complex?

In this paper we take up these questions. We also discuss how studying Floer theory from a homotopy perspective has been done in recent years, and how it has been applied with dramatic success. We state immediately that the applications that we discuss in this paper are purely the choice of the author. There are many other fascinating applications and advances that all help to make this an active and exciting area of research. We apologize in advance to researchers whose work we will not have the time or space to discuss.

This paper is organized as follows. In Section 2 we discuss the three questions raised above. We give a new take on how these questions were originally addressed in [12], and give some of the early applications of this perspective. In Section 3 we describe how Lipshitz and Sarkar used this perspective to define the notion of "Khovanov homotopy" [35][36]. This is a stable homotopy theoretic realization of Khovanov's homology theory [27] which in turn is a categorification of the Jones polynomial invariant of knots and links. In particular we describe how the homotopy perspective Lipshitz and Sarkar used give more subtle and delicate invariants than the homology theory alone, and how these invariants have been applied. In Section 3 we describe Manolescu's work on an equivariant stable homotopy theoretic view of Seiberg-Witten Floer theory. In the case of his early work [37][38], the group acting is the circle group S^1. In his more recent work [39] [41] he studied $Pin(2)$-equivariant Floer homotopy theory and used it to give a dramatic solution (in the negative) to the longstanding question about whether all topological manifolds admit triangulations. In Section 4 we describe the Floer homotopy theoretic methods of Kragh [30] and of Abouzaid-Kragh [2] in the study of the symplectic topology of the cotangent bundle of a closed manifold, and how they were useful in studying Lagrangian immersions and embeddings inside the cotangent bundle.

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10.1 The homotopy theoretic foundations

10.1.1 Realizing chain complexes

We begin with a purely homotopy theoretic question: Given a finite chain complex, C_*,

$$C_n \xrightarrow{\partial_n} C_{n-1} \xrightarrow{\partial_{n-1}} \cdots \xrightarrow{\partial_2} C_1 \xrightarrow{\partial_1} C_0$$

how can one classify the finite CW-spectra X whose associated cellular chain complex is C_*? This is question **Q3** above.

Of course one does not need to restrict this question to finite complexes, but that is a good place to start. In particular it is motivated by Morse theory, where, given a Morse function on a closed, n-dimensional Riemannian manifold, $f : M^n \to \mathbb{R}$, one has a corresponding "Morse - Smale" chain complex C^f_*

$$C^f_n \xrightarrow{\partial_n} C^f_{n-1} \xrightarrow{\partial_{n-1}} \cdots \xrightarrow{\partial_2} C^f_1 \xrightarrow{\partial_1} C^f_0.$$

Here C^f_p is the free abelian group generated by the critical points of f of index p. The boundary homomorphisms are as described above (10.0.1). Of course in this case the Morse function together with the Riemannian metric define a CW complex homotopy equivalent to X. In Floer theory that is not the case. One does start with geometric information that allows for the definition of a chain complex, but knowing if this complex comes, in a natural way from the data of the Floer theory, is not at all clear, and was the central question of study in [12].

This homotopy theoretic question was addressed more specifically in [10]. Of central importance in this study was to understand how the attaching maps of the cells in a finite CW-spectrum can be understood geometrically, via the theory of (framed) cobordism of manifolds with corners. We will review these ideas in this section and recall some basic examples of how they can be applied.

By assumption, the chain complex C_* is finite, so each C_i is a finitely generated free abelian group. Let $\mathcal{B}_i$ be a basis for C_i. Let $\mathbb{S}$ denote the sphere spectrum. For each i, consider the free $\mathbb{S}$-module spectrum generated by $\mathcal{B}_i$,

$$\mathcal{E}_i = \bigvee_{\alpha \in \mathcal{B}_i} \mathbb{S}.$$

There is a natural isomorphism

$$H_0(\mathcal{E}_i) \simeq C_i.$$

Definition 10.1.1. We say that a finite spectrum X *realizes* the complex C_* if there exists a filtration of subspectra of X,

$$X_0 \hookrightarrow X_1 \hookrightarrow \cdots \hookrightarrow X_n = X,$$

satisfying the following properties:

1. There is an equivalence of the subquotients

$$X_i/X_{i-1} \simeq \Sigma^i \mathcal{E}_i,$$

2. The induced composition map in integral homology,

$$\begin{aligned}&\tilde{H}_i(X_i/X_{i-1}) \xrightarrow{\delta_i} \tilde{H}_i(\Sigma X_{i-1}) \xrightarrow{\rho_{i-1}} H_i(\Sigma(X_{i-1}/X_{i-2}))\\ &= H_0(\mathcal{E}_i) \to H_0(\mathcal{E}_{i-1})\\ &= C_i \to C_{i-1}\end{aligned}$$

is the boundary homomorphism, ∂_i.

Here the "subquotient" X_i/X_{i-1} refers to the homotopy cofiber of the map $X_{i-1} \to X_i$, the map $\rho_i : X_i \to X_i/X_{i-1}$ is the projection map, and the map $\delta_i : X_i/X_{i-1} \to \Sigma X_{i-1}$ is the Barratt-Puppe extension of the homotopy cofibration sequence $X_{i-1} \to X_i \xrightarrow{\rho_i} X_i/X_{i-1}$.

To classify finite spectra that realize a given finite complex, in [12] the authors introduced a topological category $\mathcal{J}$ whose objects are the nonnegative integers $\mathbb{Z}^+$, and whose non-identity morphisms from i to j is empty for $i \leq j$, for $i > j+1$ it is defined to be the one point compactification,

$$Mor_{\mathcal{J}}(i,j) \cong (\mathbb{R}_+)^{i-j-1} \cup \infty$$

and $Mor_{\mathcal{J}}(j+1,j)$ is defined to be the two point space, S^0. Here $\mathbb{R}_+$ is the space of nonnegative real numbers. Composition in this category can be viewed in the following way. Notice that for $i > j+1$ $Mor_{\mathcal{J}}(i,j)$ can be viewed as the one point compactification of the space $J(i,j)$ consisting of sequences of real numbers $\{\lambda_k\}_{k\in\mathbb{Z}}$ such that

$$\begin{aligned}\lambda_k &\geq 0 \quad \text{for all } k\\ \lambda_k &= 0 \quad \text{unless } i > k > j.\end{aligned}$$

For consistency of notation we write $Mor_{\mathcal{J}}(i,j) = J(i,j)^+$. Composition of morphisms $J(i,j)^+ \wedge J(j,k)^+ \to J(i,k)^+$ is then induced by addition of sequences. In this smash product the basepoint is taken to be ∞. Notice that this map is basepoint preserving. Given integers $p > q$, then there are subcategories $\mathcal{J}_q^p$ defined to be the full subcategory generated by integers $q \geq m \geq p$. The category $\mathcal{J}_q$ is the full subcategory of $\mathcal{J}$ generated by all integers $m \geq q$.

The following is a recasting of a discussion in [12].

Theorem 10.1.2. *The realizations of the chain complex C_* by finite spectra are classified by extensions of the association $j \to \mathcal{E}_j$ to basepoint preserving functors $Z : \mathcal{J}_0 \to Spectra$, with the property that for each $j \geq 0$, the map obtained by the application of morphisms,*

$$\begin{aligned} Z_{j+1,j} : J(j+1,j)^+ \wedge \mathcal{E}_{j+1} &\to \mathcal{E}_j \\ S^0 \wedge \mathcal{E}_{j+1} &\to \mathcal{E}_j \end{aligned}$$

induces the boundary homomorphism ∂_{j+1} on the level of homology groups. Here Spectra is a symmetric monoidal category of spectra (e.g. the category of symmetric spectra), and by a "basepoint preserving functor" we mean one that maps ∞ in $J(p,q)^+$ to the constant map $Z(p) \to Z(q)$. By "classified" we mean that the filtered homotopy type of any realization of C_* determines a basepoint preserving functor, $Z : \mathcal{J}_0 \to Spectra$ with these properties, and conversely every such functor determines a realization of the chain complex C_*.

We recall from [12] how a functor $Z : \mathcal{J}_0 \to Spectra$ satisfying the properties described in the theorem, defines a realization of the chain complex C_*. As described in [12], given a functor to the category of spaces, $Z : \mathcal{J}_q \to Spaces_*$, where $Spaces_*$ denotes the category of based topological spaces, one can take its geometric realization,

$$|Z| = \coprod_{q \leq j} Z(j) \wedge J(j, q-1)^+ / \sim \tag{10.1.1}$$

where one identifies the image of $Z(j) \wedge J(j,i)^+ \wedge J(i,q-1)^+$ in $Z(j) \wedge J(j,q-1)^+$ given by composition of morphisms, with its image in $Z(i) \wedge J(i,q-1)$ defined by application of morphisms $Z(j) \wedge J(j,i)^+ \to Z(i)$.

For a functor whose value is in *Spectra*, we replace the above construction by a coequalizer, in the following way:

Let $Z : \mathcal{J}_q \to Spectra$. Define two maps of spectra,

$$\iota, \mu : \bigvee_{q \leq j} Z(j) \wedge J(j,i)^+ \wedge J(i,q-1)^+ \longrightarrow \bigvee_{q \leq j} Z(j) \wedge J(j,q-1)^+. \tag{10.1.2}$$

The first map ι is induced by the composition of morphisms in $\mathcal{J}_q$, $J(j,i)^+ \wedge J(i,q-1)^+ \hookrightarrow J(j,q-1)^+$. The second map μ is the given by the wedge of maps,

$$Z(j) \wedge J(j,i)^+ \wedge J(i,q-1)^+ \xrightarrow{\mu_q \wedge 1} Z(i) \wedge J(i,q-1)^+$$

where $\mu_q : Z(j) \wedge J(j,i)^+ \to Z(i)$ is given by application of morphisms.

Definition 10.1.3. Given a functor $Z : \mathcal{J}_q \to \mathit{Spectra}$ we define its *geometric realization* $|Z|$ to be the coequalizer (in the category *Spectra*) of the two maps,

$$\iota, \mu : \bigvee_{q \leq j} \bigvee_{i=q}^{j-1} Z(j) \wedge J(j,i)^+ \wedge J(i,q-1)^+ \longrightarrow \bigvee_{q \leq j} Z(j) \wedge J(j,q-1)^+.$$

Technically, so that the homotopy type of the coequalizer is well-defined, the functor may have to be modified so that it takes each morphism to a cofibration.

So a functor $Z : \mathcal{J}_0 \to \mathit{Spectra}$ satisfying the properties specified in Theorem 10.1.2 defines a geometric realization $|Z|$ which is a finite spectrum. Consider how the data of the functor Z defines the CW-structure of $|Z|$. Clearly $|Z|$ will have one cell of dimension i for every element of $\pi_0(Z(i)) = \pi_0(\mathcal{E}_i) = \mathcal{B}_i$. The attaching maps were described in [12], [9], [10] in the following way.

In general assume that X be a finite CW-spectrum with skeletal filtration

$$X_0 \hookrightarrow X_1 \hookrightarrow \cdots \hookrightarrow X_n = X.$$

In particular each map $X_{i-1} \hookrightarrow X_i$ is a cofibration, and we call its cofiber $K_i = X_i/(X_{i-1})$. This is a wedge of (suspension spectra) of spheres of dimension i.

$$K_i \simeq \bigvee_{\mathrm{cd}_i} \Sigma^i \mathbb{S}$$

where cd_i is a finite indexing set.

As was discussed in [12] one can then "rebuild" the homotopy type of the n-fold suspension, $\Sigma^n X$, as the union of iterated cones and suspensions of the K_i's,

$$\Sigma^n X \simeq \Sigma^n K_0 \cup c(\Sigma^{n-1} K_1) \cup \cdots \cup c^i(\Sigma^{n-i} K_i) \cup \cdots \cup c^n K_n. \tag{10.1.3}$$

This decomposition can be described as follows. Define a map $\delta_i : \Sigma^{n-i} K_i \to \Sigma^{n-i+1} K_{i-1}$ to be the iterated suspension of the composition,

$$\delta_i : K_i \to \Sigma X_{i-1} \to \Sigma K_{i-1}$$

where the two maps in this composition come from the cofibration sequence, $X_{i-1} \to X_i \to K_i \to \Sigma X_{i-1} \cdots$. As was pointed out in [9], this induces a "homotopy chain complex",

$$\begin{aligned} K_n \xrightarrow{\delta_n} \Sigma K_{n-1} \xrightarrow{\delta_{n-1}} \cdots \xrightarrow{\delta_{i+1}} \Sigma^{n-i} K_i \xrightarrow{\delta_i} \Sigma^{n-i+1} K_{i-1} \xrightarrow{\delta_{i-1}} \\ \cdots \xrightarrow{\delta_1} \Sigma^n K_0 = \Sigma^n X_0. \end{aligned} \tag{10.1.4}$$

We refer to this as a homotopy chain complex because examination of the defining cofibrations leads to canonical null homotopies of the compositions,

$$\delta_j \circ \delta_{j+1}.$$

This canonical null homotopy defines an extension of δ_j to the mapping cone of δ_{j+1}:

$$c(\Sigma^{n-j-1}K_{j+1}) \cup_{\delta_{j+1}} \Sigma^{n-j}K_j \longrightarrow \Sigma^{n-j+1}K_{j-1}.$$

More generally, for every q, using these null homotopies, we have an extension to the iterated mapping cone,

$$c^q(\Sigma^{n-j-q}K_{j+q}) \cup c^{q-1}(\Sigma^{n-j-q+1}K_{j+q-1}) \cup \cdots \cup c(\Sigma^{n-j-1}K_{j+1}) \cup_{\delta_{j+1}} \Sigma^{n-j}K_j \longrightarrow \Sigma^{n-j+1}K_{j-1}. \tag{10.1.5}$$

In other words, for each $p > q$, these null homotopies define maps of spectra,

$$\phi_{p,q} : c^{p-q-1}\Sigma^{n-p}K_p \to \Sigma^{n-q}K_q. \tag{10.1.6}$$

The cell attaching data in the CW-spectrum X as in (10.1.3) is then defined via the maps $\phi_{p,q}$.

Given a functor $Z : \mathcal{J}_0 \to \mathit{Spectra}$ satisfying the hypotheses of Theorem 10.1.2, we have that

$$K_p = |Z|_p/|Z|_{p-1} \simeq \bigvee_{\mathcal{B}_p} \Sigma^p \mathbb{S}$$

and the attaching maps

$$\begin{aligned} \phi_{p,q} :& c^{p-q-1}\Sigma^{-p}K_p \to \Sigma^{-q}K_q \\ & J(p,q)^+ \wedge \Sigma^{-p}|Z|_p/|Z|_{p-1} \to \Sigma^{-q}|Z|_q/|Z|_{q-1} \\ & J(p,q)^+ \wedge Z(p) \to Z(q) \end{aligned} \tag{10.1.7}$$

are given by the application of the morphisms of the category $\mathcal{J}_0$ to the value of the functor Z.

10.1.2 Manifolds with corners and framed flow categories

In order to make use of Theorem 10.1.2 in the setting of Morse and Floer theory, one needs a more geometric way of understanding the homotopy theoretic information contained in a functor $Z : \mathcal{J}_0 \to \mathit{Spectra}$ satisfying the hypotheses of the theorem. As is common in algebraic and differential topology, the translation between homotopy theoretic information and geometric information is done via cobordism theory and the Pontrjagin-Thom construction.

In the case when the CW-structure comes from a Morse function $f : M^n \to \mathbb{R}$ on a closed n-dimensional Riemannian manifold, the attaching maps $\phi_{p,q}$ defining a functor $Z_f : \mathcal{J}_0 \to \mathit{Spectra}$, were shown in [9], [10] to come from the cobordism-type of the moduli spaces of gradient flow lines connecting critical points of index p to those of index q. The relevant cobordism theory is *framed cobordism of manifolds with corners.* Thus to extend this idea to the Floer setting, with the goal of developing a "Floer homotopy type", one needs to understand certain cobordism-theoretic properties of the corresponding moduli spaces of gradient flows of the particular Floer theory. (In [10] the author also considered how other cobordism theories give rise not to "Floer homotopy types" but rather to "Floer module spectra", or said another way, "Floer E-theory" where E is a generalized homology theory. We refer the reader to [10] for details.) In order to make this idea precise we recall some basic facts about cobordisms of manifolds with corners. The main reference for this is Laures's paper [33].

Recall that an n-dimensional manifold with corners, M, has charts which are local homeomorphisms with $\mathbb{R}^n_+$. Here $\mathbb{R}_+$ denotes the nonnegative real numbers and $\mathbb{R}^n_+$ is the n-fold cartesian product of $\mathbb{R}_+$. Let $\psi : U \to \mathbb{R}^n_+$ be a chart of a manifold with corners M. For $x \in U$, the number of zeros of this chart, $c(x)$ is independent of the chart. One defines a *face* of M to be a connected component of the space $\{m{\in}M \text{ such that } c(m) = 1\}$.

Given an integer k, there is a notion of a manifold with corners having "codimension k", or a $\langle k\rangle$-manifold. This notion was originally due to Janich [25], and we recall the definition from [33].

Definition 10.1.4. A $\langle k\rangle$-*manifold* is a manifold with corners, M, together with an ordered k-tuple $(\partial_1 M, ..., \partial_k M)$ of unions of faces of M satisfying the following properties.

1. Each $m{\in}M$ belongs to $c(m)$ faces
2. $\partial_1 M \cup \cdots \cup \partial_k M = \partial M$
3. For all $1{\leq}i{\neq}j{\leq}k$, $\partial_i M \cap \partial_j M$ is a face of both $\partial_i M$ and $\partial_j M$.

The archetypical example of a $\langle k\rangle$-manifold is $\mathbb{R}^k_+$. In this case the face $F_j \subset \mathbb{R}^k_+$ consists of those k-tuples with the j^{th}-coordinate equal to zero.

As described in [33], the data of a $\langle k\rangle$-manifold can be encoded in a categorical way as follows. Let $\underline{2}$ be the partially ordered set with two objects, $\{0, 1\}$, generated by a single nonidentity morphism $0 \to 1$. Let $\underline{2}^k$ be the product of k-copies of the category $\underline{2}$. A $\langle k\rangle$-manfold M then defines a functor from $\underline{2}^k$ to the category of topological spaces, where for an object $a = (a_1, \cdots, a_k) \in \underline{2}^k$, $M(a)$ is the intersection of the faces $\partial_i(M)$ with $a_i = 0$. Such a functor is a k-dimensional cubical diagram of spaces, which, following Laures's terminology, we refer to as a $\langle k\rangle$-diagram, or a $\langle k\rangle$-space. Notice that $\mathbb{R}^k_+(a) \subset \mathbb{R}^k_+$ consists of those k-tuples of nonnegative real numbers so that the i^{th}-coordinate is zero for every i with $a_i = 0$. More generally, consider the $\langle k\rangle$-Euclidean space, $\mathbb{R}^k_+ \times \mathbb{R}^n$, where the value on $a \in \underline{2}^k$ is $\mathbb{R}^k_+(a) \times \mathbb{R}^n$. In general we refer to a functor $\phi : \underline{2}^k \to \mathcal{C}$ as a $\langle k\rangle$-object in the category $\mathcal{C}$.

In this section we will consider embeddings of manifolds with corners into Euclidean spaces $M \hookrightarrow \mathbb{R}^k_+ \times \mathbb{R}^n$ of the form given by the following definition.

Definition 10.1.5. A *neat embedding* of a $\langle k\rangle$-manifold M into $\mathbb{R}^k_+ \times \mathbb{R}^m$ is a natural transformation of $\langle k\rangle$-diagrams

$$e : M \hookrightarrow \mathbb{R}^k_+ \times \mathbb{R}^m$$

that satisfies the following properties:

1. For each $a{\in}\underline{2}^k$, $e(a)$ is an embedding.
2. For all $b < a$, the intersection $M(a) \cap \left(\mathbb{R}^k_+(b) \times \mathbb{R}^m\right) = M(b)$, and this intersection is perpendicular. That is, there is some $\varepsilon > 0$ such that

$$M(a) \cap \left(\mathbb{R}^k_+(b) \times [0, \varepsilon)^k(a - b) \times \mathbb{R}^m\right) = M(b) \times [0, \varepsilon)^k(a - b).$$

Here $a - b$ denotes the object of $\underline{2}^k$ obtained by subtracting the k-vector b from the k-vector a.

In [33] it was proved that every $\langle k \rangle$-manifold neatly embeds in $\mathbb{R}^k_+ \times \mathbb{R}^N$ for N sufficiently large. In fact it was proved there that a manifold with corners, M, admits a neat embedding into $\mathbb{R}^k_+ \times \mathbb{R}^N$ *if and only if* M has the structure of a $\langle k \rangle$-manifold. Furthermore in [24] it is shown that the connectivity of the space of neat embeddings, $Emb_{\langle k \rangle}(M; \mathbb{R}^k_+ \times \mathbb{R}^N)$ increases with the dimension N.

An embedding of manifolds with corners, $e : M \hookrightarrow \mathbb{R}^k_+ \times \mathbb{R}^m$, has a well defined normal bundle. In particular, for any pair of objects in $\underline{2}^k$, $a > b$, the normal bundle of $e(a) : M(a) \hookrightarrow \mathbb{R}^k_+(a) \times \mathbb{R}^m$, when restricted to $M(b)$, is the normal bundle of $e(b) : M(b) \hookrightarrow \mathbb{R}^k_+(b) \times \mathbb{R}^m$.

These embedding properties of $\langle k \rangle$-manifolds make it clear that these are the appropriate manifolds to study for cobordism-theoretic information. In particular, given an embedding $e : M \hookrightarrow \mathbb{R}^k_+ \times \mathbb{R}^m$ the Thom space of the normal bundle, $Th(M, e)$, has the structure of an $\langle k \rangle$-space, where for $a \in \underline{2}^k$, $Th(M, e)(a)$ is the Thom space of the normal bundle of the associated embedding, $M(a) \hookrightarrow \mathbb{R}^k_+(a) \times \mathbb{R}^N$. We can then desuspend and define the Thom spectrum, $M^\nu_e = \Sigma^{-N} Th(M, e)$, to be the associated $\langle k \rangle$-spectrum. The Pontrjagin-Thom construction defines a map of $\langle k \rangle$-spaces,

$$\tau_e : \left(\mathbb{R}^k_+ \times \mathbb{R}^N\right) \cup \infty = ((\mathbb{R}^k_+) \cup \infty) \wedge S^N \to Th(M, e).$$

Desuspending we get a map of $\langle k \rangle$-spectra, $\Sigma^\infty((\mathbb{R}^k_+) \cup \infty) \to M^\nu_e$. Notice that the homotopy type (as $\langle k \rangle$-spectra) of M^ν_e is independent of the embedding e. We denote the homotopy type of this normal Thom spectrum as M^ν, and the Pontrjagin-Thom map, $\tau : \Sigma^\infty((\mathbb{R}^k_+) \cup \infty) \to M^\nu$.

Compact manifolds with corners, and in particular $\langle k \rangle$ - manifolds naturally occur as the moduli spaces of flow lines of a Morse function, and in some cases, of a Floer function. We first recall how they appear in Morse theory.

Consider a smooth, closed n-manifold M^n, and a smooth Morse function $f : M^n \to \mathbb{R}$. Given a Riemannian metric on M, one studies the flow of the gradient vector field ∇f. In particular a flow line is a curve $\gamma : \mathbb{R} \to M$ satisfying the ordinary differential equation,

$$\frac{d}{dt}\gamma(s) + \nabla f(\gamma(s)) = 0.$$

By the existence and uniqueness theorem for solutions to ODE's, one knows that if $x \in M$ is any point then there is a unique flow line γ_x satisfying $\gamma_x(0) = x$. One then studies unstable and stable manifolds of the critical points,

$$W^u(a) = \{x \in M \,:\, \lim_{t \to -\infty} \gamma_x(t) = a\}$$
$$W^s(a) = \{x \in M \,:\, \lim_{t \to +\infty} \gamma_x(t) = a\}.$$

The unstable manifold $W^u(a)$ is diffeomorphic to a disk $D^{\mu(a)}$, where $\mu(a)$ is the index of the critical point a. Similarly the stable manifold $W^s(a)$ is diffeomorphic to a disk $D^{n-\mu(a)}$.

For a generic choice of Riemannian metric, the unstable manifolds and stable manifolds intersect transversally, and their intersections,

$$W(a,b) = W^u(a) \cap W^s(b)$$

are smooth manifolds of dimension equal to the relative index, $\mu(a)-\mu(b)$. When the choice of metric satisfies these transversality properties, the metric is said to be "Morse-Smale". The manifolds $W(a,b)$ have free $\mathbb{R}$-actions defined by "going with the flow". That is, for $t \in \mathbb{R}$, and $x \in M$,

$$t \cdot x = \gamma_x(t).$$

The "moduli space of flow lines" is the manifold

$$\mathcal{M}(a,b) = W(a,b)/\mathbb{R}$$

and has dimension $\mu(a) - \mu(b) - 1$. These moduli spaces are not generally compact, but they have canonical compactifications which we now describe.

In the case of a Morse-Smale metric, (which we assume throughout the rest of this section), there is a partial order on the finite set of critical points given by $a \geq b$ if $\mathcal{M}(a,b) \neq \emptyset$. We then define

$$\bar{\mathcal{M}}(a,b) = \bigcup_{a=a_1>a_2>\cdots>a_k=b} \mathcal{M}(a_1,a_2) \times \cdots \times \mathcal{M}(a_{k-1},a_k). \tag{10.1.8}$$

The topology of $\bar{\mathcal{M}}(a,b)$ can be described naturally, and this is done in many references including [11]. $\bar{\mathcal{M}}(a,b)$ is the space of "piecewise flow lines" emanating from a and ending at b.

The following definition of a Morse function's "flow category " was also given in [11].

Definition 10.1.6. The *flow category* $\mathcal{C}_f$ is a topological category associated to a Morse function $f : M \to \mathbb{R}$ where M is a closed Riemannian manifold. Its objects are the critical points of f. If a and b are two such critical points, then $Mor_{\mathcal{C}_f}(a,b) = \bar{\mathcal{M}}(a,b)$. Composition is determined by the maps

$$\bar{\mathcal{M}}(a,b) \times \bar{\mathcal{M}}(b,c) \hookrightarrow \bar{\mathcal{M}}(a,c),$$

which are defined to be the natural embeddings into the boundary.

The moduli spaces $\mathcal{M}(a,b)$ have natural framings on their stable normal bundles (or equivalently, their stable tangent bundles) that play an important role in this theory. These framings are defined in the following manner. Let a and b be critical points with $a > b$. Let $\varepsilon > 0$ be chosen so that there are no critical values in the half open interval $[f(a)-\varepsilon, f(a))$. Define the *unstable sphere* to be the level set of the unstable manifold,

$$S^u(a) = W^u(a) \cap f^{-1}(f(a) - \varepsilon).$$

The sphere $S^u(a)$ has dimension $\mu(a) - 1$. Notice there is a natural diffeomorphism,

$$\mathcal{M}(a,b) \cong S^u(a) \cap W^s(b).$$

This leads to the following diagram,

$$\begin{array}{ccc} W^s(b) & \xrightarrow{\hookrightarrow} & M \\ \cup\uparrow & & \uparrow\cup \\ \mathcal{M}(a,b) & \underset{\hookrightarrow}{\longrightarrow} & S^u(a). \end{array} \qquad (10.1.9)$$

From this diagram one sees that the normal bundle ν of the embedding $\mathcal{M}(a,b) \hookrightarrow S^u(a)$ is the restriction of the normal bundle of $W^s(b) \hookrightarrow M$. Since $W^s(b)$ is a disk, and therefore contractible, this bundle is trivial. Indeed an orientation of $W^s(b)$ determines a homotopy class of trivialization, or a framing. In fact this framing determines an isotopy class of diffeomorphism of the bundle to the product, $W^s(b) \times W^u(b)$. Thus these orientations give the moduli spaces $\mathcal{M}(a,b)$ canonical normal framings, $\nu \cong \mathcal{M}(a,b) \times W^u(b)$.

As was pointed out in [11], these framings extend to the boundary of the compactifications, $\bar{\mathcal{M}}(a,b)$. In order to describe what it means for these framings to be "coherent" in an appropriate sense, the following categorical approach was used in [10]. The first step is to abstract the basic properties of a flow category of a Morse function.

Definition 10.1.7. A *smooth, compact category* is a topological category $\mathcal{C}$ whose objects form a discrete set, and whose whose morphism spaces, $Mor(a,b)$ are compact, smooth $\langle k \rangle$ -manifolds, where $k = dim\, Mor(a,b)$. If $a > b > c$, the composition map, $\nu : Mor(a,b) \times Mor(b,c) \to Mor(a,c)$, is a smooth codimension one embedding (of manifolds with corners) whose image lies in the boundary. Moreover every point in the boundary of $Mor(a,c)$ is in the image under ν of a maximal sequence in $Mor(a,b_1) \times Mor(b_1,b_2) \times \cdots \times Mor(b_{k-1},b_k) \times Mor(b_k,c)$ for some objects $\{b_1,\ldots,b_k\}$.

A smooth, compact category $\mathcal{C}$ is said to be a "Morse-Smale" category if the following additional properties are satisfied.

1. The objects of $\mathcal{C}$ are partially ordered by the condition

$$a \geq b \quad \text{if} \quad Mor(a,b) \neq \emptyset.$$

2. $Mor(a,a) = \{identity\}$.

3. There is a set map, $\mu : Ob(\mathcal{C}) \to \mathbb{Z}$, which preserves the partial ordering, such that if $a > b$,

$$dim\, Mor(a,b) = \mu(a) - \mu(b) - 1.$$

   The map μ is known as an "index" map. A Morse-Smale category such as this is said to have finite type, if there are only finitely many objects of any given index, and for each pair of objects $a > b$, there are only finitely many objects c with $a > c > b$. For ease of notation we write $k(a,b) = \mu(a) - \mu(b) - 1$.

The following is a folk theorem that goes back to the work of Smale and Franks [19] although a proof of this fact did not appear in the literature until much later [51].

Proposition 10.1.8. *Let $f : M \to \mathbb{R}$ be smooth Morse function on a closed Riemannian manifold with a Morse-Smale metric. Then the compactified moduli space of piecewise flow-lines, $\bar{\mathcal{M}}(a,b)$ is a smooth $\langle k(a,b)\rangle$-manifold.*

Using this result, as well as an associativity result for the gluing maps $\bar{\mathcal{M}}(a,b) \times \bar{\mathcal{M}}(b,c) \to \bar{\mathcal{M}}(a,c)$ which was eventually proved in [51], it was proven in [12] that the flow category $\mathcal{C}_f$ of such a Morse-Smale function is indeed a Morse-Smale smooth, compact category according to Definition 10.1.7.

Remark 10.1.9. The fact that [11] was never submitted for publication was due to the fact that the "folk theorem" mentioned above, as well as the associativity of gluing, both of which the authors of [11] assumed were "well known to the experts", were indeed not in the literature, and their proofs which were eventually provided in [51], were analytically more complicated than the authors imagined.

In order to define the notion of "coherent framings" of the moduli spaces $\bar{\mathcal{M}}(a,b)$, so that we may apply the Pontrjagin-Thom construction coherently, we need to study an associated category, enriched in spectra, defined using the stable normal bundles of the moduli spaces of flows.

Definition 10.1.10. Let $\mathcal{C}$ be a smooth, compact category of finite type satisfying the Morse-Smale condition. Then a *normal Thom spectrum* of the category $\mathcal{C}$ is a category, $\mathcal{C}^\nu$, enriched over spectra, that satisfies the following properties.

1. The objects of $\mathcal{C}^\nu$ are the same as the objects of $\mathcal{C}$.
2. The morphism spectra $Mor_{\mathcal{C}^\nu}(a,b)$ are $\langle k(a,b)\rangle$-spectra, having the homotopy type of the normal Thom spectra $Mor_{\mathcal{C}}(a,b)^\nu$, as $\langle k(a,b)\rangle$-spectra. The composition maps,
$$\circ : Mor_{\mathcal{C}^\nu}(a,b) \wedge Mor_{\mathcal{C}^\nu}(b,c) \to Mor_{\mathcal{C}^\nu}(a,c)$$
have the homotopy type of the maps,
$$Mor_{\mathcal{C}}(a,b)^\nu \wedge Mor_{\mathcal{C}}(b,c)^\nu \to Mor_{\mathcal{C}}(a,c)^\nu$$
of the Thom spectra of the stable normal bundles corresponding to the composition maps in $\mathcal{C}$, $Mor_{\mathcal{C}}(a,b) \times Mor_{\mathcal{C}}(b,c) \to Mor_{\mathcal{C}}(a,c)$. Recall that these are maps of $\langle k(a,c)\rangle$-spaces induced by the inclusion of a component of the boundary.
3. The morphism spectra are equipped with "Pontrjagin-Thom maps" $\tau_{a,b} : \Sigma^\infty(J(\mu(a), \mu(b))^+) = \Sigma^\infty((\mathbb{R}_+^{k(a,b)}) \cup \infty)) \to Mor_{\mathcal{C}^\nu}(a,b)$ such that the following diagram commutes:
$$\begin{array}{ccc} \Sigma^\infty(J(\mu(a),\mu(b))^+) \wedge \Sigma^\infty(J(\mu(b),\mu(c))^+) & \longrightarrow & \Sigma^\infty(J(\mu(a),\mu(c))^+) \\ \downarrow{\scriptstyle \tau_{a.b}\wedge\tau_{b,c}} & & \downarrow{\scriptstyle \tau_{a,c}} \\ Mor_{\mathcal{C}^\nu}(a,b) \wedge Mor_{\mathcal{C}^\nu}(b,c) & \longrightarrow & Mor_{\mathcal{C}^\nu}(a,c). \end{array}$$
Here the top horizontal map is defined via the composition maps in the category $\mathcal{J}$, and the bottom horizontal map is defined via the composition maps in $\mathcal{C}^\nu$.

With the notion of a "normal Thom spectrum" of a flow category $\mathcal{C}$, the notion of a coherent E^*-orientation was defined in [10]. Here E^* is a generalized cohomology theory represented by a commutative (E_∞) ring spectrum E. We recall that definition now.

First observe that a commutative ring spectrum E induces a $\langle k \rangle$-diagram in the category of spectra ("$\langle k \rangle$-spectrum"), $E\langle k \rangle$, defined in the following manner.

For $k = 1$, we let $E\langle 1 \rangle : \underline{2} \to \mathit{Spectra}$ be defined by $E\langle 1 \rangle(0) = S^0$, the sphere spectrum, and $E\langle 1 \rangle(1) = E$. The image of the morphism $0 \to 1$ is the unit of the ring spectrum $S^0 \to E$.

To define $E\langle k \rangle$ for general k, let a be an object of $\underline{2}^k$. We view a as a vector of length k, whose coordinates are either zero or one. Define $E\langle k \rangle(a)$ to be the multiple smash product of spectra, with a copy of S^0 for every every zero coordinate, and a copy of E for every string of successive ones. For example, if $k = 6$, and $a = (1, 0, 1, 1, 0, 1)$, then $E\langle k \rangle(a) = E \wedge S^0 \wedge E \wedge S^0 \wedge E$.

Given a morphism $a \to a'$ in $\underline{2}^k$, one has a map $E\langle k \rangle(a) \to E\langle k \rangle(a')$ defined by combining the unit $S^0 \to E$ with the ring multiplication $E \wedge E \to E$.

Said another way, the functor $E\langle k \rangle : \underline{2}^k \to \mathit{Spectra}$ is defined by taking the k-fold product functor $E\langle 1 \rangle : \underline{2} \to \mathit{Spectra}$ which sends $(0 \to 1)$ to $S^0 \to E$, and then using the ring multiplication in E to "collapse" successive strings of E's.

This structure allows us to define one more construction. Suppose $\mathcal{C}$ is a smooth, compact, Morse-Smale category of finite type as in Definition 10.1.7. We can then define an associated category, $E_{\mathcal{C}}$, whose objects are the same as the objects of $\mathcal{C}$ and whose morphisms are given by the spectra,

$$Mor_{E_{\mathcal{C}}}(a, b) = E\langle k(a, b) \rangle$$

where $k(a, b) = \mu(a) - \mu(b) - 1$. Here $\mu(a)$ is the index of the object a as in Definition 10.1.7. The composition law is the pairing,

$$\begin{aligned} E\langle k(a,b)\rangle \wedge E\langle k(b,c)\rangle &= E\langle k(a,b)\rangle \wedge S^0 \wedge E\langle k(b,c)\rangle \\ &\xrightarrow{1\wedge u \wedge 1} E\langle k(a,b)\rangle \wedge E\langle 1\rangle \wedge E\langle k(b,c)\rangle \\ &\xrightarrow{\mu} E\langle k(a,c)\rangle. \end{aligned}$$

Here $u : S^0 \to E = E\langle 1 \rangle$ is the unit. This category encodes the multiplication in the ring spectrum E.

Definition 10.1.11. An E^*-*orientation* of a smooth, compact category of finite type satisfying the Morse-Smale condition, $\mathcal{C}$, is a functor, $u : \mathcal{C}^\nu \to E_{\mathcal{C}}$, where $\mathcal{C}^\nu$ is a normal Thom spectrum of $\mathcal{C}$, such that on morphism spaces, the induced map

$$Mor_{\mathcal{C}^\nu}(a, b) \to E\langle k(a, b) \rangle$$

is a map of $\langle k(a, b) \rangle$-spectra that defines an E^* orientation of $Mor_{\mathcal{C}^\nu}(a, b) \simeq \bar{\mathcal{M}}(a, b)^\nu$.

This notion of E^*-orientation was discussed more fully in [10]. In particular it is described there how the functor $u : \mathcal{C}^\nu \to E_{\mathcal{C}}$ should be thought of as a coherent family of E^*- Thom classes for the normal bundles of the morphism spaces of $\mathcal{C}$. When $E = \mathbb{S}$, the sphere spectrum, then an E^*-orientation, as defined here, defines a coherent family of framings of the morphism spaces, and is equivalent to the notion of a *framing* of the category $\mathcal{C}$, as defined in [12].

In [12] the following was proved modulo the results of [51] which appeared much later.

Theorem 10.1.12. *Let $f : M \to \mathbb{R}$ be a Morse function on a closed Riemannian manifold satisfying the Morse-Smale condition. Then the flow category $\mathcal{C}_f$ has a canonical structure as a "$\mathbb{S}$-oriented, smooth, compact Morse-Smale category of finite type". That is, it is a "framed, smooth compact Morse-Smale category". The induced framings of the morphism manifolds $\bar{\mathcal{M}}(a, b)$ are canonical extensions of the framings of the open moduli spaces $\mathcal{M}(a, b)$ described above (10.1.9).*

The main use of the notion of compact, smooth, framed flow categories is the following result.

Theorem 10.1.13. [12, 10] *Let $\mathcal{C}$ be a compact, smooth, framed category of finite type satisfying the Morse-Smale property. Then there is an associated, naturally defined functor $Z_\mathcal{C} : \mathcal{J}_0 \to Spectra$ whose geometric realization $|Z_\mathcal{C}|$ realizes the associated "Floer complex"*

$$\to \cdots \to C_{i+1} \xrightarrow{\partial_i} C_i \xrightarrow{\partial_{i-1}} \cdots \xrightarrow{\partial_0} C_0.$$

Here C_j is the free abelian group generated by the objects $a \in Ob(\mathcal{C})$ with $\mu(a) = j$, and the boundary homomorphisms are defined by

$$\partial_{j-1}([a]) = \sum_{\mu(b)=j-1} \#Mor(a, b) \cdot [b]$$

where $\#Mor(a, b)$ is the framed cobordism type of the compact framed manifold $Mor(a, b)$ which zero dimensional (since its dimension $= \mu(a) - \mu(b) - 1 = 0$). Therefore this cobordism type is simply an integer, which can be viewed as the oriented count of the number of points in $Mor(a, b)$.

Proof. Sketch. The proof of this is sketched in [12] and is carried out in [10] in the setting of E^*-oriented compact, smooth categories, for E any ring spectrum. The idea for defining the functor $Z_\mathcal{C}$ is to use the Pontrjagin-Thom construction in the setting of framed manifolds with corners (more specifically, framed $\langle k \rangle$-manifolds). Namely, one defines

$$Z_\mathcal{C}(j) = \bigvee_{a \in Obj(\mathcal{C})\,:\,\mu(a)=j} \mathbb{S}.$$

On the level of morphisms one needs to define, for every $i > j$, a map of spectra

$$Z_\mathcal{C}(i, j) : J(i, j)^+ \wedge Z_\mathcal{C}(i) \to Z_\mathcal{C}(j).$$

This is defined to be the wedge, taken over all $a \in Obj(\mathcal{C})$ with $\mu(a) = i$, and $b \in Obj(\mathcal{C})$ with $\mu(b) = j$, of the maps

$$Z_\mathcal{C}(a, b) : J(\mu(a), \mu(b))^+ \wedge \mathbb{S}_a \to \mathbb{S}_b$$

defined to be the composition

$$\Sigma^\infty(J(\mu(a), \mu(b)^+) \xrightarrow{\tau_{a,b}} Mor_{\mathcal{C}^\nu}(a,b) \xrightarrow{u} \mathbb{S} \qquad (10.1.10)$$

where $\mathbb{S}_a$ and $\mathbb{S}_b$ are copies of the sphere spectrum indexed by a and b respectively in the definition of $Z_{\mathcal{C}}(i)$ and $Z_{\mathcal{C}}(j)$. $\tau_{a,b}$ is the Pontrjagin-Thom map, and u is $\mathbb{S}$-normal orientation class (framing). Details can be found in [10]. □

Thus to define a *"Floer homotopy type"* one is looking for a compact, smooth, framed category of finite type satisfying the Morse-Smale property. The compact framed manifolds with corners that constitute the morphism spaces, define, via the Pontrjagin-Thom construction, the attaching maps of the CW-spectrum defining this (stable) homotopy type.

We see from Theorem 10.1.12 that given a Morse-Smale function $f : M \to \mathbb{R}$, then its flow category satisfies these properties, and it was proved in [12] that, not surprisingly, its Floer homotopy type is the suspension spectrum $\Sigma^\infty(M_+)$. The CW-structure is the classical one coming from Morse theory, with one cell of dimension k for each critical point of index k. The fact that the compactified moduli spaces of flow lines, which constitute the morphism spaces in this category, together with their structure as framed manifolds with corners, define the attaching maps in the CW-structure of $\Sigma^\infty M$, can be viewed as a generalization of the well-known work of Franks in [19].

As pointed out in [12], a distinguishing feature in the flow category of a Morse-Smale function $f : M \to \mathbb{R}$ is that the framing is *canonical*. See (10.1.9) above. As also was pointed out in [12], if one chooses a *different* framing of the flow category $\mathcal{C}_f$, then the "difference" between the new framing and the canonical framing defines a functor

$$\Phi : \mathcal{C}_f \to \mathcal{G}L_1(\mathbb{S}) \qquad (10.1.11)$$

where $\mathcal{G}L_1(\mathbb{S})$ is the category corresponding to the group-like monoid $GL_1(\mathbb{S})$ of "units" of the sphere spectrum (see [5]). This monoid has the homotopy time of the colimit

$$GL_1(\mathbb{S}) \simeq \operatorname{colim}_{n\to\infty} \Omega^n_{\pm 1} S^n.$$

Here the subscript denotes the path components of $\Omega^n S^n$ consisting of based self-maps of the sphere S^n of degree ± 1. By a minor abuse of notations, we let Φ denote the framing of $\mathcal{C}_f$ that defines the map (10.1.11).

Passing to the geometric realizations of these categories, one gets a map

$$\phi : M \to BGL_1(\mathbb{S}).$$

which we think of as the isomorphism type of a spherical fibration over M. The following is also a result of [12].

Proposition 10.1.14. *If $f : M \to \mathbb{R}$ is a Morse-Smale function on a closed Riemannian manifold, and its flow category $\mathcal{C}_f$ is given a framing Φ, then the Floer homotopy type of $(\mathcal{C}_f, \Phi)$ viewed as a compact, smooth, framed category, is the Thom spectrum of the corresponding stable spherical fibration, M^ϕ.*

10.1.3 The free loop space of a symplectic manifold and symplectic Floer theory

Let (N^{2n}, ω) be a symplectic manifold. Here ω is a closed, nondegenerate, skew symmetric bilinear form on the tangent bundle, TN. Let LN be its free loop space, where here and throughout this section we are taking smooth loops. One of the earliest applications of Floer theory [18] was to the (perturbed) symplectic action functional on LN.

Let $L_0N \subset LN$ be the path component of contractible loops in N. Let $\tilde{L}_0N \xrightarrow{p} L_0N$ be its universal cover. Explicitly,

$$\tilde{L}_0N = \{(\gamma, \theta) \in L_0N \times C^\infty(D^2, N) \,:\, \text{the restriction of } \theta \text{ to } S^1 = \partial D^2 \text{ is equal to } \quad \gamma\}/\sim$$

where the equivalence relation is given by $(\gamma, \theta_1) \sim (\gamma, \theta_2)$ if, when we combine θ_1 and θ_2 to define a map of the 2-sphere,

$$\theta_{1,2} = \theta_1 \cup \theta_2 : D^2 \cup_{S^1} D^2 = S^2 \to N$$

then $\theta_{1,2}$ is null homotopic. In other words, (γ, θ_1) is equivalent to (γ, θ_2) if θ_1 and θ_2 are homotopic maps relative to the boundary.

One can then define the "symplectic action" functional,

$$\mathcal{A} : \tilde{L}_0N \to \mathbb{R}. \tag{10.1.12}$$

$$(\gamma, \theta) \to \int_{D^2} \theta^*(\omega). \tag{10.1.13}$$

There are important situations when the symplectic action descends to define an $\mathbb{R}/\mathbb{Z}$-valued function

$$\mathcal{A} : L_0N \to \mathbb{R}/\mathbb{Z}.$$

This happens, for example, when ω is an integral symplectic form. By an "integral symplectic form", we mean a symplectic form ω with the property that the real two-dimensional cohomology class it defines is in the image of integral cohomology. See [43] for details.

One needs to perturb the symplectic action functional by use of a Hamiltonian vector field in order to achieve nondegeneracy of critical points. A Morse-type complex, generated by critical points, is then studied, and the resulting symplectic Floer homology has proved to be an important invariant. We refer the reader to [43] for a thorough treatment of this theory.

We describe the situation when $N = T^*M^n$, the cotangent bundle of a closed, n-dimensional manifold, in more detail. In particular this is a situation where one has a corresponding "Floer homotopy type", defined via a compact, smooth flow category as described above. This was studied by the author in [9], making heavy use of the analysis of Abbondandolo and Schwarz in [1].

Coming from classical mechanics, the cotangent bundle of a smooth manifold has a canonical symplectic structure. It is defined as follows.

Let $p : T^*M \to M$ be the projection map. For $x \in M$ and $u \in T^*_x M$, let $z = (y, u) \in T^*M$, and consider the composition

$$\theta_z : T_z(T^*M) \xrightarrow{dp} T_y M \xrightarrow{u} \mathbb{R}.$$

This defines a 1-form θ on T^*M, called the "Liouville" 1-form, and the symplectic form ω is defined to be the exterior derivative $\omega = d\theta$. It is easy to check that ω is nondegenerate.

Let

$$H : \mathbb{R}/\mathbb{Z} \times T^*M \to \mathbb{R}$$

be a smooth function. Such a map is called a "time-dependent periodic Hamiltonian". Using the nondegeneracy of the symplectic form, this allows one to define the corresponding "Hamiltonian vector field" X_H by requiring it to satisfy the equation

$$\omega(X_H(t, z), v) = -dH_{t,z}(v)$$

for all $t \in \mathbb{R}/\mathbb{Z}$, $z \in T^*M$, and $v \in T_z(T^*M)$. We will be considering the space of 1-periodic solutions, $\mathcal{P}(H)$, of the Hamiltonian equation

$$\frac{dz}{dt} = X_H(t, z(t))$$

where $z : \mathbb{R}/\mathbb{Z} \to T^*M$ is a smooth function.

Using a periodic time-dependent Hamiltonian one can define the perturbed symplectic action functional

$$\begin{aligned} \mathcal{A}_H : L(T^*M) &\to \mathbb{R} \\ z &\to \int z^*(\theta - H dt) = \int_0^1 (\theta(\frac{dz}{dt}) - H(t, z(t))dt. \end{aligned} \tag{10.1.14}$$

This is a smooth functional [1], and its critical points are the periodic orbits of the Hamiltonian vector field, $\mathcal{P}(H)$. Now let J be a 1-periodic, smooth almost complex structure on T^*M, so that for each $t \in \mathbb{R}/\mathbb{Z}$,

$$\langle \zeta, \xi \rangle_{J_t} = \omega(\zeta, J(t, z)\xi), \quad \zeta, \xi \in T_z T^*M,\ z \in T^*M,$$

is a loop of Riemannian metrics on T^*M. One can then consider the gradient of $\mathcal{A}_H$ with respect to the metric, $\langle \cdot, \cdot \rangle$. It is given by

$$\nabla_J \mathcal{A}_H(z) = -J(z, t)(\frac{dz}{dt} - X_H(t, z)).$$

The (negative) gradient flow equation on a smooth curve $u : \mathbb{R} \to L(T^*M)$,

$$\frac{du}{ds} + \nabla_J \mathcal{A}_H(u(s))$$

can be rewritten as a perturbed Cauchy-Riemann PDE, if we view u as a smooth map $\mathbb{R}/\mathbb{Z} \times \mathbb{R} \to T^*M$, with coordinates, $t \in \mathbb{R}/\mathbb{Z}$, $s \in \mathbb{R}$:

$$\partial_s u - J(t, u(t,s))(\partial_t u - X_H(t, u(t,s)) = 0. \tag{10.1.15}$$

Let $a, b \in \mathcal{P}(H)$. Abbondandolo and Schwarz defined the space of solutions

$$\begin{aligned} W(a,b;H,J) = \{u : \mathbb{R} \to L(T^*M) \text{ a solution to (10.1.15), such that} \\ \lim_{s\to-\infty} u(s) = a, \text{ and } \lim_{s\to+\infty} u(s) = b\}. \end{aligned} \tag{10.1.16}$$

As in the case of Morse theory, we then let $\mathcal{M}(a,b)$ the "moduli space" obtained by dividing out by the free $\mathbb{R}$-action,

$$\mathcal{M}(a,b) = W(a,b;H,J))/\mathbb{R}. \tag{10.1.17}$$

It was shown in [1] that with respect to a generic choice of Hamiltonian an almost complex structure, the spaces $W(a,b;H,J)$ and $\mathcal{M}(a,b)$ are smooth manifolds, whose dimensions are given by $\mu(a) - \mu(b)$ and $\mu(a) - \mu(b) - 1$ respectively, where μ represents the "Conley - Zehnder index" of the periodic Hamiltonian orbits a and b. Furthermore it was shown in [1] that in analogy with Morse theory, one can compactify these moduli spaces as

$$\bar{\mathcal{M}}(a,b) = \bigcup_{a=a_1>a_2>\cdots>a_k=b} \mathcal{M}(a_1,a_2) \times \cdots \times \mathcal{M}(a_{k-1},a_k).$$

The fact that that these compact moduli spaces have canonical framings was shown in [9] in part by showing that the obstruction to framing, described originally in [12], is zero in this case. This obstruction was called the *polarization class* in [12] and is defined as follows.

Let N be an almost complex manifold whose tangent bundle is classified by a map $\tau : N \to BU(n)$. Applying loop spaces, one has a composite map,

$$\rho : LN \xrightarrow{L(\tau)} L(BU(n)) \hookrightarrow LBU \simeq BU \times U \to U \to U/O. \tag{10.1.18}$$

Here the homotopy equivalence $L(BU) \simeq BU \times U$ is well defined up to homotopy, and is given by a trivialization of the fibration

$$U \simeq \Omega BU \xrightarrow{\iota} L(BU) \xrightarrow{ev} BU$$

where $ev : LX \to X$ evaluates a loop a $0 \in \mathbb{R}/\mathbb{Z}$. The trivialization is the composition

$$U \times BU \xrightarrow{\iota\times\sigma} L(BU) \times L(BU) \xrightarrow{mult} L(BU).$$

Here $\sigma : BU \to L(BU)$ is the section of the above fibration given by assigning to a point $x \in BU$ the constant loop at that point, and the "multiplication" map in this composition is induced by the infinite loop space structure of BU.

The reason we refer the map ρ as the "polarization class" of the loop space LN, is because when viewed as an infinite dimensional manifold, the tangent bundle $T(LN)$ has the structure of a "polarization" as defined in [46]. This means that this infinite dimensional tangent bundle has structure group given by the "restricted general linear group of a Hilbert space", $GL_{res}(H)$ as originally defined in [46]. As shown there, $GL_{res}(H)$ has the homotopy type (as infinite loop spaces) of $\mathbb{Z} \times BO$, and so its classifying space, $BGL_{res}(H)$ has the homotopy type of $B(\mathbb{Z} \times BO) \simeq U/O$ by Bott periodicity. The classifying map $LN \to U/O$ has the homotopy type of the "polarization class" ρ defined above. See [46] , [12], and [9] for details.

When $N = T^*M$ two things were shown in [9]. First, when one views $\bar{\mathcal{M}}(a,b)$ as a space of paths, one sees that there is a natural inclusion map into the full space of paths in $L(T^*M)$ which begin at a and end at b. This path space is equivalent to the based loop space $\Omega L(T^*M)$. The result is a map $\iota_{a,b} : \bar{\mathcal{M}}(a,b) \to \Omega L(T^*M)$, well defined up to homotopy, such that the composition

$$\tau_{a,b} : \bar{\mathcal{M}}(a,b) \xrightarrow{\iota_{a,b}} \Omega L(T^*M) \xrightarrow{\Omega\rho} \Omega U/O \simeq \mathbb{Z} \times BO \tag{10.1.19}$$

classifies the stable tangent bundle of $\bar{\mathcal{M}}(a,b)$. Second, it was shown that in this case, i.e. when $N = T^*M$, the polarization class $\rho : L(T^*M) \to U/O$ is trivial. This is essentially because the almost complex structure (i.e. $U(n)$-structure) of the tangent bundle of T^*M is the complexification of the n-dimensional real bundle (i.e. $O(n)$-structure) of the tangent bundle of M pulled back to T^*M via the projection map $T^*M \to M$. By (10.1.19) this leads to a coherent family of framings on the moduli spaces, which in turn lead to a smooth, compact, framed structure on the flow category $\mathcal{C}_H$ of the symplectic action functional, as shown in [9].

Using the methods and results of [1], which is to say, comparing the flow category of the perturbed symplectic action functional $\mathcal{C}_H$ to the Morse flow category of an energy functional on LM, the following was shown in [9].

Theorem 10.1.15. *Let M^n be a closed spin manifold of dimension n. For appropriate choices of a Hamiltonian H and a generic choice of almost complex structure J on the cotangent bundle T^*M, the Floer homotopy type determined by the smooth, compact flow category $\mathcal{C}_H$ is given by the suspension spectrum of the free loop space,*

$$Z_{\mathcal{C}_H}(T^*M) \simeq \Sigma^\infty(LM_+).$$

Remark 10.1.16. 1. This theorem generalized a result of Viterbo [58] stating that the symplectic Floer homology, $SFH_*(T^*M)$ is isomorphic to $H_*(LM)$.

2. If M is not spin, one needs to use appropriately twisted coefficients in both Viterbo's theorem and in Theorem 10.1.15 above. This was first observed by Kragh in [30], and was overlooked in all or most of the discussions of the relation between the symplectic Floer theory of the cotangent bundle and homotopy type of the free loop space before Kragh's work, including the author's work in [9].

10.2 The work of Lipshitz and Sarkar on Khovanov homotopy theory

A recent dramatic application of the ideas of Floer homotopy theory appeared in the work of Lipshitz and Sarkar on the homotopy theoretic foundations of Khovanov's homological invariants of knots and links. This work appeared in [35] and [36]. Another version of Khovanov homotopy appears in [26]. It was proved to be equivalent to the Lipshitz-Sarkar construction in [34].

The Khovanov homology of a link L is a bigraded abelian group, $Kh^{i,j}(L)$. It is computed from a chain complex denoted $KhC^{i,j}(L)$ that is defined in terms of a link diagram. However the Khovanov homology was shown to be independent of the choice of link diagram, up to isomorphism. This invariant was originally defined by Khovanov in [27] in which he viewed these homological invariants as a "categorification" of the Jones polynomial $V(L)$ in the sense that the graded Euler characteristic of this homology theory recovers $V(L)$ via the formula

$$\begin{aligned}\chi(Kh^{i,j}(L)) &= \sum_{i,j}(-1)^i q^j \, rank \, Kh^{i,j}(L)\\ &= (q+q^{-1})\, V(L).\end{aligned}$$

The goal of the work of Lipshitz and Sarkar was to associate to a link diagram L a family of spectra $X^j(L)$ whose homotopy types are invariants of the isotopy class of the link (and in particular do not depend on the particular link diagram used), and such that the Khovanov homology $Kh^{i,j}(L)$ is isomorphic to the reduced singular cohomology $\tilde{H}^i(X^j(L))$. Their basic idea is to construct a compact, smooth, framed flow category from the moduli spaces associated to a link diagram. Their construction is entirely combinatorial, and the cells of the spectrum $X(L) = \bigvee_j X^j(L)$ correspond to the standard generators of the Khovanov complex $KhC^{*,*}(L)$. That is, $X(L)$ *realizes* the Khovanov chain complex in the sense described above. $X(L)$ is referred to as the "Khovanov homotopy type" of the link L, and it has had several interesting applications.

Notice that by virtue of the existence of a Khovanov homotopy type, the Khovanov homology, when reduced modulo a prime, carries an action of the Steenrod algebra $\mathcal{A}_p$. In [36] the authors show that the Steenrod operation Sq^2 acts nontrivially on the Khovanov homology for many knots, and in particular for the torus knot, $T_{3,4}$. It is also known by work of Seed [53] that there are pairs of links with isomorphic Khovanov's homology, but distinct Khovanov homotopy types. Also, Rasmussen constructed a slice genus bound, called the s-invariant, using Khovanov homology [47]. Using the Khovanov homotopy type, Lipshitz and Sarkar produced a family of generalizations of the s-invariant, and used them to obtain even stronger slice genus bounds. Stoffregen and Zhang [54] used Khovanov homotopy theory to describe rank inequalities for Khovanov homology for prime-periodic links in S^3.

We now give a sketch of the construction of compact framed flow category of Lipshitz and Sarkar, which yields the Khovanov homotopy type.

By a "link diagram", one means the projection onto $\mathbb{R}^2$ of an embedded disjoint union of circles in $\mathbb{R}^3$ that has transverse crossings. One keeps track of the resulting "over" and "under crossings", and usually one orients the link (i.e. puts an arrow in each component). One can "resolve" a crossing in two ways. These are referred to as a 0-resolution and a 1 resolution and are shown in Figure 10.2.1.

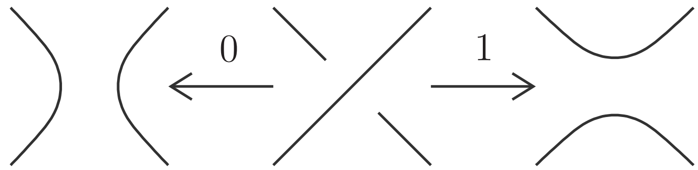


Figure 10.2.1. The 0-resolution and the 1-resolution

The Khovanov chain complex is generated by all possible configurations of resolutions of the crossings of a link diagram. We recall the Lipshitz-Sarkar view of its definition more carefully.

Definition 10.2.1. A *resolution configuration* D is a pair $(Z(D), A(D))$, where $Z(D)$ is a set of pairwise disjoint embedded circles in $S^2 = \mathbb{R}^2 \cup \infty$, and $A(D)$ is an ordered collection of arcs embedded in S^2 with

$$A(D) \cap Z(D) = \partial A(D).$$

The number of arcs in the resolution configuration D is called its index denoted by $ind(D)$.

Definition 10.2.2. Given a link diagram L with n crossings, an ordering of the crossings, and a vector $v \in \{0,1\}^n$, there is an associated resolution configuration $D_L(v)$ obtained by taking the resolution of L corresponding to v. That is, one takes the 0-resolution of the i^{th} crossing if $v_i = 0$, and the 1-resolution of the i^{th} crossing if $v_i = 1$. One then places arcs corresponding to each of the crossings labeled by zeros in v.

Figure 10.2.2 illustrates the resolution configuration corresponding to a diagram of the trefoil knot.

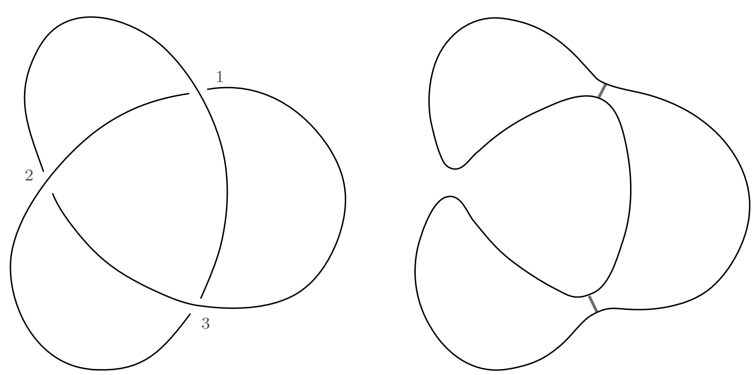


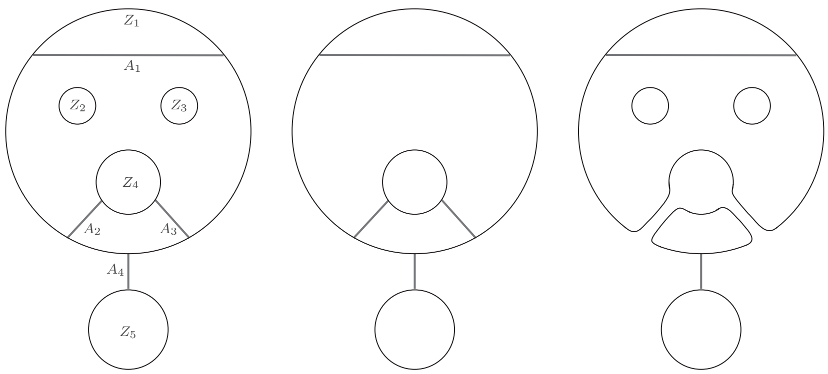
Figure 10.2.2. A knot diagram K for the trefoil with ordered crossings, and the resolution configuration $D_K((0,1,0))$

The following terminology is also useful.

Definition 10.2.3. 1. The *core* $c(D)$ of a resolution configuration is the resolution configuration obtained from D by deleting all the circles in $Z(D)$ that are disjoint from all arcs in $A(D)$.

2. A resolution configuration is *basic* if $D = c(D)$, i.e. every circle in $Z(D)$ intersects an arc in $A(D)$.

One can also do a *surgery* along a subset $A \subset A(D)$. The resulting resolution configuration is denoted $s_A(D)$. The surgery procedure is best illustrated in Figure 10.2.3.



a: The starting resolution configuration D. The circles and arcs are numbered.

b: The core $c(D)$.

c: The surgery $E = s_{\{A_2,A_3\}}(D)$.

FIGURE 10.2.3. A knot diagram K for the trefoil with ordered crossings and the resolution configuration $D_K((0,1,0))$

A *labeling* $x = \{x_+, x_-\}$ of a resolution configuration D is a labeling of each circle in $Z(D)$ by either x_+ or x_-. Labeled resolutions configurations have a partial ordering defined to be the transitive closure of the following relations.

We say that $(E, y) < (D, x)$ if

1. the labelings agree on $D \cap E$.
2. D is obtained from E by surgering along a single arc of $A(E)$, In particular, either:

   (a) $Z(E \backslash D)$ contains exactly one circle, say Z_i and $Z(D \backslash E)$ contains exactly two circles, say Z_j and Z_k, or

   (b) $Z(E \backslash D)$ contains exactly two circles, say Z_i and Z_j, and $Z(D \backslash E)$ contains exactly one circle, say Z_k.
3. In case (2a), either $y(Z_i) = x(Z_j) = x(Z_k) = x_-$ or $y(Z_i) = x_+$ and $\{x(Z_j), x(Z_k)\} = \{x_+, x_-\}$.

   In case (2b), either $y(Z_i) = y(Z_j) = x(Z_k) = x_+$ or $y(Z_i) = x_+$ or $\{y(Z_i), y(Z_j)\} = \{x_+, x_-\}$ and $x(Z_k) = x_+$.

One can now define the *Khovanov chain complex* $KhC(L)$ as follows.

Definition 10.2.4. Given an oriented link diagram L with n crossings and an ordering of the crossings in L, $KhC(L)$ is defined to be the free abelian group generated by labeled resolution configurations of the form $(D_L(u), x)$ for $u \in \{0,1\}^n$. $KhC(L)$ carries two gradings, a *homological grading* gr_h and a *quantum grading* gr_q, defined as follows:

$$\begin{aligned} gr_h((D_L(u), x)) &= -n_- + |u| \\ gr_q((D_L(u), x)) &= n_+ - 2n_- + |u| + \#\{Z \in Z(D_L(u)) \,:\, x(Z) = x_+\} \\ &\quad - \#\{Z \in Z(D_L(u)) \,:\, x(Z) = x_-\}. \end{aligned}$$

Here n_+ denotes the number of positive crossings in L, and n_- denotes the number of negative crossings.

The differential preserves the quantum grading, increases the homological grading by 1, and is defined as

$$\delta(D_L(v), y) = \sum (-1)^{s_0(\mathcal{C}_{u,v})}(D_L(u), x)$$

where the sum is taken over all labeled resolution configurations $(D_L(u), x)$ with $|u| = |v| + 1$ and $(D_L(v), y) < (D_L(u), x)$. The sign $s_0(\mathcal{C}_{u,v}) \in \mathbb{Z}/2$ is defined as follows: If $u = (\varepsilon_1, \ldots, \varepsilon_{i-1}, 1, \varepsilon_{i+1}, \ldots, \varepsilon_n)$ and $v = (\varepsilon_1, \ldots, \varepsilon_{i-1}, 0, \varepsilon_{i+1}, \ldots, \varepsilon_n)$, then $s_0(\mathcal{C}_{u,v}) = \varepsilon_1 + \cdots + \varepsilon_{i-1}$.

The homology of this chain complex is the *Khovanov homology* $Kh^{*,*}(L)$. To define the *Khovanov homotopy type* of the link L, Lipshitz and Sarkar define higher dimensional moduli spaces which have the structure of framed manifolds with corners so that they in turn define a compact, smooth, framed flow category, which by the theory described above defines the associated (stable) homotopy type.

These moduli spaces are defined as a certain covering of the moduli spaces occurring in a "framed flow category of a cube". We now sketch these constructions, following Lipshitz and Sarkar [35].

Let $f_1 : \mathbb{R} \to \mathbb{R}$ be a Morse function with one index zero critical point and one index 1 critical point. For concreteness one can use the function

$$f_1(x) = 3x^2 - 2x^3.$$

Define $f_n : \mathbb{R}^n \to \mathbb{R}$ by

$$f_n(x_1, \ldots, x_n) = f_1(x_1) + \cdots + f_1(x_n).$$

f_n is a Morse function, and we let $\mathcal{C}(n)$ denote its flow category . It is a straightforward exercise to see that the geometric realization $|\mathcal{C}(n)|$ is the n-cube $[0,1]^n$. The vertices of this cube $u \in \{0,1\}^n$ correspond to the critical points of f_n and have a grading which corresponds to the Morse index: $gr(u) = |u| = \sum_i u_i$. They also have a partial ordering coming from the ordering of $\{0,1\}$. We say that $v \leq_i u$ if $v \leq u$ and $gr(u) - gr(v) = i$. That is i is the relative index of u and v. Let $\bar{1} = (1, 1, \cdots, 1)$ and $\bar{0} = (0, \cdots, 0)$. The following is not difficult and is verified in [35].

Lemma 10.2.5. *The compactified moduli space of piecewise flows* $\bar{\mathcal{M}}_{\mathcal{C}(n)}(\bar{1}, \bar{0})$ *is*

- *a single point if* $n = 1$,
- *a closed interval if* $n = 2$,

- *a closed hexagonal disk if $n = 3$ (as in Figure 10.2.4), and is*
- *homeomorphic to a closed disk D^{n-1} for general n.*

Furthermore, given any $v < u$, $\bar{\mathcal{M}}_{\mathcal{C}(n)}(u, v) \cong \bar{\mathcal{M}}_{\mathcal{C}(gr(u)-gr(v))}(\bar{1}, \bar{0})$.

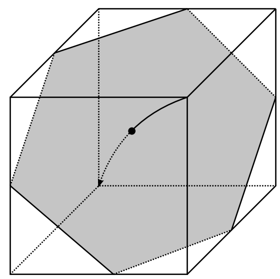


Figure 10.2.4. The cube moduli space $\bar{\mathcal{M}}_{\mathcal{C}(3)}(\bar{1}, \bar{0})$

Lipshitz and Sarkar then proceed to define moduli spaces of "decorated resolution configurations" that cover the moduli spaces occurring in a framed flow category of a cube.

Definition 10.2.6. A *decorated resolution configuration* is a triple (D, x, y) where D is a resolution configuration, x is a labeling of each component of $Z(s(D))$, y is a labeling of each component of $Z(D)$ such that

$$(D, y) < (s(D), x).$$

Here $s(D) = s_{A(D)}(D)$ is the maximal surgery on D.

In [35] Lipshitz and Sarkar proceed to construct moduli spaces $\mathcal{M}(D, x, y)$ for every decorated resolution configuration (D, x, y). These will be compact, framed manifolds with corners. Indeed they are $< n-1 >$-manifolds where n is the index of D. They also produce covering maps

$$\mathcal{F} : \mathcal{M}(D, x, y) \to \bar{\mathcal{M}}_{\mathcal{C}(n)}(\bar{1}, \bar{0})$$

which are maps of $< n-1 >$-spaces, trivial as a covering maps on each component of $\mathcal{M}(D, x, y)$, and are local diffeomorphisms. The framings of the moduli spaces $\mathcal{M}(D, x, y)$ are then induced from the framings of $\bar{\mathcal{M}}_{\mathcal{C}(n)}(\bar{1}, \bar{0})$. They produce composition or gluing maps for every labeled resolution configuration (E, z) with

$$(D, y) < (E, z) < (s(D), x)$$

$$\circ : \mathcal{M}(D \backslash E, z|, y|) \times \mathcal{M}(E \backslash s(D), x|, z|) \longrightarrow \mathcal{M}(D, x, y)$$

that are embeddings into the boundary of $\mathcal{M}(D, x, y)$. These moduli spaces are constructed recursively using a clever, but not very difficult argument. We refer the reader to [35] for details.

These constructions allow for the definition of a Khovanov flow category for the link L, and it is shown to be a compact, smooth, framed flow category as defined in section

1.2 above. Using the theory introduced in [12] and described in section 1.2, one obtains a spectrum realizing the Khovanov chain complex. This is the Lipshitz-Sarkar "Khovanov homotopy type" of the link L.

10.3 Manolescu's equivariant Floer homotopy and the triangulation problem

In this section the author is relying heavily on the expository article by Manolescu [40]. We refer the reader to this beautiful survey of recent topological applications of Floer theory.

10.3.1 Monopole Floer homology and equivariant Seiberg-Witten stable homotopy

One example of a dramatic application of Floer's original instanton homology theory, and in particular its topological field theory relationship to the Donaldson invariants of closed 4-manifolds, was to the study of the group of cobordism classes of homology 3-spheres. Define

$$\Theta^3_H = \{\text{oriented homology 3-spheres}\}/\sim$$

where $X_0 \sim X_1$ if and only if there exists a smooth, compact, oriented 4-manifold W with

$$\partial W = (-Y_0) \sqcup Y_1$$

and $H_1(W) = H_2(W) = 0$. The group operation is represented by connected sum, and the inverse is given by reversing orientation. The standard unit 3-sphere S^3 is the identity element. The Rokhlin homomorphism [50], [14] is the map

$$\mu : \Theta^H_3 \to \mathbb{Z}/2$$

defined by sending a homology sphere X to $\mu(X) = \sigma(X)/8\,(mod\,2)$, where W is any compact spin 4-manifold with $\partial W = X$. It is a theorem that this homomorphism is well-defined. Furthermore, using this homomorphism one knows that the group Θ^H_3 is nontrivial, since, for example, the Poincaré sphere P bounds the E_8 plumbing which has signature -8. Therefore $\mu(P) = 1$.

This result has been strengthened using Donaldson theory and Instanton Floer homology. For example Furuta and Fintushel-Stern proved that Θ^H_3 is infinitely generated [21][15]. And using the $SU(2)$ equivariance of Instanton Floer homology, in [20] Froyshov defined a surjective homomorphism

$$h : \Theta^H_3 \to \mathbb{Z}. \tag{10.3.1}$$

Monopole Floer homology is similar in nature to Floer's Instanton homology theory, but it is based on the Seiberg-Witten equations rather than the Yang-Mills equations. More precisely, let Y be a 3-manifold equipped with a $Spin^c$ structure σ. One considers the configuration space of pairs (A, ϕ), where A is a connection on the trivial $U(1)$ bundle over Y, and ϕ is a spinor. There is an action of the gauge group of the bundle on this configuration space, and one considers the orbit space of this action. One can then define the Chern-Simons-Dirac functional on this space by

$$CSD(A,\phi) = -\frac{1}{8}\int_Y (A^t - A_0^t) \wedge (F_{A^t} + F_{A_0^t}) + \frac{1}{2}\int_Y \langle D_A\phi, \phi\rangle \, dvol.$$

Here A_0 is a fixed base connection, and the superscript t denotes the induced connection on the determinant line bundle. The symbol F denotes the curvature of the connection and the symbol D denotes the covariant derivative.

Monopole Floer homology is the Floer homology associated to this functional. To make this work precisely involves much analytic, technical work, due in large part to the existence of reducible connections. Kronheimer and Mrowka dealt with this (and other issues) by considering a blow-up of this configuration space [31]. They actually defined three versions of Monopole Floer homology for every such pair (Y,σ).

Monopole Floer homology can also be used to give an alternative proof of Froyshov's theorem about the existence of a surjective homomorphism.

$$\delta : \Theta_3^H \to \mathbb{Z}.$$

This uses the S^1-equivariance of these equations. Monopole Floer homology has also been used to prove important results in knot theory [32] and in contact geometry [57].

In [37] Manolescu defined a "Monopole", or "Seiberg-Witten" Floer stable homotopy type. He did not follow the program defined by the author, Jones and Segal in [12] as outlined above, primarily because the issue of smoothness in defining a framed, compact, smooth flow category is particularly difficult in this setting. Instead, he applied Furuta's technique of "finite dimensional approximations" [22]. More specifically, the configuration space X of connections and spinors (A,ϕ) is a Hilbert space that he approximated by a nested sequence of finite dimensional subspaces X_λ. He considered the Conley index associated to the flow induced by CSD on a large ball $B_\lambda \subset X_\lambda$. Roughly, if $L_\lambda \subset \partial B_\lambda$ is that part of the boundary where the flow points in an outward direction, then Manolescu views the Conley index as the quotient space

$$I_\lambda = B_\lambda / L_\lambda.$$

The homology $H_*(I_\lambda)$ is the Morse-homology of the approximate flow on B_λ, assuming that the flow satisfies the Morse-Smale transversality condition. However Manolosecu did not need to assume the Morse-Smale condition in his work. He simply defined the Seiberg-Witten Floer homology directly as the relative homology of I_λ, with a degree shift that depends on λ. The various I_λ's fit together to give a spectrum $SWF(Y,\sigma)$ defined for every rational homology sphere Y with $Spin^c$-structure σ. Since the Seiberg-Witten equations have an S^1 symmetry, the spectrum $SWF(Y,\sigma)$ carries an S^1 action. This defines Manolescu's "S^1-equivariant Seiberg-Witten Floer stable homotopy type".

An important case is when Y is a homology sphere. In this setting there is a unique $Spin^c$ structure σ coming from a spin structure. The conjugation and S^1-action together yield an action by the group

$$Pin(2) = S^1 \oplus S^1 j \subset \mathbb{C} \oplus \mathbb{C}j = \mathbb{H}$$

where $\mathbb{H}$ is the quarternion skew-field. In [39] Manolescu defined the $Pin(2)$-equivariant Floer homology of Y to be the (Borel) equivariant homology of the spectrum,

$$SWFH_*^{Pin(2)}((Y) = \tilde{H}_*^{Pin(2)}(SWF(Y)). \tag{10.3.2}$$

This theory played a crucial role in Manolescu's resolution of the triangulation question as we will describe below. But before doing that we point out that by having the equivariant stable homotopy type, he was able to define a corresponding $Pin(2)$-equivariant Seiberg-Witten Floer K-theory, which he used in [39] to prove an analogue of Furuta's "10/8-conjecture" for 4-manifolds with boundary. That is, Furuta proved that if W is a closed, smooth spin 4-manifold then

$$b_2(W) \geq \frac{10}{8}|sign(W)| + 2$$

where b_2 is the second Betti number and $sign(W)$ is the signature. (The "11/8-conjecture" states that $b_2 \geq \frac{11}{8} sign(W)$.) Using $Pin(2)$-equivariant Floer K-theory, Manolescu proved that if Y is a homology 3-sphere, there is a number $\kappa(Y) \in \mathbb{Z}$ such that if W is a smooth, spin compact 4-manifold with boundary equal to Y, then the following analogue of Furuta's inequality holds:

$$b_2(W) \geq \frac{10}{8}|sign(W)| + 2 - 2\kappa(Y).$$

10.3.2 The triangulation problem

A famous question asked in 1926 by Kneser [29] is the following:

Question. Does every topological manifold admit a triangulation?

By "triangulation" (or "simplicial triangulation"), one means a homeomorphism to a simplicial complex. We refer to this question as the "simplicial triangulation question (or conjecture)". One can also ask the stronger question regarding whether every manifold admits a *combinatorial triangulation,* which is one in which the links of the simplices are spheres. Such a triangulation is equivalent to a piecewise linear (PL) structure on the manifold.

In the 1920s Rado proved that every surface admits a combinatorial triangulation, and in the early 1950s, Moise showed that any topological three manifold also admits a combinatorial triangulation. In the 1930s Cairns and Whitehead showed that *smooth* manifolds of any dimension admit combinatorial triangulations. And in celebrated work in the late 1960s, Kirby and Siebenmann [28] showed that there exist topological manifolds without PL-structures in every dimension greater than four. Furthermore they showed that in these dimensions, the existence of PL-structures is determined by an obstruction class $\Delta(M) \in H^4(M; \mathbb{Z}/2)$. The first counterexample to the simplicial triangulation conjecture was given by Casson [4] who showed that Freedman's four dimensional E_8-manifold, which he had proven did not have a PL-structure, did not admit a simplicial triangulation. In dimensions five or greater, a resolution of Kneser's triangulation question was not achieved until Manolescu's recent work [39] using equivariant Seiberg-Witten Floer homotopy theory.

We now give a rough sketch of Manolescu's work on this.

Let M be a closed, oriented topological n-manifold, with $n \geq 5$ that is equipped with a homeomorphism to a simplicial complex K (i.e. a simplicial triangulation). As mentioned above, the Kirby-Siebenmann obstruction to M having a combinatorial (PL) triangulation is a class $\Delta(M) \in H^4(M;\mathbb{Z}/2)$. A related cohomology class is the Sullivan-Cohen-Sato class $c(K)$ [56], [8], [52] defined by

$$c(K) = \sum_{\sigma \in K^{(n-4)}} [link_K(\sigma)] \cdot \sigma \in H_{n-4}(M;\Theta_3^H) \cong H^4(M;\Theta_3^H).$$

Here the sum is taken over all codimension 4 simplices in K. The link of each such simplex is known to be a homology 3-sphere. If this were a combinatorial triangulation the link would be an actual 3-sphere and so this class would vanish.

Consider the short exact sequence given by the Rokhlin homomorphism

$$0 \to ker(\mu) \to \Theta_3^H \xrightarrow{\mu} \mathbb{Z}/2 \to 0. \tag{10.3.3}$$

This induces a long exact sequence in cohomology

$$\cdots \to H^4(M;\Theta_3^H) \xrightarrow{\mu_*} H^4(M;\mathbb{Z}/2) \xrightarrow{\delta} H^5(M;\mathrm{Ker}(\mu_*)) \to \cdots .$$

In this sequence it is known that $\mu_*(c(K)) = \Delta(M) \in H^4(M;\mathbb{Z}/2)$. One concludes that if M is a manifold that admits a simplicial triangulation, the Kirby-Siebenmann obstruction $\Delta(M)$ is in the image of μ_* and therefore in the kernel of δ. An important result was that this necessary condition for admitting a triangulation is also a sufficient condition.

Theorem 10.3.1 (Galewski-Stern [23], Matumoto [42])**.** *A closed topological manifold M of dimension greater or equal to 5 is admits a triangulation if and only if $\delta(\Delta(M)) = 0$.*

Now observe that the connecting homomorphism δ in this long exact sequence would be zero if the short exact sequence (10.3.3) were split. If this were the case then by the Galewski-Stern-Matumoto theorem (10.3.1), every closed topological manifold of dimension ≥ 5 would be triangulable. The following theorem states that this is in fact a sufficient condition.

Theorem 10.3.2 (Galewski-Stern [23], Matumoto [42])**.** *There exist non-triangulable manifolds of every dimension greater or equal to 5 if and only if the exact sequence (10.3.3) does not split.*

Notice that this theorem reduces an important question about topological n-manifolds for $n \geq 5$ to a question in 3-manifold topology. In settling the triangulation problem Manolescu proved the following.

Theorem 10.3.3. (Manolescu [39]). *The short exact sequence (10.3.3) does not split.*

The strategy of his proof was the following. Suppose $\eta : \mathbb{Z}/2 \to \Theta_3^H$ is a splitting of the above sequence. The image under η of the nonzero element would represent a homology 3-sphere Y of order 2 in Θ_3^H with nonzero Rokhlin invariant. Notice that the fact that Y has order 2 means that $-Y$ ($= Y$ with the opposite orientation) and Y represent the same element in Θ_3^H. Thus to show that this cannot happen, Manolescu defined a lift of the Rokhlin invariant to the integers

$$\beta : \Theta_3^H \to \mathbb{Z}.$$

Given such a lift and any element $X \in \Theta_3^H$ of order 2,

$$\begin{aligned}\beta(X) &= \beta(-X) \quad \text{because } X \text{ has order two}\\ &= -\beta(X) \in \mathbb{Z}.\end{aligned}$$

Thus $\beta(X) = -\beta(X) \in \mathbb{Z}$ and therefore must be zero. Thus X has zero Rokhlin invariant.

Manolescu's strategy was therefore to construct such a lifting $\beta : \Theta_3^H \to \mathbb{Z}$ of the Rokhlin invariant $\mu : \Theta_3^H \to \mathbb{Z}/2$. His construction was modeled on Froyshov's invariant (10.3.1). But to construct β he used $Pin(2)$-equivariant Seiberg-Witten Floer homology $SWFH^{Pin(2)}$ defined using the $Pin(2)$-equivariant Seiberg-Witten Floer homotopy type (spectrum) as described above (10.3.2). We refer to [39] for the details of the construction and resulting dramatic solution of the triangulation problem.

10.4 Floer homotopy and Lagrangian immersions: the work of Abouzaid and Kragh

Another beautiful example of an application of a type of Floer homotopy theory was found by Abouzaid and Kragh [2] in their study of Lagrangian immersions of a Lagrangian manifold L into the cotangent bundle of a smooth, closed manifold, T^*N. Here, L and N must be the same dimension. The basic question is whether a Lagrangian immersion is Lagrangian isotopic to a Lagrangian embedding. This is particularly interesting when $L = N$. In this case the "Nearby Lagrangian Conjecture" of Arnol'd states that every closed exact Lagrangian submanifold of T^*N is Hamiltonian isotopic to the zero section, $\eta : N \hookrightarrow T^*N$. In [2] the authors use Floer homotopy theory and classical calculations by Adams and Quillen [49] of the J-homomorphism in homotopy theory to give families of Lagrangian immersions of S^n in T^*S^n that are regularly homotopic to the zero section embedding as smooth immersions, but are *not* Lagrangian isotopic to any Lagrangian embedding.

We now describe these constructions and results in a bit more detail. Let (M^{2n}, ω) be a symplectic manifold of dimension $2n$. Recall that a Lagrangian submanifold $L \subset M$ is a smooth submanifold of dimension n such the restriction of the symplectic form ω to the tangent space of L is trivial. That is, each tangent space of L is an isotropic subspace of the tangent space of M. A Lagrangian embedding $e : L \hookrightarrow M$ is a smooth embedding whose image is a Lagrangian submanifold. Similarly, a Lagrangian immersion $\iota : L \to M$ is a smooth immersion so that the image of each tangent space is a Lagrangian subspace of the tangent space of M.

If M is a cotangent bundle T^*N for some smooth, closed n-manifold N, then a Lagrangian immersion $\iota : L \to T^*N$ determines a map $\tau_L : L \to U/O$, which is well-defined up to homotopy. This map is defined as follows.

Let $j : N \subset \mathbb{R}^K$ be any smooth embedding with normal bundle ν. Complexifying, we get

$$\mathbb{C}^K \times N \cong (\nu \otimes \mathbb{C}) \oplus (TN \otimes \mathbb{C}) \cong (\nu \otimes \mathbb{C}) \oplus T(T^*N)_{|_N}.$$

Here $T(T^*N)$ has a Hermitian structure induced by its symplectic structure and the Riemannian structure on N coming from the embedding j. So for each $x \in L$, the tangent space T_xL defines, via the immersion ι, a Lagrangian subspace of $T_{\iota(x)}(T^*N)$. By taking the direct sum with the Lagrangian $\nu = \nu \otimes \mathbb{R} \subset \nu \otimes \mathbb{C}$, one obtains a Lagrangian subspace of $\mathbb{C}^K$. The Grassmannian of Lagrangian subspaces of $\mathbb{C}^K$ is homeomorphic to $U(K)/O(K)$. One therefore has a map

$$\tau_L : L \to U(K)/O(K) \xrightarrow{\subseteq} U/O.$$

Now the h-principle for Lagrangian immersions states that the set of Lagrangian isotopy classes of immersions $L \to T^*N$ in the homotopy class of a fixed map $\iota_0 : L \to T^*N$ can be identified with the set of connected components of the space of injective maps of vector bundles

$$TL \to \iota_0^*(T(T^*N))$$

that have Lagrangian image. Let $Sp(T(T^*N))$ be the bundle over T^*N with fiber over $(x, u) \in T^*N$ the group of linear automorphisms of $T_{(x,u)}(T^*N))$ preserving the symplectic form. This is a principal $Sp(n, \mathbb{R})$-bundle. Then the space of all such maps of vector bundles is either empty or a principal homogeneous space over the space of sections of $\iota_0^*(Sp(T(T^*N)))$. That is, the space of sections acts freely and transitively.

Abouzaid and Kragh then consider the following special case:

$$L = N,\ TN \otimes \mathbb{C}\text{ is a trivial complex vector bundle, and} \qquad (10.4.1)$$
$$\iota_0 \text{ is the inclusion of the zero section.}$$

In this case the the bundle $Sp(T(T^*N))$ is the trivial $Sp(n; \mathbb{R})$ bundle, and so the corresponding space of sections is the (based) mapping space $Map(N, Sp(n; \mathbb{R}))$, and since $U(n) \simeq Sp(n; \mathbb{R})$ it is equivalent to $Map(N, U(n))$. Since the inclusion of $U(n) \hookrightarrow U$ is $(2n - 1)$- connected, one concludes the following.

Lemma 10.4.1. *The equivalence classes of Lagrangian immersions in the homotopy class of the zero section of T^*N are classified by homotopy classes of maps from N to U.*

One therefore has the following homotopy theoretic characterization of the Lagrangian isotopy classes of Lagrangian immersions of the sphere S^n into its cotangent space.

Corollary 10.4.2. *Isotopy classes of Lagrangian immersions of the sphere S^n in T^*S^n in the homotopy class of the standard embedding of the zero section, are classified by $\pi_n(U)$.*

Of course these homotopy groups are known to be the integers $\mathbb{Z}$ when n is odd and zero when n is even. Using a type of Floer homotopy theory, Abouzaid and Kragh proved the following in [2].

Theorem 10.4.3. *Whenever n is congruent to 1, 3, or 5 modulo 8, there is a class of Lagrangian immersions of S^n in T^*S^n in the homotopy class of the zero section, that does not admit an embedded representative.*

We now sketch the Abouzaid-Kragh proof of this theorem, and in particular point out the Floer homotopy theory they used.

They first observed that given a Lagrangian immersion $j : N \to T^*N$ in the homotopy class of the zero section, satisfying condition (10.4.1), then one has a well defined (up to homotopy) classifying map $\gamma_j : N \to U$ that lifts the map $\tau_N : N \to U/O$ described above:

$$\tau_N : N \xrightarrow{\gamma_j} U \xrightarrow{project} U/O. \tag{10.4.2}$$

Now given any Lagrangian embedding $L \hookrightarrow M$ as above, Kragh [30] defined a "Maslov" (virtual) bundle η_L over the component of the free loop space consisting of contractible loops, $\mathcal{L}_0L$. [1]

The bundle η_L is classified by the following map (which by abuse of notation we also call η_L)

$$\eta_L : \mathcal{L}_0L \xrightarrow{\mathcal{L}\tau_L} \mathcal{L}_0U/O \xrightarrow{\simeq} U/O \times \Omega_0U/O \xrightarrow{project} \Omega_0U/O \simeq BO. \tag{10.4.3}$$

Here the equivalence $\mathcal{L}_0U/O \simeq U/O \times \Omega_0U/O$ comes from considering the evaluation fibration

$$\Omega U/O \to LU/O \xrightarrow{ev} U/O$$

where ev evaluates a loop at the basepoint. This fibration has a canonical (up to homotopy) trivialization as infinite loop spaces because of the existence of a section as constant loops, and using the infinite loop structure of U/O. The equivalence $\Omega_0U/O \simeq BO$ comes from Bott periodicity.

Note. Given a map of any space $f : X \to U/O$, the corresponding virtual bundle over the free loop space

$$\mathcal{L}_0X \to BO$$

defined as this composition $\mathcal{L}_0X \xrightarrow{\mathcal{L}f} \mathcal{L}_0U/O \xrightarrow{\simeq} U/O \times \Omega_0U/O \xrightarrow{project} \Omega_0U/O \simeq BO$. also appeared in the work of Blumberg, the author, and Schlichtkrull [6] in their work on the topological Hochschild homology of Thom spectra.

As above let $GL_1(\mathbb{S})$ be the group-like monoid of units in $\mathbb{S}$. There is a natural map $J : BO \to BGL_1(\mathbb{S})$ coming from the inclusion $O(n) \hookrightarrow \Omega^n_{\pm 1}S^n$ defined by considering the based self equivalence of $S^n = \mathbb{R}^n \cup \infty$ given by an orthogonal matrix. We call this map "J" as it induces the classical J homomorphism on the level of homotopy groups, $J : \pi_q(O) \to \pi_q(\mathbb{S})$. On the level of fibrations, J associates to a k-dimensional vector bundle $\zeta \to X$ the associated spherical fibration $S(\zeta) \to X$ defined by taking the one-point compactification of each fiber.

The following is an important result about Lagrangian *embeddings* in [2].

[1] We have changed the notation for the free loop space by using a script $\mathcal{L}$, so as not to get confused by the use of an "L" to denote a Lagrangian.

Theorem 10.4.4. *If $j : L \to T^*N$ is an exact Lagrangian embedding then the stable spherical fibration of the Maslov bundle η_L is trivial. That is, the composition*

$$\mathcal{L}_0 L \xrightarrow{\eta_L} BO \xrightarrow{J} BGL_1(\mathbb{S})$$

is null homotopic.

Before sketching how this theorem was proven in [2], we indicate how Abouzaid and Kragh used this result to detect the Lagrangian immersions of S^n in T^*S^n yielding Theorem 10.4.3.

A Lagrangian immersion $\iota : S^n \to T^*S^n$ is classified by a class $\alpha_\iota \in \pi_n(U)$ by Corollary 10.4.2. By Theorem 10.4.4 if ι is Lagrangian isotopic to a Lagrangian embedding, then the composition

$$\mathcal{L}S^n = \mathcal{L}_0 S^n \xrightarrow{\mathcal{L}_0\alpha_\iota} \mathcal{L}_0 U \to \mathcal{L}_0(U/O) \xrightarrow{\simeq} U/O \times \Omega_0 U/O \to \Omega_0 U/O \\ \simeq BO \xrightarrow{J} BGL_1(\mathbb{S})$$

is null homotopic. If ι is such that $\alpha_\iota \in \pi_n(U)$ is a nonzero generator (here n must be odd), then Abouzaid and Kragh precompose this map with the composition

$$S^{n-1} \hookrightarrow \Omega S^n \to \mathcal{L}S^n$$

and show that in the dimensions given in the statement of the theorem, then classical calculations of the J-homomorphism in homotopy theory imply that this composition $S^{n-1} \to BGL_1(\mathbb{S})$ is nontrivial. Therefore by Theorem 10.4.4, the Lagrangian immersions of S^n into T^*S^n that these classes represent cannot be Lagrangian isotopic to embeddings, even though as smooth immersions, they are isotopic to the zero section embedding.

The proof of Theorem 10.4.4 is where Floer homotopy theory is used. This was based on earlier work by Kragh in [30]. The Floer homotopy theory used was a type of Hamiltonian Floer theory for the cotangent bundle. They did not directly use the constructions in [12] described above, but the spirit of the construction was similar. More technically they used finite dimensional approximations of the free loop space, not unlike those used by Manolescu [37]. We refer the reader to [30], [2] for details of this construction. In any case this construction was used to give a spectrum-level version of the Viterbo transfer map, when one has an exact Lagrangian embedding $j : L \to T^*N$. This transfer map is given on the spectrum level by a map

$$\mathcal{L}_0 j^! : \mathcal{L}_0 N^{-TN} \to (\mathcal{L}_0 L)^{-TL\oplus\eta}$$

and similarly a map

$$\mathcal{L}_0 j^! : \Sigma^\infty(\mathcal{L}_0 N_+) \to (\mathcal{L}_0 L)^{TN-TL\oplus\eta}. \tag{10.4.4}$$

An important result in [2] is that $\mathcal{L}_0 j^!$ is an equivalence. Then they make use of the result, essentially due to Atiyah, that given a finite CW-complex X with a stable spherical fibration classified by a map $\rho : X \to BGL_1(\mathbb{S})$, then the Thom spectrum X^ρ is equivalent to the suspension spectrum $\Sigma^\infty(X_+)$ if and only if ρ is null homotopic. Applying this to finite dimensional approximations to $\mathcal{L}_0 L$, they are then able to show that the equivalence (10.4.4) implies that the Maslov bundle η, when viewed as a stable spherical fibration is trivial.

We end this discussion by remarking on the recasting of the Abouzaid-Kragh results by the author and Klang in [13]. Given a Lagrangian immersion, $j : L \to T^*N$, consider the resulting class

$$\tau_L : L \to U/O$$

described above. Taking based loop spaces, one has a loop map

$$\Omega\tau_L : \Omega_0 L \to \Omega_0 U/O \simeq BO.$$

The resulting Thom spectrum $(\Omega_0 L)^{\Omega\tau_L}$ is a ring spectrum. Thus one can apply topological Hochshild homology to this ring spectrum, $THH((\Omega_0 L)^{\Omega\tau_L})$ and one obtains a homotopy theoretic invariant of the Lagrangian isotopy type of the Lagrangian immersion j. This topological Hochshild was computed by Blumberg, the author, and Schlichtkrull in [6]. It was shown that

$$THH((\Omega_0 L)^{\Omega\tau_L}) \simeq (\mathcal{L}_0 L)^{\ell(\tau_L)}$$

where $\ell(\tau_L)$ is a specific stable bundle over the free loop space $\mathcal{L}L$. In particular, as was observed in [13], in the case when τ_L factors through a map to U, as is the case when $L = N$ and it satisfies the condition (10.4.1), then there is an equivalence of stable bundles, $\ell(\tau_L) \cong \eta_L$, where η_L is the Maslov bundle as above. As a consequence of Theorem 10.4.4 of Abouzaid and Kragh, one obtains the following:

Proposition 10.4.5. *Assume $j : N \to T^*N$ is a Lagrangian immersion in the homotopy class of the zero section, and that the complexification $TN \otimes \mathbb{C}$ is stably trivial. Then if j is Lagrangian isotopic to a Lagrangian embedding, then $THH(\Omega_0 N)^{\Omega\tau_N}))$ is equivalent to $THH(\Sigma^\infty(\Omega N_+))$.*

Using the results of [6] the authors in [13] used this proposition to give a proof of Abouzaid and Kragh's Theorem 10.4.3.

Bibliography

[1] A. Abbondandolo and M. Schwarz, *On the Floer homology of cotangent bundles*, Comm. Pure Appl. Math **59,** 254–316 (2006).

[2] M. Abouzaid and T. Kragh, *On the immersion classes of nearby Lagrangians*, J. Topol. **9** (2016), no. 1, 232–244.

[3] M. Abouzaid and I. Smith, *Khovanov homology from Floer cohomology*, J. Amer. Math. Soc. **32** (2019), no. 1, 1–79.

[4] S. Akbulut and J. McCarthy, *Casson's invariant for oriented homology* 3*-spheres*, Math. Notes, vol. 36, Princeton University Press, 1990.

[5] M. Ando, A. J. Blumberg, D. J. Gepner, M. J. Hopkins, and C. Rezk, *Units of ring spectra and Thom spectra*, arXiv:0810.4535, 2008.

[6] A. Blumberg, R. L. Cohen, and C. Schlichtkrull *Topological Hochschild homology of Thom spectra and the free loop space*, Geom. Topol. **14** no.2 2010, 1165–1242.

[7] M. Chas and D. Sullivan, *String topology*, arXiv:math.GT/9911159, 1999.

[8] M. Cohen, *Homeomorphisms between homotopy manifolds and their resolutions*, Invent. Math. **10** (1970), 239–250.

[9] R. L. Cohen, *The Floer homotopy type of the cotangent bundle*, Pure Appl. Math. Q. **6** (2010), no. 2, Special Issue: In honor of Michael Atiyah and Isadore Singer, 391–438.

[10] R. L. Cohen, *Floer homotopy theory, realizing chain complexes by module spectra, and manifolds with corners,* in Proc. of Fourth Abel Symposium, Oslo, 2007, ed. N. Baas, E.M. Friedlander, B. Jahren, P. Ostvaer, Springer Verlag (2009), 39–59.

[11] R. L. Cohen, J. D. S. Jones, and G. B. Segal, *Morse theory and classifying spaces*, available at http://math.stanford.edu/~ralph/morse.ps (1995).

[12] R. L. Cohen, J. D. S. Jones, and G. B. Segal, *Floer's infinite-dimensional Morse theory and homotopy theory*, The Floer Memorial Volume, Progr. Math., vol. 133, Birkhäuser, 1995, pp. 297–325.

[13] R. L. Cohen and I. Klang, *Twisted Calabi-Yau ring spectra, string topology, and gauge symmetry*, Tunis. J. Math. 2 (2020), no. 1, 147–196.

[14] J. Eels, Jr. and N. Kuiper, *An invariant for certain smooth manifolds* Ann. Mat. Pura Appl. (4) **60** (1962), 93-110.

[15] R. Fintushel and R. Stern, *Instanton homology of Seifert fibred homology three spheres*, Proc. London Math. Soc. (3) **61** (1990), no. 1, 109–137.

[16] A. Floer, *Morse theory for Lagrangian intersections,* J. Differential Geom. **28** (1988), 513–547.

[17] A. Floer, *An instanton invariant for 3-manifolds,* Comm. Math. Phys., **118** (1988), 215–240.

[18] A. Floer, *Symplectic fixed points and holomorphic spheres,* Comm. Math. Phys., **120** (4) (1989), 575–611.

[19] J. M. Franks, *Morse-Smale flows and homotopy theory*, Topology **18** (1979), 119–215.

[20] K. A. Froyshov, *Equivariant aspects of Yang-Mills Floer theory*, Topology **41** (2002), no. 3, 525–552.

[21] M. Furuta, *Homology cobordism group of homology* 3*-spheres,* Invent. Math. **100** (1990), no. 2, 339–355.

[22] M. Furuta, *Monopole equation and the* $\frac{11}{8}$*-conjecture*, Math. Res. Lett. **8** (2001), no. 3, 279–291.

[23] D. Galewski and R. Stern, *Classification of simplicial triangulations of topological manifolds*, Ann. Math. (2) **111** (1980), no.1, 1–34.

[24] J. Genauer, *Cobordism categories of manifolds with corners*, Stanford University PhD thesis, (2009), arXiv.org/abs/0810.0581, 2008.

[25] K.Janich, *On the classification of* $O(n)$*-manifolds*, Math. Ann. **176** (1968), 53–76.

[26] P. Hu, D. Kriz, and I. Kriz, *Field theories, stable homotopy theory, and Khovanov homology*, Topology Proc. 48 (2016), 327–360.

[27] M. Khovanov, *A categorification of the Jones polynomial*, Duke Math. J. **101** no. 3, (2000), 359–426.

[28] R. Kirby and L. Siebenmann, *Foundational essays on topological manifolds, smoothings, and triangulations*, Ann. of Math. Stud., No. 88. Princeton University Press, 1977. With notes by J. Milnor and M. Atiyah,

[29] H. Kneser, *Die Topologie der Mannigfaltigkeiten*, Jahresbericht der Deut. Math. Verein. **34** (1926), 1–13.

[30] T. Kragh, *Parameterized ring-spectra and the nearby Lagrangian conjecture*, Geom. Topol. **17** (2013), 639–731 (Appendix by M. Abouzaid).

[31] P. Kronheimer and T. Mrowka, *Monopoles and three-manifolds,* New Mathematical Monographs, vol. 10, Cambridge University Press, 2007.

[32] P. Kronheimer, T. Mrowka, P. Oszvath, and Z. Szabo, *Monopoles and lens space surgeries*, Ann. of Math. (2) **165** (2007), no. 2, 457–546.

[33] G. Laures, *On cobordism of manifolds with corners*, Trans. Amer. Math. Soc. **352** no. 12, (2000), 5667–5688.

[34] T. Lawson, R. Lipshitz, and S. Sarkar, *Khovanov homotopy type, Burnside category, and products*, arXiv: 1505.00213v1, 2015.

[35] R. Lipshitz and S. Sarkar, *A Khovanov stable homotopy type*, J. Amer. Math. Soc. 27 (2014), no. 4, 983–1042.

[36] R. Lipshitz and S. Sarkar, *A Steenrod square on Khovanov homology*, J. Topology, 7 (2014), no. 3, 817–848.

[37] C. Manolescu, *Seiberg-Witten-Floer stable homotopy type of three-manifolds with* $b_1 = 0$, Geom. Topol. 7 (2003), 889–932.

[38] C. Manolescu, *A gluing theorem for the relative Bauer-Furuta invariants*, J. Differential Geom. 76 (2007), no. 1, 117–153.

[39] C. Manolescu, *Pin (2)-equivariant Seiberg-Witten Floer homology and the triangulation conjecture*, J. Amer. Math. Soc. 29 (2016), no. 1, 147–176.

[40] C. Manolescu, *Floer theory and its topological applications*, Jpn. J. Math. 10 (2015), no. 2, 105–133.

[41] C. Manolescu, *The Conley index, gauge theory, and triangulations*, J. Fixed Point Theory Appl. 13 (2013), no. 2, 431–457.

[42] T. Matumoto, *Triangulations of manifolds*, Algebraic and Geometric Topology (Proc. Symp. Pure Math., Stanford Univ., 1976), Part 2, Proc. Symp. Pure Math., XXXII, Amer. Math. Soc., 1978, pages 3–6.

[43] D. McDuff and D. Salamon, *Introduction to symplectic topology*, Oxford Math Monographs, Clarendon Press, 1998.

[44] P. Ozsváth and Z. Szabó, *Holomorphic disks and topological invariants for closed three-manifolds*, Ann. of Math. **159** (3) (2004) 1027–1158.

[45] P. Ozsváth and Z. Szabó, *Holomorphic disks and knot invariants*, arXiv:math/0209056v4, 2003.

[46] A. Pressley and G. Segal, *Loop groups*, Oxford Math. Monogr., Clarendon Press, 1986.

[47] J. Rasmussen, *Floer homology and knot complements*, arXiv:math/0306378, 2003.

[48] J. Rasmussen, *Khovanov homology and the slice genus*, Invent. Math. 182 (2010), no. 2, 419–447.

[49] D. Ravenel, *Complex cobordism and stable homotopy groups of spheres,* Pure Appl. Math., vol. 121, Academic Press, Inc., 1986.

[50] V. A. Rokhlin, *New results in the theory of four-dimensional manifolds*, Doklady Akad. Nauk SSSR (N.S) **84** (1952), 221–224.

[51] L. Qin, *On the associativity of gluing*, J. Topol. Anal. 10 (2018), no. 3, 585–604.

[52] H. Sato, *Constructing manifolds by homotopy equivalences, I. An obstruction to constructing PL manifolds from homology manifolds*, Ann. Inst. Fourier (Grenoble) **22** (1972), no. 1, 271–286.

[53] C. Seed, *Computations of the Lipshitz-Sarkar Steenrod square on Khovanov homology*, arXiv:1210.1882, 2012.

[54] M. Stoffregen and M. Zhang, *Localization in Khovanov homology*, arXiv:1810.04769, 2018.

[55] P. Seidel and I. Smith, *A link invariant from the symplectic geometry of nilpotent slices*, Duke Math. J. **134** (2006), 453–514.

[56] D. Sullivan, *Triangulating and smoothing homotopy equivalences and homeomorphisms*, Geometric Topology Seminar notes, The Hauptvermutung book, K-Monogr. Math., vol. 1, Kluwer Acad. Publ., 1996, pp. 69–103.

[57] C. Taubes *The Seiberg-Witten equations and the Weinstein conjecture*, Geom. Topol. **11** (2007), 2117–2202.

[58] C. Viterbo, *Functors and computations in Floer homology with applications, Part II*, arXiv:1805.01316, 2018.

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