9

Complex Convexity

Christer Oscar Kiselman

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9.1 Introduction to This Chapter

This chapter is devoted to complex convexity. Which are the most significant results presented here? This is the question that I answer in this first section of the chapter. But first I shall explain why real convexity is of interest, why complex convexity is important, and why mathematical morphology is a useful tool in the study of convexity.

What makes the approach in the present chapter different from other presentations of the subject? Also, this question will receive an answer.

9.1.1 Why is real convexity of interest?

A subset $A$ of $\mathbb{R}^n$ is defined to be convex if for any pair $\{a, b\}$ of points in $A$ the whole interval $[a, b]$ is also contained in $A$. A function $f : \mathbb{R}^n \to [-\infty, +\infty]$ is said to be convex if its finite epigraph

$$\text{epi}^\text{finite}(f) = \{(x, t) \in \mathbb{R}^n \times \mathbb{R}; f(x) \leq t\},$$

(9.1)

is a convex set.
Convex functions possess a property of great importance in optimization theory: A local minimum of a convex function $\mathbb{R}^n \to \mathbb{R}$ is automatically a global minimum. In other words, if $f: \mathbb{R}^n \to \mathbb{R}$ is convex and $f(x) \geq c$ for all points $x$ in a neighborhood of $a$, however small, then $f(x) \geq c$ for all $x$ in $\mathbb{R}^n$.

Real convexity appears naturally also in complex analysis: the indicator function of the Fourier transform (defined in $\mathbb{C}^n$) of a function or distribution in $\mathbb{R}^n$ of compact support is convex.

We collect in Section 9.2 definitions and basic properties of convex sets and functions in vector spaces over the field of real numbers or over the field of complex numbers.

### 9.1.2 Why is complex convexity important?

In one complex variable, complex convexity of sets is not of great importance. Given any open set $\Omega$ of the complex plane $\mathbb{C}$ and a point $p$ not belonging to $\Omega$, we define a rational function $z \mapsto 1/(z - p)$ which cannot be extended as a holomorphic function across $p$.

But in two variables, the set of singularities of a rational function, indeed of any meromorphic function, cannot be just a singleton set. There are easy examples of two sets $\omega \subset \Omega$ such that any holomorphic function in $\omega$ can be extended to a holomorphic function in $\Omega$.

This phenomenon gives rise to the concept of domains of holomorphy, which are domains such that there are holomorphic functions that cannot be continued to a larger domain, in a sense to be made precise. Related to these is the definition of a pseudoconvex domain. That a domain of holomorphy is pseudoconvex was proved by Eugenio Elia Levi (1883–1917); the converse, at the time an unsolved problem, came to be known as the Levi problem. It was solved by Kiyoshi Oka (1901–1978) in two variables, and later in any finite dimension by Oka, François Norguet (1929–2010) and Hans-Joachim Bremermann (1926–1996)—for a survey, see (Slatyer 2016).

So these phenomena point to the fact that there are great differences between the geometry of $\mathbb{C}$ and the geometry of $\mathbb{C}^2$. We can draw two-dimensional figures on a paper, and we can visualize objects in three dimensions. Nowadays there are even nice programs that create figures on the screen that can be rotated to exhibit all properties of an object in three-space. But two complex variables is a challenge because they correspond to four real coordinates.

Can you see in four dimensions? Yes, it is indeed possible to train one’s inner eyes to see in four dimensions. A nontrivial but most rewarding sport. We can actually arrive at true stereoscopic vision … However, if you are not yet a master of four-dimensional landscapes, you will appreciate the Hartogs sets, named for Friedrich Moritz Hartogs (1874–1943), where we can be content with three real variables $(Re z_1, Im z_1, |z_2|)$ instead of the four $(Re z_1, Im z_1, Re z_2, Im z_2)$. An example is Figure 9.1 on page 281. To view
Reinhardt domains, named for Karl Reinhardt (1895–1941), we need only $(|z_1|, |z_2|) \in \mathbb{R}^2$.

**9.1.3 Why is mathematical morphology a useful tool in the study of convexity?**

Mathematical morphology can be superficially described as applied lattice theory. As such it is about the operations $(x, y) \mapsto x \land y = \min(x, y)$ and $(x, y) \mapsto x \lor y = \max(x, y)$ in an ordered set. These operations replace addition and multiplication in a ring, and an important example is the Boolean ring of all subsets of a given set, with $\land$ as intersection and $\lor$ as union.

In convexity theory, we see that the intersection of two convex sets is convex, while the union is, in general, not. But we still have a lattice, in that the convex hull of the union of two convex sets is the smallest convex set containing the two, and therefore is the supremum of the two. This can then be done analogously for convex functions, and, more generally for plurisubharmonic functions. It turns out that complete lattices are important; they are the ordered sets which allow infima and suprema also of infinite families.

Mathematical morphology provides us with important concepts in the theory of ordered sets that are helpful in understanding several related phenomena in mathematics. See Section 9.3 for more details.

**9.1.4 Which are the most significant results reported in the present chapter?**

An important observation is the non-local character of lineal convexity for general sets. As always, properties such that the local and global variants are different create difficulties—which may be challenging.

Because of the non-local character just mentioned, it is of importance to know that for bounded sets with a smooth boundary, the property of being locally lineally convex actually implies the global property. This is proved in Section 9.6.

**9.1.5 What makes the approach in the present chapter different from other presentations of the subject?**

Complex convexity is quite a well-studied field, and the ways to approach it are not many. However, we shall view convexity from the inside as well as from the outside, and this gives perhaps interesting perspectives. A set is concave if and only if its complement is convex, and the two notions should be studied together.

As mentioned, a subset $A$ of $\mathbb{R}^n$ is defined to be convex if for any pair $\{a, b\}$ of points in $A$, the whole segment $[a, b]$ is also contained in $A$. This is what we can call convexity from the inside, i.e., looking at subsets of the given
set. But we can also look at the set from the outside: If \( p \) does not belong to \( A \) and \( A \) is open or closed, then there is a half-space that contains \( A \) but not \( p \). This is the Hahn–Banach theorem, of utmost important in convexity theory, both in finite dimension and infinite dimension. Explicitly, we say that a set in a vector space over \( \mathbb{R} \) or \( \mathbb{C} \) is \textbf{lineally concave} if it is a union of hyperplanes, and \textbf{lineally convex} if its complement is lineally concave. In one dimension, hyperplanes are just points, so every set is both lineally concave and lineally convex. In higher dimensions a convex set need not be lineally convex, but if it is open or closed, this is true. All this is true both in the real and the complex settings. For more details, see Section 9.4.

\section*{Acknowledgments}

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9.2 Introduction to Convexity

9.2.1 Introduction to this section

The theory of convexity of sets and functions in vector spaces is a highly developed and very rich theory. We collect in this section definitions and basic properties of convex sets and functions in vector spaces over the field of real numbers or over the field of complex numbers. However, we present but a bare minimum of what is needed for the rest of this chapter. For fuller accounts, see the classical book by R. Tyrrell Rockafeller (1970, 1997), the books by Jean-Baptiste Hiriart-Urruty and Claude Maréchal (1993, 2002), and also my book manuscript (ms 2021).

9.2.2 Sets, mappings, and order relations

9.2.2.1 Notation for numbers, norms, and derivatives

We write \(N = \{0, 1, 2, \ldots\}\) for the set of natural numbers, following Bourbaki (1963:67), \(Z, \mathbb{R}, \mathbb{C}\) for the ring of integers, the fields of real and complex numbers.

We shall use the \(l^p\)-norm \(\|z\|_p = \left(\sum |z_j|^p\right)^{1/p}, 1 \leq p < +\infty\), and the \(l^\infty\)-norm \(\|z\|_\infty = \sup_j |z_j|\) for \(z \in \mathbb{C}^n\). When any norm can serve, we write only \(\|z\|\).

The bilinear inner product of two vectors in \(\mathbb{R}^m\) or \(\mathbb{C}^n\) shall be denoted by a dot:

\[
x \cdot y = x_1y_1 + \cdots + x_my_m; \quad z \cdot w = z_1w_1 + \cdots + z_nw_n, \quad x, y \in \mathbb{R}^m, z, w \in \mathbb{C}^n.
\]

The Euclidean norm will be written like this:

\[
\|x\|_2 = \sqrt{x \cdot x}; \quad \|z\|_2 = \sqrt{z \cdot \bar{z}}, \quad x \in \mathbb{R}^m, z \in \mathbb{C}^n.
\]

We shall denote by \(B_<(c, r)\) and \(B_<(c, r)\) the open ball and the closed ball, respectively, with center at \(c \in \mathbb{C}^n\) and radius \(r \in \mathbb{R}\) for any norm, thus

\[
B_<(c, r) = \{z \in \mathbb{C}^n; \|z - c\| < r\} \quad \text{and} \quad B_<(c, r) = \{z \in \mathbb{C}^n; \|z - c\| \leq r\}.
\]

If \(n = 1\), we shall write instead \(D_<(c, r)\) and \(D_<(c, r)\) for the disks.

The closure, interior and boundary of a subset \(A\) of a topological space will be denoted by \(\overline{A}\), \(A^\circ\) and \(\partial A\), respectively. Thus \(\overline{B_<(c, r)} = \overline{B_<(c, r)}\) if \(r\) is positive, and \(\overline{B_<(c, r)} = \overline{B_<(c, r)}\) for all real \(r\).

For derivatives of functions we shall use the notation

\[
f_x = \frac{\partial f}{\partial x_i}, \quad f_y = \frac{\partial f}{\partial y_j}, \quad f_z = \frac{\partial f}{\partial z_j} = \frac{1}{2}(f_{x_j} - if_{y_j}).
\]

\(^{1}\)Alfred Tarski (1956:121) calls 0 a natural number.
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\[ f_{z_j} = \frac{\partial f}{\partial z_j} = \frac{1}{2}(f_{x_j} + i f_{y_j}), \quad f_{z_j \bar{z}_k} = \frac{\partial^2 f}{\partial z_j \partial \bar{z}_k}, \quad j, k = 1, \ldots, n. \]

Differentials are written as

\[ df = d'f + d''f, \]

where

\[ d'f = \sum f_{z_j} dz_j \quad \text{and} \quad d''f = \sum f_{\bar{z}_k} d\bar{z}_k. \] (9.2)

9.2.2.2 Counting with infinities

We shall also use a notation for the extended real line

\[ \mathbb{R}_1 = \mathbb{R} \cup \{-\infty, +\infty\} = [-\infty, +\infty], \]

thus creating the two-point compactification of \( \mathbb{R} \) by adding two infinities, \(+\infty\) and \(-\infty\). We also add these infinities to the integers,

\[ \mathbb{Z}_1 = \mathbb{Z} \cup \{-\infty, +\infty\} = [-\infty, +\infty]. \]

Similarly we write \( \mathbb{Y}_1 = \mathbb{Y} \cup \{-\infty, +\infty\} \) for any subset \( \mathbb{Y} \) of \( \mathbb{R} \).

How shall we define a sum like \((+\infty) + (-\infty)\)? Can addition \( \mathbb{R} \times \mathbb{R} \ni (x, y) \mapsto x + y \in \mathbb{R}\) be extended to an operation \( \mathbb{R}_1 \times \mathbb{R}_1 \ni (x, y) \mapsto x + y \in \mathbb{R}_1 \) in a reasonable way? A convenient solution, pioneered by Jean-Jacques Moreau (1923–2014) in his paper (1970), is to define two extensions, upper addition and lower addition. The first is an upper semicontinuous mapping from \( \mathbb{R}_1 \times \mathbb{R}_1 \) into \( \mathbb{R}_1 \); the second a lower semicontinuous mapping. They are denoted by \(+\) and \(\downarrow\) and are defined by the requirements of being commutative and to satisfy

\[ x + ( + \infty ) = + \infty \quad \text{for all } x \in \mathbb{R}_1; \]

\[ x + ( - \infty ) = - \infty \quad \text{for all } x \in [-\infty, +\infty]; \]

and

\[ x \downarrow y = -(( -x ) + ( -y )) \quad \text{for all } x, y \in \mathbb{R}_1. \] (9.3)

When there are several terms we may use the summation symbol with a dot:

\[ \sum_{j=1}^{m} t_j = t_1 \downarrow \cdots \downarrow t_m, \quad t_j \in \mathbb{R}_1. \] (9.4)

A convenient rule is the following.

**Lemma 9.2.1** For any element \( c \in \mathbb{R}_1 \) and any function \( f : X \to \mathbb{R}_1 \) defined on an arbitrary set \( X \) we have

\[ \inf_{x \in X} (c \downarrow f(x)) = c \downarrow \inf_{x \in X} f(x). \] (9.5)

Note that there are no exceptions to this formula.
Proof We just need to check all possibilities where our intuition is less reliable than usual, i.e., when \( c = \pm \infty \) or \( X \) is empty.

We also note the equivalence
\[
a + b \geq c \iff a \geq c + (-b), \quad a, b, c \in \mathbb{R}.
\]  
(9.6)

9.2.2.3 Sets

The **empty set**, the unique set with no elements, will be denoted by \( \emptyset \). If \( A \) and \( B \) are sets such that every element of \( A \) belongs also to \( B \), then \( A \) is said to be a **subset** of \( B \) and \( B \) a **superset** of \( A \). This is written \( A \subset B \) and \( B \supset A \).

The family of all subsets of a set \( W \) is called the **power set** of \( W \), and will denoted by \( \mathcal{P}(W) \). Thus \( A \in \mathcal{P}(W) \) if and only if \( A \subset W \). We denote by \( \mathcal{P}_\text{finite}(W) \) the family of all finite subsets of \( W \).

We shall use the usual symbols for the **intersection** and **union** of a family \((A_j)_{j \in J}\) of sets:
\[
\bigcap_{j \in J} A_j \quad \text{and} \quad \bigcup_{j \in J} A_j, \quad A_j \in \mathcal{P}(W).
\]  
(9.7)

When the index set \( J \) has only two elements, we write \( A_1 \cap A_2 \) and \( A_1 \cup A_2 \).

The set of all elements in \( A \) that are not elements of \( B \) is called the **set-theoretical difference of \( A \) and \( B \)**, written \( A \setminus B \). When \( A \) is equal to the whole set \( W \) we can write \( \emptyset \setminus B \), the **complement** of a subset \( B \) of \( W \).

The **Cartesian product** of two sets \( X \) and \( Y \) is the set of all pairs \((x, y)\) with \( x \in X \) and \( y \in Y \). The Cartesian product of \( n \) sets \( X_j, j = 1, \ldots, n \), denoted by
\[
X_1 \times \cdots \times X_n = \prod_{j=1}^{n} X_j,
\]
is the set of all \( n \)-tuples \((x_1, \ldots, x_n)\) with \( x_j \in X_j \).

To any set \( A \subset X \) we associate its **characteristic function** \( \chi_A \) defined to take the value 1 in \( A \) and the value 0 in \( X \setminus A \). We also define its **indicator function** \( \text{ind}_A \), which takes the value 0 in \( A \) and the value \(+\infty\) in its complement.

9.2.2.4 Graphs, epigraphs, and hypographs

**Definition 9.2.2** To any mapping \( f : X \to Y \) we define its **graph**:

graph(f) = \{(x, y) : f(x) = y\} \subset X \times Y,

a subset of the Cartesian product \( X \times Y \).
Definition 9.2.3 To any function \( f : X \to \mathbb{R} \), we associate its epigraph
\[
\text{epi}(f) = \{(x,t) \in X \times \mathbb{R}; \ t \geq f(x)\}; \tag{9.8}
\]
and its strict epigraph
\[
\text{epi}_{\text{strict}}(f) = \{(x,t) \in X \times \mathbb{R}; \ t > f(x)\}. \tag{9.9}
\]
We define its finite epigraph as
\[
\text{epi}_{\text{finite}}(f) = \{(x,t) \in X \times \mathbb{R}; \ t \geq f(x)\}; \tag{9.10}
\]
and its strict finite epigraph as
\[
\text{epi}_{\text{finite}}^{\text{strict}}(f) = \{(x,t) \in X \times \mathbb{R}; \ t > f(x)\}. \tag{9.11}
\]
The first two being subsets of the Cartesian product \( X \times \mathbb{R} \), the following two of the product \( X \times \mathbb{R} \).

It is easy to pass from the finite epigraph to the strict finite epigraph as well as in the other direction:
\[
\text{epi}_{\text{finite}}^{\text{strict}}(f) = \bigcup_{c > 0} \text{epi}_{\text{finite}}(f + c); \quad \text{epi}_{\text{finite}}^{\text{strict}}(f) = \bigcap_{c > 0} \text{epi}_{\text{finite}}^{\text{strict}}(f - c). \tag{9.12}
\]

Similarly for \( \text{epi}_{\text{strict}} \) and \( \text{epi} \).

The finite epigraph of a function \( f : \mathbb{R}^n \to \mathbb{R} \) is contained in the closure (taken for the usual topology in \( \mathbb{R}^n \times \mathbb{R} \)) of \( \text{epi}_{\text{finite}}^{\text{strict}}(f) \), maybe strictly.

Definition 9.2.4 Analogously we define the hypograph of a function:
\[
\text{hypo}(f) = \{(x,t) \in X \times \mathbb{R}; \ t \leq f(x)\}, \tag{9.13}
\]
as well as the strict hypograph, \( \text{hypo}_{\text{strict}}(f) \), the finite hypograph, \( \text{hypo}_{\text{finite}}(f) \), and the strict finite hypograph, \( \text{hypo}_{\text{finite}}^{\text{strict}}(f) \).

It is often convenient to express properties of mappings in terms of their epigraphs or hypographs.

9.2.2.5 Inverse and direct images
To any mapping \( f : X \to Y \) we define two mappings on a higher level,
\[
f^* : \mathcal{P}(Y) \to \mathcal{P}(X)\quad \text{and}\quad f_* : \mathcal{P}(X) \to \mathcal{P}(Y).
\]
The first is defined by
\[
f^*(B) = \{x \in X; \ f(x) \in B\}, \quad B \in \mathcal{P}(Y). \tag{9.14}
\]
Here \( f^*(B) \) is called the inverse image of \( B \). The second is defined by
\[
f_*(A) = \{f(x); \ x \in A\}, \quad A \in \mathcal{P}(X). \tag{9.15}
\]
The set $f_*(A)$ is called the (direct) image of $A$. We write $\text{im}(f)$, the image of $f$, for $f_*(X)$.

The mappings $f^*$ and $f_*$ are the simplest examples of the pullbacks and pushforwards used in differential geometry, and similarly in homology theory and distribution theory.

**Definition 9.2.5** To any function $f: X \to \mathbb{R}$, we associate its effective domain, denoted by $\text{dom}(f)$ and defined as the set where the function takes values less than $+\infty$:

$$\text{dom}(f) = \{x \in X; f(x) < +\infty\} = f^*([-\infty, +\infty]),$$

the inverse image of $\mathbb{R} \cup \{-\infty\}$. □

### 9.2.3 Defining convex sets

It is most convenient to define convex functions with the help of convex sets. This also has the advantage that we can treat functions with infinite values without difficulty. That is why we start now with convex sets.

**Definition 9.2.6** A rectilinear segment in a vector space $E$ is the set

$$[a, b] = \{(1 - t)a + tb; t \in \mathbb{R}, 0 \leq t \leq 1\},$$

where $a$ and $b$ are its endpoints. A subset $A$ of a vector space is said to be convex if $\{a, b\} \subset A$ implies $[a, b] \subset A$. □

Every segment, every straight line, and all affine subspaces are convex sets.

It is worth noticing that to check convexity of a set, only its intersections with one-dimensional subspaces need to be considered. In other words, a set $A \subset E$ is convex if and only if the inverse image $f^*(A)$ is an interval (bounded or unbounded) in $\mathbb{R}$ for every mapping $f: \mathbb{R} \to E$ of the form $f(t) = ta + b$, $t \in \mathbb{R}, a, b \in E$.

A convex set $A$ in $\mathbb{R}^n$ need not have interior points, but it does so if we consider it as a subset of the smallest affine space that contains it. We define the relative interior of $A$, denoted by $\text{relint}(A)$, as the interior taken with respect to the topology in this affine subspace that is induced by the usual topology on $\mathbb{R}^n$. In this way every convex set, even if only a singleton set, has a nonempty relative interior. The set $\overline{A} \setminus \text{relint}(A)$ is the boundary of $A$ taken in the smallest affine space containing $A$ and is more interesting than the boundary of $A$ if the space mentioned is not all of $\mathbb{R}^n$.

### 9.2.3.1 The convex hull

**Definition 9.2.7** Given a subset $A$ of a vector space $E$, we define the convex hull of $A$ as the smallest convex set containing $A$. It will be denoted by $\text{cvxh}(A)$. □

The convex hull is well defined since any intersection of convex sets is convex.
9.2.4 The Hahn–Banach theorem

Among the affine subspaces we will pay attention to the hyperplanes, those of codimension 1, which means that they are defined by a single equation $\xi(x) = c$, where $\xi$ is a nonzero linear form on the vector space. Other important sets are the half-spaces, which are defined by an inequality $\xi(x) \geq c$ or $\xi(x) > c$.

A topological vector space is a vector space $E$ equipped with a topology such that both addition

$$E \times E \ni (x, y) \mapsto x + y \in E$$

and multiplication by scalars

$$C \times E \ni (t, x) \mapsto tx \in E$$

are continuous.

After having restricted some variables, we see that all translations $x \mapsto x + a$ are continuous as well as all mappings $t \mapsto ta$ for a fixed $a$. The latter property implies that the inverse image of an open set in $E$ under the mapping $t \mapsto ta$ is open for the usual topology in $C$. Similarly of the field of scalars is $\mathbb{R}$.

Theorem 9.2.8 (The Hahn–Banach theorem) Every open convex set in a topological vector space $E$ is the intersection of a family of open half-spaces.

Every closed convex set in a topological vector space is the intersection of a family of closed half-spaces—and also the intersection of a family of open half-spaces.

There are topological vector spaces with dual equal to zero. The separation can then seem like a paradox. It is resolved by the fact that in these vector spaces, the only open convex sets are the empty set and the whole space.

Here we shall accept this important theorem and refer to its proof in any of the many books on functional analysis.

Werner Fenchel (1952) characterized the sets that are intersections of a family of open half-spaces and called them evenly convex. As the Hahn–Banach theorem states, all open convex sets, and all closed convex sets are evenly convex. So are all strictly convex sets and a set like the closed triangle in $\mathbb{R}^2$ with one vertex removed defined by $0 < x_1 + x_2 \leq 1$, $x_1 \geq 0$, $x_2 \geq 0$. An open triangle with one boundary point added is not evenly convex. See Section 9.7, page 325 for the definition of refined half-spaces, which serve to represent general convex sets.

9.2.5 Supporting hyperplanes

Definition 9.2.9 Given any set $A$ in a topological vector space $E$, a supporting hyperplane is a hyperplane $Y$ such that $A$ is contained in one of the closed half-spaces defined by $Y$ and such that the closure of $A$ meets $Y$. □
Theorem 9.2.10 Let $A$ be a convex subset of a topological vector space $E$ with $\emptyset \neq A \neq E$ and let $a$ be any boundary point of $A$. Then there exists a supporting hyperplane of $A$ passing through $a$.

Proof In view of the Hahn–Banach theorem, there is a closed hyperplane $Y$ that passes through $a$ and such that $A$ is in one of the closed half-spaces defined by $Y$.

A set $A$ in a topological vector space is said to be bounded if for any neighborhood $U$ of the origin there exists a number $\lambda_0$ such that $\lambda U$ contains $A$ for all $\lambda$ with $|\lambda| \geq \lambda_0$.

Corollary 9.2.11 Let $A$ be a bounded nonempty subset of a topological vector space and let $a$ be any boundary point of $\text{cvxh}(A)$. Then there exists a supporting hyperplane of $A$ passing through $a$.

Proof We apply the theorem to $\text{cvxh}(A)$. A supporting hyperplane of $\text{cvxh}(A)$ must also be a supporting hyperplane of $A$.

9.2.6 Defining convex functions

Definition 9.2.12 A function $f : E \to \mathbb{R}$ defined in a vector space $E$ is said to be convex if its finite epigraph is convex as a subset of $E \times \mathbb{R}$. We shall write $\text{CVX}(E, \mathbb{R})$ for the set of these functions.

The functions that take one of the values $-\infty$, $+\infty$ identically are convex, since their finite epigraphs are, respectively, the whole space $E \times \mathbb{R}$ and the empty set. An affine function is convex, since its finite epigraph is a closed half-space.

It is easy to see that a function $f : E \to \mathbb{R}$ is convex if and only if it satisfies Jensen’s inequality

$$f((1-t)x + ty) \leq (1-t)f(x) + tf(y), \quad x, y \in E, \quad 0 \leq t \leq 1. \quad (9.16)$$

Here we define $0 \cdot (+\infty) = 0 \cdot (-\infty) = 0$ (or else consider only $0 < t < 1$). There are more general inequalities to be described now.

9.2.7 Strict and strong convexity

Definition 9.2.13 We shall say that a set $A$ in a vector space is strictly convex if every supporting hyperplane cuts its closure in only one point. This means that the boundary of $A$ does not contain any intervals of nonzero length. A function is said to be strictly convex if its finite epigraph is strictly convex.
Introduction to Convexity

Definition 9.2.14 A function $f: \mathbb{R}^n \to \mathbb{R}$ is said to be strongly convex if for every point $a \in E$ there is a positive number $s$ such that $x \mapsto f(x) - s \|x\|^2$ is convex in a neighborhood of $a$ (we use a Euclidean norm here).

A subset $A$ of finite-dimensional vector space $E$ is said to be strongly convex if for every point $a \in E$ there is strongly convex function $f$ so that $A$ agrees with the set $\{x \in \mathbb{R}^n; f(x_1, \ldots, x_{n-1}) < x_n\}$ near $a$ for some choice of coordinates in $E$. □

The function $\mathbb{R} \ni x \mapsto \sqrt{1+x^2}$ is strongly convex, but we see that the number $s$ cannot be chosen independently of $a$.

The function $\mathbb{R} \ni x \mapsto x^4$ is strictly convex but not strongly convex.

9.2.8 The convex envelope

Definition 9.2.15 Given a function $f: A \to \mathbb{R}$, where $A$ is a subset of a vector space $E$, the largest convex function $F: E \to \mathbb{R}$ such that $F|_A \leq f$ is called the convex envelope of $f$ and will be denoted by $\text{cvxe}(f)$. □

Remark 9.2.16 In general we have

$$\text{cvxe}(f)(x) = \inf_{t \in \mathbb{R}} \left[t; (x,t) \in \text{cvxh}(\text{epi}^{\text{finite}}(f))\right]. \quad (9.17)$$

The convex envelope of $f: E \to \mathbb{R}$ evaluated at a point $x$ is equal to the infimum of all expressions

$$\sum_{j=1}^{N} \lambda_j f(x + a^{(j)}),$$

where the $\lambda_j$ and the $a^{(j)}$ satisfy $\lambda_j \geq 0$, $\sum \lambda_j = 1$ and $\sum \lambda_j a^{(j)} = 0$. We note that the points $(x + a^{(j)}, f(x + a^{(j)}))$ belong to $\text{epi}^{\text{finite}}(f)$, implying that the point

$$\left(\sum \lambda_j (x + a^{(j)}), \sum \lambda_j f(x + a^{(j)})\right)$$

belongs to $\text{cvxh}(\text{epi}^{\text{finite}}(f))$ and that therefore $(x, (\text{cvxe}(f))(x) + t)$ belongs to $\text{cvxh}(\text{epi}^{\text{finite}}(f))$ for every positive number $t$. □

Remark 9.2.17 We have inclusions

$$\text{epi}^{\text{finite}}(\text{cvxe}(f)) = \text{cvxh}(\text{epi}^{\text{finite}}(f)) \quad \subset \text{cvxh}(\text{epi}^{\text{finite}}(f)) \subset \text{epi}^{\text{finite}}(\text{cvxe}(f)). \quad (9.18)$$

The two inclusion relations here can be strict.

Definition 9.2.18 Let $A$ be any subset of a vector space $E$ and let $f: A \to \mathbb{R}$ be any function defined on $A$. We shall say that $f$ is convex extensible if it is the restriction to $A$ of a convex function defined on all of $E$, i.e., if there is a convex function $F: E \to \mathbb{R}$ such that $F|_A = f$. □
There may exist more than one convex extension of a given function \( f \). For example \( F(x) = |x| - 1 \) and \( F^+(x) = \max(|x| - 1, 0) \), \( x \in \mathbb{R} \), have the same restriction to \( A = \mathbb{Z} \setminus \{0\} \). If there exists a convex extension, then \( \text{cvxe}(f) \) is the largest one.

### 9.2.9 Normed spaces

To any vector space \( E \) over the field of real numbers, we associate its **algebraic dual** \( E^* \), the vector space of all linear forms on \( E \), i.e., the functions \( \xi : E \to \mathbb{R} \) satisfying \( \xi(x + ty) = \xi(x) + t\xi(y) \) for all \( x, y \in E \) and all \( t \in \mathbb{R} \).

We say that a function \( E \ni x \mapsto \|x\| \in \mathbb{R} \) is a **norm** if \( \|x\| \geq 0 \) with equality if and only if \( x = 0 \); \( \|x + y\| \leq \|x\| + \|y\| \) for all \( x \) and \( y \); and finally \( \|tx\| = t\|x\| \) for all \( x \in E \) and all positive numbers \( t \). This means that the function \( E^2 \ni (x, y) \mapsto \|x - y\| \) is a metric with the extra property of being positively homogeneous. The subadditivity and the homogeneity together imply that \( x \mapsto \|x\| \) is convex.

The space \( \mathbb{R}^n \) of all \( n \)-tuples can be normed by the \( p \) **norm** \( \| \cdot \|_p \), \( 1 \leq p \leq +\infty \), which is defined for \( 1 \leq p < +\infty \) by

\[
\|x\|_p = \left( \sum |x_j|^p \right)^{1/p}, \quad x = (x_1, \ldots, x_n) \in \mathbb{R}^n.
\]

When \( p = +\infty \) this has to be interpreted as a limit. More explicitly one defines

\[
\|x\|_\infty = \max_j |x_j|, \quad x \in \mathbb{R}^n.
\]

In addition to the algebraic dual \( E^* \), which is defined for any vector space, we consider for any normed vector space \( E \) also its **dual space**, denoted by \( E' \) and consisting of all continuous linear forms on \( E \). These are the linear mappings \( \xi : E \to \mathbb{R} \) such that \( |\xi(x)| \leq C\|x\| \) for some constant \( C \). On the dual we define the norm **dual** to \( \| \cdot \| \) by

\[
\|\xi\|' = \sup_{\|x\| \leq 1} |\xi(x)|, \quad \xi \in E'.
\]

(9.19)

It follows that \( |\xi(x)| \leq \|\xi\|' \cdot \|x\| \) for all \( x \in E \) and all \( \xi \in E' \).

When \( E = \mathbb{R}^n \), we may identify both \( E^* \) and \( E' \) with \( \mathbb{R}^n \), and the evaluation of \( \xi \) at the point \( x \), i.e., the number \( \xi(x) \), is then the **inner product**, defined by \( \xi \cdot x = \xi_1x_1 + \cdots + \xi_nx_n \). The Euclidean norm \( \| \cdot \|_2 \), defined by \( \|x\|_2^2 = x \cdot x \), is dual to itself:

\[
\|\xi\|_2 = \sup_{\|x\|_2 \leq 1} \xi(x) = \|\xi\|_2 = \sqrt{\sum \xi_j^2}.
\]

It is not difficult to prove that the norm dual to \( \| \cdot \|_1 \) is \( \| \cdot \|_\infty \) and vice versa. More generally, one can prove that the norm dual to \( \| \cdot \|_p \) is \( \| \cdot \|_q \), where \( q = p/(p - 1) \), \( 1 < p < +\infty \), with a natural interpretation also when \( p = 1, +\infty \). This statement follows from Hölder’s inequality and its converse.

A vector space provided with a Euclidean norm is called a **Euclidean space**.
9.2.10 Duality in convex analysis

By the term *duality* we aim at properties and results that involve a vector space and its dual. The most important examples are the support function and the Fenchel transformation, to be defined now.

9.2.10.1 The support function

**Definition 9.2.19** Given any subset $A$ of a vector space $E$ we define its support function $H_A$ by

$$H_A(\xi) = \sup_{x \in A} \xi(x), \quad \xi \in E^*.$$  

Here $E^*$ is the algebraic dual of $E$. □

**Example 9.2.20** If $A$ is a ball, $A = B_{\leq}(c, r)$ with $r \geq 0$, or $B_{<}(c, r)$ with $r > 0$, then $H_A(\xi) = \xi(c) + r\|\xi\|'$, $\xi \in E^*$, where $\|\cdot\|$ is an arbitrary norm in $E$ and $\|\cdot\|$ its dual norm, defined on $E^*$ in (9.19) above. □

9.2.10.2 The Fenchel transformation

**Definition 9.2.21** To any function $\varphi : E \to \mathbb{R}$, we define its Fenchel transform $\tilde{\varphi}$ by

$$\tilde{\varphi}(\xi) = \sup_{x \in E} (\xi(x) - \varphi(x)),$$

defined for $\xi \in E^*$, the algebraic dual of $E$. □

It follows that

$$\xi(x) \leq \varphi(x) + \tilde{\varphi}(\xi), \quad x \in E, \quad \xi \in E^*,$$

called *Fenchel’s inequality.*

It is evident that $\tilde{\varphi}$ is the smallest function $g$ such that $\xi(x) \leq \varphi(x) + g(\xi)$ holds.

For any family of functions $(\varphi_j)_{j \in J}$, $\varphi_j \in \mathcal{F}(E, \mathbb{R})$, we clearly have

$$\sup_{j \in J} \tilde{\varphi} = \left(\inf_{j \in J} \varphi\right)^\sim.$$  

(9.21)

We see that the support function of a set is the Fenchel transform of its indicator function: $H_A = \text{indf}_{A}$. □

**Example 9.2.22** If the graph of a function $\varphi$ is a paraboloid, $\varphi(x) = a + \beta \cdot x + \frac{1}{2}c \|x\|^2$, $x \in \mathbb{R}^n$, where $a \in \mathbb{R}$, $\beta \in \mathbb{R}^n$ and $c > 0$, then the same is true of the graph of its transform: $\tilde{\varphi}(\xi) = -a + \frac{1}{2}c^{-1}\|\xi - \beta\|^2$. □
We now ask what happens if we apply the transformation again. We let $\Xi$ be any nonempty subset of $E^*$, and define the Fenchel transform of any function $f$ defined on $\Xi$ by

$$\tilde{f}(x) = \sup_{\xi \in \Xi} (\xi(x) - f(\xi)), \quad x \in E.$$ 

Here we can take $\Xi = \{0\}$ as well as $\Xi = E^*$, in a normed space it is customary to take $\Xi = E'$, the dual of $E$.

The equivalence $\tilde{\varphi} \leq f$ if and only if $\tilde{f} \leq \varphi$, $\varphi \in \mathcal{F}(E, \mathbb{R})$, $f \in \mathcal{F}(\Xi, \mathbb{R})$, (9.22) follows easily from the definition.

We may form the second transform $\tilde{\tilde{\varphi}}$ of a function defined on $E$. The main result in the theory of the Fenchel transformation is the following.

**Theorem 9.2.23 (Fenchel’s theorem)** Let $\varphi$ be a function defined on a vector space $E$ and let $\Xi$ be any nonempty subset of its algebraic dual $E^*$. Then we always have $\tilde{\tilde{\varphi}} \leq \varphi$. Equality holds if and only if

- (A) $\varphi$ is convex;
- (B) $\varphi$ is lower semicontinuous for the weakest topology for which all linear forms in $\Xi$ are continuous; and
- (C) $\varphi$ does not take the value $-\infty$ unless it is identically equal to $-\infty$.

Property (B) here means that if $\varphi(a) > s$, then there are linear forms $\xi_1, \ldots, \xi_m$ in $\Xi$ and a number $\theta > 0$ such that $\varphi(x) > s$ when $|\xi_j(x - a)| \leq \theta$, $j = 1, \ldots, m$. We shall denote this topology by $\sigma(E, \Xi)$. (We get the chaotic topology when $\Xi = \{0\}$, implying that the only lower semicontinuous functions are the constants.)

In $\mathbb{R}^n$ we usually choose $\Xi = \mathbb{R}^n$; the semicontinuity is then semicontinuity with respect to the usual topology of $\mathbb{R}^n$.

Before we go on, let us consider other expressions of this semicontinuity. The topology $\sigma(E, \Xi)$ on $E$ gives rise to a topology in $E \times \mathbb{R}$, viz. the product topology, for which a basis for the neighborhoods of a point $(a, s) \in E \times \mathbb{R}$ are given by the sets

$$\{(x, t); |\xi_j(x - a)| < \theta, \quad j = 1, \ldots, N, \quad |t - s| < \theta\}, \quad \xi_j \in \Xi, \quad \theta > 0.$$ 

Let us call this topology $\tau(E \times \mathbb{R}, \Xi)$.

So a function $\varphi$ that satisfies the three conditions can be represented as the supremum of a family of affine functions $\varphi(x) = \sup_\xi (\xi(x) - \tilde{\varphi}(\xi))$. This can be most helpful in proving that certain functions are convex.

We accept Fenchel’s Theorem without proof here.
It is now clear that Fenchel’s inequality can be improved to
\[ \xi(x) \leq \tilde{\varphi}(x) + \tilde{\varphi}(\xi). \]

For Werner Fenchel’s pioneering work on duality in convexity theory, see (1949, 1952, 1953, 1983).

9.2.11 Introduction to complex convexity

Every vector space over the field of complex numbers is at the same time a real vector space, obtained by simply restricting the multiplication by complex scalars to real scalars. So everything what we have said here about convexity applies also to complex vector spaces. But the presence of complex numbers gives birth to new phenomena. Let us list here several variants of complex convexity, in increasing order of strength. Most of them will be considered in detail in later sections of this chapter, some of them in other chapters.

1. An open set \( \Omega \) in \( \mathbb{C}^n \) is \textit{pseudoconvex} if there is a continuous plurisubharmonic function defined in \( \Omega \) that tends to \( +\infty \) at the boundary of \( \Omega \). See, e.g., (Hörmander 1990: Theorem 2.6.7 and Definition 2.6.8.)

2. An open set \( \Omega \) in \( \mathbb{C}^n \) is a \textit{domain of holomorphy} if there exists a holomorphic function defined in \( \Omega \) that cannot be continued, in a precise sense, over the boundary of \( \Omega \). See, e.g., (Hörmander 1990: Definition 2.5.1).

3. A connected open set \( \Omega \) in \( \mathbb{C}^n \) is called \textit{hyperconvex} if there is a continuous negative plurisubharmonic function \( u \) defined in \( \Omega \) such that the sublevel set \( \{ z \in \Omega : u(z) \leq c \} \) is compact for every negative number \( c \). See, e.g., (Kerzman & Rosay 1981).

There are several different notions of convexity related to lineal convexity. In increasing order of strength we have:

1. \textit{Local weak lineal convexity in the sense of Yužakov & Krivokolesko}; see Definition 9.5.8 on page 303;
2. \textit{Local weak lineal convexity}; see Definition 9.5.7 on page 302;
3. \textit{Weak lineal convexity}, originally introduced as \textit{Planarkonvexität} by Behnke & Peschl (1935:158, 162); see Definition 9.5.3 on page 301;
4. \textit{Lineal convexity}, introduced as \textit{convexité linéelle} by André Martineau (1966: 73; 1977:228); see Definition 9.5.1 on page 300;

Hörmander (1994: Definition 4.6.6) defines an open subset of \( \mathbb{C}^n \) to be \textit{C convex} if \( \Omega \cap L \) is a connected and simply connected subset of \( L \) for every affine complex line \( L \).
Andersson, Passare & Sigurdsson (2004: Definition 2.2.1) first defines a subset $E$ of $\mathbb{P}$ to be $\mathbb{C}$-convex if $E \neq \mathbb{P}$ and both $E$ and its complement $\mathbb{P} \setminus E$ are connected. A subset $E$ of $\mathbb{P}^n$ is called $\mathbb{C}$-convex if all its intersections with complex lines are $\mathbb{C}$-convex.

6. Then we have the usual convex sets already discussed; see Definition 9.2.6) on page 256.

7. The strict convex sets; see Definition 9.2.13 on page 258.

8. The strong convex sets; see Definition 9.2.14 on page 259.

9.2.12 Notes on the history of the concepts discussed in this chapter

I learned about lineal convexity from André Martineau during the academic year 1967–1968 when I was in Nice with him. His premature death on 1972 May 04 was a great loss to world mathematics. He introduced also the notion of strong lineal convexity (1968), which, however, was not geometrically defined. Later Znamenskij (1979) found a geometric characterization; the property is now called $\mathbb{C}$-convexity. Nowadays the most important sources for $\mathbb{C}$-convexity are the book by Hörmander (1994) and the survey by Andersson, Passare & and Sigurdsson (2004). My earlier contributions to the field are to be found in (1978, 1996, 1997, 2016, 2019).

9.2.13 A note on terminology

Heinrich Behnke and Ernst Peschl (1935) introduced the notion which is now known as weak lineal convexity. They called it *Planarkonvexität*.

André Martineau used the terms *convexité linéelle* and *linéellement convexe*—see Martineau (1966:73) and (1968:427), reprinted in (Œuvres de André Martineau 1977:228) and (1977:323), respectively. In French there are two adjectives, *linéaire*, corresponding to the English *linear*; and *linéel*, which I rendered as *lineal*. (There is also an adjective *linéal*.) Martineau obviously wanted a distinctive term in order to signal the special meaning of his convexity, not to be misunderstood as ordinary convexity. Diederich & Fornæss (2003) and Diederich & Fischer (2006) write “lineally convex.”

In Russian, the adjective *линейный* is most often used for both French terms *linéel* and *linéaire*, and this is the term used by Alizenberg, Krivokolesko, Yužakov, and others who write in Russian. In the translations into English of these Russian texts, there appears most often *linear convexity* and *linearly convex*.


Later Znamenskij (2001) used *линейчатый* (usually translated as ‘ruled’; a common term is *линейчатая поверхность* ‘ruled surface’). He thus
established the distinction between lineal, linéel and linear, linéaire in Russian (Yuri˘ı Zelinskij, personal communication 2013 March 26).

Hörmander (1994:290, Definition 4.6.1), Andersson, Passare & Sigurdsson (2004:16, Definition 2.1.2), and Jacquet (2008:8, Definition 2.1.2) used linear and linearly and thus did not keep the distinction introduced by Martineau. In my opinion, these authors unnecessarily copied the usage in the translations from Russian and did not pay attention to the pioneering work of Martineau. It should also be noted that the English lineal is actually the older of the two words, being attested since the fourteenth century, while linear is attested from 1706 (Webster 1983).

Another term is hypoconvex. The first appearance in this context that I have found is in Helton & Marshall (1990:182), where it is used for sets with a boundary of class $C^2$ and has the meaning of ‘strongly lineally convex’ (satisfying the strict Behnke–Peschl differential condition as I called it in my paper (1998:3); later it was weakened to a synonym of lineally convex by Whittelsey (2000:678), and used in this sense by Agler & Young (2004:379). The term helpfully reminds us that it signifies a property weaker than convexity.

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2Norberto Salinas (1976:144, 1979:327) used the term hypoconvex in a different sense.
9.3 Introduction to Mathematical Morphology

9.3.1 Introduction to this section

Lattice theory is a mature mathematical theory thanks to the pioneering work by Garrett Birkhoff (1911–1996), Øystein Ore (1899–1968), and others in the first half of the twentieth century. A standard reference is still Birkhoff’s book (1995), first published in 1940.

Mathematical morphology is a branch of science that was created in the 1960s by Georges Matheron (1930–2000) and Jean Serra. It thrives in complete lattices.

Mathematical morphology can be described as lattice theory applied to several branches of science, in particular, to image analysis and image processing, where many concepts and procedures can be successfully described with concepts from mathematical morphology. In this section we give the most basic definitions and the simplest properties—those that will be useful in the coming parts of the chapter. For more complete treatments, see the books by Matheron and Serra mentioned in the list of references as well as my papers (2007, 2010) and my book manuscript (MS 2021).

With the arrival of tropical geometry, lattice theory can (now) be viewed as a tropicalization of other mathematical theories. Developments originate in several branches of mathematics, for instance algebra (Blyth & Janowitz 1972, Blyth 2005), logic (Stoltenberg-Hansen et al. 1994, Gierz et al. 2003), general topology and functional analysis (Gierz et al. 2003:xxx–xxxii), convexity theory (Singer 1997), and, for mathematical morphology with applications in image processing, books by Matheron (1975), Serra (1982), Serra, Ed. (1988), and Heijmans (1994); articles by Heijmans & Ronse (1990), Ronse (1990), Ronse & Heijmans (1991, 1998), Heijmans (1995), Serra (2006), and Ronse & Serra (2008, 2010). Other areas where concepts from lattice theory are used include semantics (abstract interpretation) of programming, the theory of fuzzy sets, fuzzy logic, and formal concept analysis (Ganter & Wille 1999).

For general lattice theory a standard reference is Grätzer (1998).

This variety of sources for fundamental concepts has led to varying terminology and hence to difficulties in tracing history.

The two concepts of lattice and complete lattice must be carefully distinguished. This becomes obvious when we see that a complete lattice \( L \) can contain another complete lattice \( M \) with \( M \) as a sublattice of \( L \) ... but \( M \) is not a sub-complete-lattice of \( L \) (see Example 9.3.15).

A useful tool in the sequel will be generalized inverses and generalized quotients. They come in two versions, lower and upper.

To define an inverse of a general mapping seems to be a hopeless task. However, if the mapping is between preordered sets, there is some hope of constructing mappings that can serve in certain contexts just like inverses do.
There is an analogy between lattice theory and the theory of vector spaces. The theory of topological vector spaces was developed to a large extent because of the theory of distributions, which in turn was motivated by applications in partial differential equations. Developments in image processing motivated a renewed interest in lattice theory, in particular in complete lattices. Lattice theory was applied to switching circuits, and it was then enough, because of general finiteness conditions, to form models using lattices, but in image processing it is more convenient to assume completeness; for a motivation, see (Ronse 1990).

While vector spaces are useful in modelling linear problems, lattices seem to be more adapted to nonlinear problems. Auditory phenomena are often additive: all the instruments of an orchestra can be heard, while with visual phenomena this is not so: one object can block another from our view. This indicates that linear models may suffice for the first kind of phenomena (Fourier analysis and synthesis are successful for sound waves), while the visual ones are more in agreement with nonlinear operators, like maximum and minimum.

There are also analogies between topological spaces and preordered sets, in particular lattices. The continuous linear mappings in the first case correspond to increasing mappings in the second.

A comparison of the equations $a + x = b$ and $a \lor x = b$ shows that the second is more complicated than the first: The first has the unique solution $x = b - a$ for $a, b \in \mathbb{R}$, while the second has no solution if $a, b \in \mathbb{R}$ with $a > b$; a unique solution $x = b$ if $a < b$ and infinitely many solutions $x \leq a$ if $a = b$. In our schools we tend to prefer problems with a unique solution but in real life problems are more like $a \lor x = b$.

### 9.3.2 Preorders and orders

For the morphological operations on the computer screen, we need to consider families of sets. The family of all subsets of a given set is ordered by the inclusion relation, which is an example of an order relation. It is therefore convenient to introduce concepts that will be useful in the general theory of order relations. In this subsection we shall do so.

**Definition 9.3.1** A **preorder** in a given set $X$ is a relation (a subset of $X \times X = X^2$) which is reflexive and transitive. A **preordered set** is a set together with a preorder.

An **order** is a preorder which is antisymmetric. An **ordered set** is a set together with an order.

Two elements $x$ and $y$ are said to be **comparable** if either $x \leq y$ or $y \leq x$.

An order is said to be **total** if any two elements are comparable.

The definition of preorder means, if we denote the relation by $\leq$, that

$$x \leq x, \quad x \in X; \text{ and that}$$

(9.23)
\[ x \leq y \text{ and } y \leq z \text{ implies } x \leq z, \quad x, y, z \in X. \] (9.24)

The definition of order means that the relation shall in addition satisfy
\[ x \leq y \&\ y \leq x \Rightarrow x = y, \quad x, y \in X. \] (9.25)

We shall write \( x < y \) if \( x \leq y \) and \( x \neq y \). We shall also write \( x \geq y \) and \( x > y \) for \( y \leq x \) and \( y < x \), respectively.

As already mentioned, a basic example of an ordered set is the power set of a set \( W \) with the order relation given by inclusion, thus \( A \leq B \) being defined as \( A \subset B \) for \( A, B \in \mathcal{P}(W) \).

Suppose that we have two preorders defined in a set \( X \); denote them by \( \leq \) and \( \preceq \). The preorder \( \leq \) is said to be finer than the preorder \( \preceq \), and \( \preceq \) is said to be coarser than \( \leq \), if \( x \leq y \) implies \( x \preceq y \) for all \( x, y \).

There is a finest preorder in a set, viz. when we define \( x \leq y \) to mean that \( x = y \). This preorder is an order; let us call it the discrete order. There is also a coarsest preorder in any set \( X \), when we declare that \( x \leq y \) for all \( x, y \in X \). Let us call this the chaotic preorder. The set of all preorders on any set is thus an ordered set with a largest and a smallest element.

In a preordered set \( L \), given \( a, b \in L \), the interval \([a, b]\) is the set
\[ \{ x \in L; a \leq x \leq b \}, \]
in particular to be used when \( L = \mathbb{R} \). We shall write \([a, b]_\mathbb{Z}\) for an interval of integers, thus
\[ [a, b]_\mathbb{Z} = [a, b] \cap \mathbb{Z} = \{ x \in \mathbb{Z}; a \leq x \leq b \}, \quad a, b \in \mathbb{Z}. \] (9.26)

**Definition 9.3.2** An equivalence relation is a preorder which is symmetric, i.e., such that \( x \preceq y \) if and only if \( y \preceq x \). \( \square \)

This means that \( x \leq y \) implies \( y \leq x \) for all \( x, y \in X \).

### 9.3.3 Mappings between preordered sets

In preordered spaces the increasing mappings are of importance:

**Definition 9.3.3** If \( f: X \to Y \) is a mapping from a preordered set \( X \) to a preordered set \( Y \), then we say that \( f \) is increasing if

for all \( x, x' \in X \), the relation \( x \leq_X x' \) implies \( f(x) \leq_Y f(x') \).

We shall write \( \text{incr}(X, Y) \) for the set of all increasing mappings \( X \to Y \).

We shall say that \( f \) is decreasing if

the relation \( x \leq_X x' \) implies \( f(x) \geq_Y f(x') \)

for all elements \( x, x' \in X \). \( \square \)
The increasing mappings play the same role in the context of ordered sets as the linear mappings in the theory of vector spaces and as the continuous mappings in the theory of topological spaces.

A preorder $\leq$ is finer than another preorder $\preceq$ if and only if the identity mapping $(X, \leq) \to (X, \preceq)$ is increasing.

If $f, g: X \to Y$ are increasing, then so are $f \wedge g$ and $f \vee g$. If also $h: Y \to Z$ is increasing, then so is $h \circ f: X \to Z$. In particular, when $X = Y$, we have the three operations $(f, g) \mapsto f \wedge g, f \vee g, g \circ f$, which all preserve the property of being increasing.

A comparison with topology is in order here. If $f: X \to Y$ is a mapping of a topological space $X$ into a topological space $Y$ with topologies (families of open sets) $\tau_X$ and $\tau_Y$, we can define a new topology $\tau_f$ in $X$ as the family of all sets

$$\{x \in X; f(x) \in V\}, \quad V \in \tau_Y.$$

Then $f$ is continuous if and only if $\tau_X$ is finer than $\tau_f$.

For mappings $f: X \to X$ with target set equal to the domain we can form the iterations $f \circ f, f \circ f \circ f$ and so on, and among these mappings those that satisfy $f \circ f = f$ are of interest:

**Definition 9.3.4** We shall say that a mapping $f: X \to X$ is *idempotent* if $f \circ f = f$, i.e., if $f(f(x)) = f(x)$ for all $x \in X$. A mapping which is both increasing and idempotent will be called an *ethmomorphism*.

**Definition 9.3.5** We shall say that a mapping $f: X \to X$ is *extensive* if it is larger than the identity, i.e., $f(x) \geq x$ for all elements $x \in X$. We shall say that it is *antiextensive* if it is smaller than the identity, i.e., $f(x) \leq x$ for all $x \in X$.

We define the *invariance set of a function* $f: X \to X$ as the set of all $x \in X$ such that $f(x) = x$. We denote it by $\text{invar}(f)$. For extensive mappings $f$ the invariance set is decreasing in $f$, while it is increasing in $f$ for antiextensive mappings.

### 9.3.4 Cleistomorphisms and anoiktomorphisms

**Definition 9.3.6** A *cleistomorphism* in an ordered set $X$ is an ethmomorphism (see Definition 9.3.4) $\kappa: X \to X$ which is extensive (see Definition 9.3.5); in other words, which satisfies the following three conditions.

$$x \leq x' \text{ implies } \kappa(x) \leq \kappa(x'); \quad (9.27)$$

$$\kappa(\kappa(x)) = \kappa(x); \quad (9.28)$$

$$x \leq \kappa(x), \quad x \in X. \quad (9.29)$$

for all elements $x, x' \in X$. □
The element \( \kappa(x) \) is said to be the **closure** of \( x \). Elements \( x \) such that \( \kappa(x) = x \) are called **invariant** or **closed** (for this operator). An element is closed if and only if it is the closure of some element (and then it is the closure of itself).

In many applications the set \( X \) is the power set \( \mathcal{P}(W) \) of some set \( W \). Then the cleistomorphism is given as an intersection:

\[
\overline{A} = \bigcap_{Y} (Y; Y \text{ is closed and } Y \supset A).
\]

When \( W \) is a topological space, a basic example is the topological closure operator which associates to a set in a topological space its topological closure, i.e., the smallest closed set containing the given set, denoted by \( A \mapsto \overline{A} \). In fact a cleistomorphism in \( \mathcal{P}(W) \) defines a topology in \( W \) if and only if it satisfies, in addition to (9.27), (9.28), (9.29) above, two extra conditions, viz.

\[
\emptyset = \emptyset \text{ and } A \cup B = \overline{A} \cup \overline{B} \text{ for all } A, B \subset W,
\]

(9.30)

where \( \emptyset \) denotes the empty set.

Another cleistomorphism of great importance is the operator which associates to a set \( A \) in \( \mathbb{R}^n \) its convex hull, the smallest convex set containing the given set, denoted by \( \text{cvxh}(A) \). The composition \( A \mapsto \text{cvxh}(A) \) is a cleistomorphism, whereas the composition in the other order, \( A \mapsto \text{cvxh}(\overline{A}) \) is not idempotent if \( n \geq 2 \). We see that the composition of two cleistomorphisms is sometimes, but not always, a cleistomorphism.

Dual to the concept of cleistomorphism is the concept of anoiktomorphism.

**Definition 9.3.7** An ethnomorphism \( \alpha : X \to X \) is said to be an **anoiktomorphism** if it is antieextensive; in other words, if it satisfies the following three conditions.

\[
x \leq x' \text{ implies } \alpha(x) \leq \alpha(x');
\]

\[
\alpha \circ \alpha = \alpha;
\]

\[
\alpha(x) \leq x,
\]

for all elements \( x, x' \in X \). \( \square \)

The composition of a cleistomorphism and an anoiktomorphism is always idempotent:

**Proposition 9.3.8** Let \( \alpha, \kappa : L \to L \) be an anoiktomorphism and a cleistomorphism. Then \( \eta = \alpha \circ \kappa \) and \( \theta = \kappa \circ \alpha \) are ethnomorphisms.

**Proof** That \( \eta \) and \( \theta \) are increasing is obvious. Since \( \kappa \) is extensive, we get

\[
\eta \circ \eta = \alpha \circ \kappa \circ \alpha \circ \kappa \geq \alpha \circ \alpha \circ \kappa = \alpha \circ \kappa = \eta.
\]

Since \( \alpha \) is antiextensive, we get

\[
\eta \circ \eta = \alpha \circ \kappa \circ \alpha \circ \kappa \leq \alpha \circ \kappa \circ \kappa = \alpha \circ \kappa = \eta,
\]

so \( \eta \) is idempotent. The proof for \( \theta \) is similar.
Example 9.3.9 A typical example is when we take $\alpha$ and $\kappa$ as the operations of taking the interior and the topological closure in a topological space, respectively; $\alpha(A) = A^\circ$, $\kappa(A) = \overline{A}$. Then a fixed point of the composition $\alpha \circ \kappa$ is called a regular open set and a fixed point of $\kappa \circ \alpha$ is called a regular closed set. These operations are neither extensive nor antiextensive in general. □

Proposition 9.3.10 The infimum of a family of cleistomorphisms in a complete lattice $L$ is a cleistomorphism. The supremum of any family of anoiktomorphisms in $L$ is an anoiktomorphism.

Proof Let $\kappa_j, j \in J$, be cleistomorphisms and define $\kappa = \bigwedge_{j \in J} \kappa_j$, meaning that the value of $\kappa$ at $x \in L$ equals $\bigwedge_{j \in J} \kappa_j(x)$. Clearly $\kappa$ is increasing and larger than the identity. It follows that $\kappa \circ \kappa \geq \kappa$. To prove the opposite inequality, we note that $\kappa \circ \kappa \leq \kappa_j \circ \kappa_j = \kappa_j$. Taking the infimum over all $j$ we get what we want.

The result for anoiktomorphisms follows by duality.

9.3.5 Lattices and complete lattices

Let $L$ be an ordered set and $A$ a subset of $L$. An element $b \in L$ is said to be the infimum of all elements $a \in A$ if $b$ is the largest minorant of all $a \in A$. This means that $b \leq a$ for all elements $a \in A$, and that if $b' \leq a$ for all $a \in A$, then $b' \leq b$. The infimum, if it exists, is necessarily unique. The infimum of the empty set exists if and only if $L$ possesses a largest element, and if so, the infimum is this largest element.

We shall write

$$b = \inf_{a \in A} a = \inf(a; a \in A) = \bigwedge_{a \in A} a$$

for the infimum of all elements in $A$; if $A$ has only $n$ elements we write $b = a_1 \land \cdots \land a_n$. If the infimum belongs to $A$, we call it a minimum. As an example, the set of all positive real numbers has 0 as its infimum, but 0 is not a positive number and therefore not a minimum.

Similarly we define the supremum

$$c = \sup_{a \in A} a = \sup(a; a \in A) = \bigvee_{a \in A} a$$

as the smallest majorant of all elements in $A$. If the supremum belongs to $A$, we call it a maximum. The supremum of all elements in the empty set, $\sup_{x \in \emptyset} x$, exists if and only if $L$ has a smallest element.

Definition 9.3.11 Let $L$ be a nonempty set. If any subset consisting of two elements in $L$ has an infimum, we shall call $L$ an inf-semilattice; similarly, if any two-set of $L$ has a supremum, we shall call $L$ a sup-semilattice. If $L$ is both an inf-semilattice and a sup-semilattice we shall call $L$ a lattice. □
Definition 9.3.12 If any nonempty subset (finite or infinite) of a nonempty set $L$ has an infimum, $L$ will be said to be a complete inf-semilattice; analogously we define complete sup-semilattice and complete lattice. □

We may denote the smallest element in a complete lattice by $0$ and the largest by $1$. We have

$$\sup x = \inf x = 1; \quad \inf x = \sup x = 0;$$

the infimum of the empty set exists and is $1$, and the supremum of the empty set is $0$.

A sublattice is defined just like a subgroup with respect to the operations $\land$ and $\lor$: that $M$ is a sublattice of $L$ means that for all $x, y \in M$, $x \land y$ and $x \lor y$, when calculated in $L$, are elements of $M$. A sublattice is therefore something more than a subset with the induced order; see the following examples.

Example 9.3.13 The space of real-valued continuous functions on a topological space is a lattice with the usual order: $f \leq g$ if and only if $f(x) \leq g(x)$ for all $x$. The space $C^1(R^n, R)$ of real-valued continuously differentiable functions on $R^n$ is not a sublattice of $C(R^n, R)$ if $n \geq 1$. It is not even a lattice on its own. (The functions $R \ni x \mapsto \sqrt{t^2 + x^2}, t > 0$, converge to $x \mapsto |x|$ as $t \to 0$, but there is no infimum in $C^1(R, R)$.) □

Example 9.3.14 The family $\mathcal{P}(W)$ of all subsets of a set $W$ is a complete lattice, with $\bigwedge A_j = \bigcap A_j$ and $\bigvee A_j = \bigcup A_j$. The compact sets in $R^n$ form a sublattice $\mathcal{K}(R^n)$ of $\mathcal{P}(R^n)$. This lattice is a complete inf-semilattice but not a complete sup-semilattice. The family $\mathcal{K}_{cvx}(R^n)$, $n \geq 1$, of all convex compact sets is a lattice but not a sublattice of $\mathcal{K}(R^n)$: the supremum of two convex compact sets is not always the same in the two lattices. □

Example 9.3.15 The family of all closed sets in $R^n$, denoted by $\mathcal{C}(R^n)$, is a sublattice of $\mathcal{P}(R^n)$: the union and intersection of two closed sets are closed. But, although $\mathcal{C}(R^n)$ is a complete lattice, it is not a sub-complete-lattice of the complete lattice $\mathcal{P}(R^n)$ when $n \geq 1$. The union of a family of closed sets is not always closed, but there is a supremum, viz. the closure of the union. Thus, finite suprema agree with those in $\mathcal{P}(R^n)$ while infinite suprema do not. □

Example 9.3.16 The set $\mathcal{F}(R^n, R_1)$ of all functions defined on $R^n$ and with values in the extended real line $R_1$ is a lattice under the usual order for real numbers, extended in an obvious way to the two infinities. The subset of all convex functions is ordered in the same way and is also a lattice under this order. However, the convex functions $CVX(R^n, R_1)$ do not form a sublattice of $\mathcal{F}(R^n, R_1)$ if $n \geq 1$. The supremum of two convex functions is equal to the pointwise supremum of them:

$$(f \lor g)(x) = f(x) \lor g(x) = \max(f(x), g(x)),$$
but the infima are different in the two lattices: the infimum in the lattice of convex functions is
\[ f \land_{cvx} g = \sup \{ h \in CVX(\mathbb{R}^n, \mathbb{R}_+); h \leq f, g \leq f \land g = \min(f, g) \}, \]
where the supremum of all \( h \leq f, g \) is calculated in \( \mathcal{P}(\mathbb{R}^n, \mathbb{R}_+) \) and has a sense because that lattice is complete. That the two infima may be different is shown by easy examples like \( f(x) = e^x, g(x) = e^{-x}, x \in \mathbb{R} \). Here \( \min(f, g)(x) = e^{-|x|} \), while \( f \land_{cvx} g = 0 \).

9.3.5.1 Dilations and erosions in complete lattices
Mappings of the form \( P(\mathbb{R}^n) \ni A \mapsto A + B \in P(\mathbb{R}^n) \) with a fixed set \( B \) are called dilations. It can be proved that a mapping \( P(\mathbb{R}^n) \to P(\mathbb{R}^n) \) which commutes with translations and the formation of infinite unions is necessarily of this form. In lattice theory it is therefore natural to take the latter property as a definition:

**Definition 9.3.17** We say that a mapping \( \delta: L \to M \), where \( L \) and \( M \) are complete lattices, is a **dilation** if it commutes with the formation of suprema, i.e.,
\[ \delta(\bigvee_{x \in A} x) = \bigvee_{x \in A} \delta(x) \]
for all subsets \( A \) of \( L \).

In particular we get \( \delta(0_L) = 0_M \) (take \( A \) empty), while
\[ \delta(1_L) = \bigvee_{x \in L} \delta(x) \leq 1_M. \] (9.31)

**Definition 9.3.18** Similarly we shall say that \( \varepsilon \) is an **erosion** if it commutes with the formation of infinite infima,
\[ \varepsilon(\bigwedge_{x \in A} x) = \bigwedge_{x \in A} \varepsilon(x) \]
for all subsets \( A \) of \( L \).

We note that \( \varepsilon(1_L) = 1_M \) (take \( A \) empty), while \( \varepsilon(0_L) = \bigwedge_{x \in L} \varepsilon(x) \geq 0_M \).

Dilations and erosions are always increasing. Indeed, we have \( \delta(x \lor y) = \delta(x) \lor \delta(y) \). If \( x \leq_L y \), this equation simplifies to \( \delta(y) = \delta(x) \lor \delta(y) \geq_M \delta(x) \), which shows that \( \delta \) is increasing, \( \delta \in \text{incr}(L, M) \). A similar argument shows that erosions are increasing.

In \( \mathbb{R}^n \) or \( \mathbb{C}^n \) we define \( \delta_B, \varepsilon_B: \mathcal{P}(G) \to \mathcal{P}(G) \) by \( \delta_B(A) = A + B \) and \( \varepsilon_B(A) = \{ x \in G; B \subseteq A \} \). It is easily seen that \( \delta_B(A) \subseteq C \) if and only if \( A \subseteq \varepsilon_B(C) \). In a lattice this may be written as \( \delta(x) \leq y \) if and only if \( x \leq \varepsilon(y) \),
equivalently as $\text{epi} \delta = (\text{hypo} \varepsilon)$, where the symbol\` means that we swap the components: for a subset $A$ of a Cartesian product $X \times Y$ we define

$$A = \{(y, x); (x, y) \in A\} \subset Y \times X.$$  

(9.32)

May we use this as a model to define erosions from dilations and conversely in the more general lattice situation? Indeed this is the case, and we shall do so in the next subsection.

**Example 9.3.19** The mapping $f^*: \mathcal{P}(Y) \to \mathcal{P}(X)$, defined by (9.14), is both a dilation and an erosion, while $f_*: \mathcal{P}(X) \to \mathcal{P}(Y)$, defined in (9.15), is a dilation but in general not an erosion.

\[ \square \]

### 9.3.6 Inverses and quotients of mappings

#### 9.3.6.1 Introduction

In this subsection we shall study inverses and quotients of mappings between ordered sets which are analogous to inverses $1/y$ and quotients $x/y$ of positive numbers. The theory of lower and upper inverses defined in Subsubsection 9.3.6.2 generalizes the theory of Galois connections as well as residuation theory and the theory of adjunctions. An interesting question is to what extent a generalized inverse can serve as a left inverse, as a right inverse, and how an inverse of an inverse relates to the identity mapping. These inverses and quotients can be used to create a convenient formalism for a unified treatment of dilations $\delta: L \to M$ and erosions $\varepsilon: M \to L$ as well as of cleistomorphisms $\kappa = \varepsilon \circ \delta: L \to L$ and anoiktomorphisms $\alpha = \delta \circ \varepsilon: M \to M$.

In general a mapping $g: X \to Y$ between sets does not have an inverse. If $g$ is injective, we may define a left inverse $u: Y \to X$, thus with $u \circ g = \text{id}_X$, where $\text{id}_X$ denotes the identity mapping in $X$, defining $u(y)$ in an arbitrary way when $y$ is not in the image of $g$. If $g$ is surjective, we may define a right inverse $v: Y \to X$, thus with $g \circ v = \text{id}_Y$. We then need to define $v(y)$ as an element of the preimage $\{x; g(x) = y\}$. In the general situation this has to be done using the axiom of choice. In a complete lattice, however, it could be interesting to define $v(y)$ as the supremum or infimum of all points $x$ such that $g(x) = y$, even though this supremum or infimum need not belong to the set of points that are mapped to $y$. At any rate, the preimage of $y$ is contained in the interval defined by the infimum and the supremum. However, for various reasons it is convenient to take instead the infimum of all $x$ such that $g(x) \geq y$ or the supremum of all $x$ such that $g(x) \leq y$. This yields better monotonicity properties. (The case $g(x) = y$ is allowed, since we can take the preorder in $Y$ be the discrete order.)

If, given a mapping $g: L \to M$ from an ordered set $L$ into an ordered set $M$, we can find mappings $u, v: M \to L$ such that $\text{hypo} u = (\text{epi} g)^\circ$ or $\text{epi} v = (\text{hypo} g)^\circ$, we would be content to have some kinds of inverses to $g$. However, usually the best we can do is to study mappings satisfying either
hypo \cup (\text{epi}g) \text{ or } \text{epi}v \supset (\text{hypo}g)^{-}. This will be the approach in what follows, where we shall define not one but two inverses, viz. the lower (to be denoted by \(g\)\(^{-1}\)) and the upper (written as \(g\)\(^{-1}\)).

9.3.6.2 Defining inverses of mappings

**Definition 9.3.20** Let \(L\) be a complete lattice, \(M\) a preordered set, and \(g: L \to M\) any mapping. We then define the **lower inverse** \(g^{-1}: M \to L\) and the **upper inverse** \(g^{-1}: M \to L\) as the mappings

\[
g^{-1}(y) = \bigvee_{x \in L} (x; g(x) \leq_M y) = \bigvee_{x \in L} (x; (x, y) \in \text{epi}g); \quad (9.33)
\]

\[
g^{\dagger}(y) = \bigwedge_{x \in L} (x; g(x) \geq_M y) = \bigwedge_{x \in L} (x; (x, y) \in \text{hypo}g), \quad (9.34)
\]

where \(y \in M\). \(\square\)

As a first observation, let us note that these inverses are always increasing. If there exists a largest element \(1_M\), then \(g^{-1}(1_M) = 1_L\). Similarly, if \(M\) possesses a smallest element \(0_M\), then \(g^{-1}(0_M) = 0_L\). If \(M\) has the chaotic preorder, then both inverses are constant, \(g^{-1} = 1_L\) and \(g^{-1} = 0_L\) identically. Here the lower inverse is larger than the upper inverse.

**Example 9.3.21** For any mapping \(f: X \to Y\) we defined in (9.14) and (9.15) the mappings \(f^*: \mathcal{P}(Y) \to \mathcal{P}(X)\) and \(f_+: \mathcal{P}(X) \to \mathcal{P}(Y)\). We note that, for all \(A \in \mathcal{P}(X)\) and all \(B \in \mathcal{P}(Y)\) we have \(f_+(A) \subseteq B\) if and only if \(A \subseteq f^*(B)\). It follows that \((f^*)^{-1} = f_+\) and \((f_+)^{-1} = f^*\). From this we see that \(((f^*)^{-1})^{-1} = f^*\) and that \(((f_+)^{-1})^{-1} = f_+\). \(\square\)

9.3.6.3 Properties of inverses

We note that we always have

\[
\text{epi}g \subseteq (\text{hypo}g^{-1})^*, \quad (9.35)
\]

in other words, if \(y \geq g(x)\), then \(x \leq g^{-1}(y)\); and

\[
\text{hypo}g \subseteq (\text{epi}g^{-1})^*, \quad (9.36)
\]

in other words, if \(y \leq g(x)\), then \(x \geq g^{-1}(y)\). Here \(R^*\) for a subset \(R\) of \(X \times Y\) is defined by (9.32). In general, these inclusions are strict.

**Example 9.3.22** If \(f: \mathcal{P}(G) \to \mathcal{P}(G)\) is the dilation \(f(A) = \delta_U(A) = A + U, A \in \mathcal{P}(G)\), where \(G\) is an abelian group and \(U\) a fixed subset of \(G\), called the **structuring element**, then

\[
(\delta_U)^{-1}(C) = \bigcup_{A \in \mathcal{P}(G)} (\delta_U(A) \subseteq C) = \varepsilon_U(C), \quad C \in \mathcal{P}(G),
\]

the erosion associated to \(\delta_U\). A most fundamental example.
The compositions $\alpha_U = \delta_U \circ \varepsilon_U$ and $\kappa_U = \varepsilon_U \circ \delta_U$ are the anoiktomorphism and the cleistomorphism associated to $\delta_U$, respectively.

When $X = Y = \mathbb{R}^n$, the upper inverse of $\delta_U$ is not interesting, since

$$\left(\delta_U \right)^{-1}(C) = \emptyset$$

for all $C$ if $U$ has interior points, and equal to $C$ if $U = \{0\}$. □

An ideal inverse $u$ would satisfy $u \circ g = \operatorname{id}_L$, $g \circ u = \operatorname{id}_M$, and the inverse of $u$ would be $g$. It is therefore natural to compare $g \circ \delta_U$ and $\delta_U \circ g$ with $\operatorname{id}_L$; and inverses of inverses of $g$ with $g$. We shall not do so here but refer the reader to the book manuscript (MS 2021).

9.3.6.4 Quotients of mappings

We shall now generalize the definitions of upper and lower inverses.

**Definition 9.3.23** Let a set $X$, a complete lattice $M$, and a preordered set $Y$, as well as two mappings $f: X \to M$ and $g: X \to Y$ be given. We define two mappings $f/\star g, f/\triangleright g: Y \to M$ by

$$
(f/\star g)(y) = \bigvee_{x \in X} (f(x); g(x) \leq_Y y), \quad y \in Y;
$$

$$
(f/\triangleright g)(y) = \bigwedge_{x \in X} (f(x); g(x) \geq_Y y), \quad y \in Y.
$$

We shall call them the **lower quotient** and the **upper quotient** of $f$ and $g$.

We shall often assume that $X$, $M$ and $Y$ are all complete lattices, but this is not necessary for the definitions to make sense.

The mappings $f/\star g, f/\triangleright g \in \mathcal{F}(Y, M)$ are always increasing.

The quotients $f/\star g$ and $f/\triangleright g$ increase when $f$ increases, and they decrease when $g$ increases—just as with the division of positive numbers:

If $f_1 \leq_M f_2$ and $g_1 \geq_Y g_2$, then $f_1/\star g_1 \leq_M f_2/\star g_2$ and $f_1/\triangleright g_1 \leq_M f_2/\triangleright g_2$.

If $g(x) \leq_Y y$, then $f(x) \leq_M (f/\star g)(y)$; if $g(x) \geq_Y y$, then $f(x) \geq_M (f/\triangleright g)(y)$. In particular,

if $g(x) = y$, then $(f/\triangleright g)(y) \leq_M f(x) \leq_M (f/\star g)(y)$.

We note some special cases.
(1) If we specialize the definitions to the situation when $X = M$ and $f = \text{id}_X$, then $f/\ast g = \text{id}_X/\ast g = g_{[-1]}$ and $f/\ast g = \text{id}_X/\ast g = g[-1]$; cf. Definition 9.3.20. So inverses are quotients.

(2) A second special case is this: Taking $Y = M$ and $g = f$ in the definition we see that, for all mappings $f : X \to M$ we have

$$f/\ast f \leq \text{id}_M \leq f/\ast f; \quad (9.37)$$

$$f/\ast f \circ f = f = (f/\ast f) \circ f. \quad (9.38)$$

### 9.3.7 Set-theoretical representation of dilations, erosions, cleistomorphisms, and anoiktomorphisms

We present here an easy result for translation-invariant operators on the family of subsets of an abelian group, putting several operations under a common roof.

**Proposition 9.3.24** Let $S$ be a subset of an abelian group $G$. Then the dilation, erosion, cleistomorphism and anoiktomorphism with structuring element $S$ can all be written in the form

$$\varphi_{S,T,U}(A) = \bigcup_{x \in G} (x + S; \ x + T \subset A + U), \quad A \in \mathcal{P}(G), \quad (9.39)$$

for special choices of the structuring elements $S$, $T$, $U$, viz.

$$\delta_S = \varphi_{S,\{0\},\{0\}}, \ \varepsilon_S = \varphi_{\{0\},S,\{0\}}, \ \alpha_S = \varphi_{S,S,\{0\}}, \ \kappa_S = \varphi_{\{0\},S,S}. \quad (9.40)$$

We can also write the mappings as

$$\varphi = (f/\ast g) \circ h,$$

where

$$f(B) = B \text{ or } B + S, \ g(B) = B + S, \ h(A) = A \text{ or } A + S,$$

for $A, B \in \mathcal{P}(G)$.

**Proof** The dilation $\delta = \delta_S$, the erosion $\varepsilon = \delta_{[-1]}$, the cleistomorphism $\kappa = \varepsilon \circ \delta$, and the anoiktomorphism $\alpha = \delta \circ \varepsilon$ can be written

$$\delta(A) = \bigcup_{B \in \mathcal{P}(G)} (B + S; \ B + S \subset A + S),$$

$$\varepsilon(A) = \bigcup_{B \in \mathcal{P}(G)} (B; \ B + S \subset A),$$

$$\kappa(A) = \bigcup_{B \in \mathcal{P}(G)} (B; \ B + S \subset A + S),$$

$$\alpha(A) = \bigcup_{B \in \mathcal{P}(G)} (B + S; \ B + S \subset A).$$
We now let

\[ f(B) = B + S, \quad h(A) = A + S \] in the first case;

\[ f(B) = B, \quad h(A) = A \] in the second case;

\[ f(B) = B, \quad h(A) = A + S \] in the third case; and

\[ f(B) = B + S, \quad h(A) = A \] in the fourth case;

while \( g(B) = B + S \) in all four cases.

We can think of the points as atoms and the sets \( x + S \) as molecules. Then \( \delta(A) \) and \( \alpha(A) \) consists of molecules, the latter of those that are contained in \( A \); whereas \( \varepsilon(A) \) and \( \kappa(A) \) consists of centers of molecules (which makes sense if the structuring set \( S \) is symmetric).
9.4 Lineally Convex Hartogs Domains

Abstract of this section

We study lineally convex domains of a special type, viz. Hartogs domains, and prove that such sets can be characterized by local conditions if they are smoothly bounded.

9.4.1 Introduction to the present section

Lineal convexity is a kind of complex convexity intermediate between usual convexity and pseudoconvexity. More precisely, if $A$ is a convex set which is either open or closed, then $A$ is lineally convex (this is true also in the real category), and if $\Omega$ is a lineally convex open set in $\mathbb{C}^n$, the space of $n$ complex variables, then $\Omega$ is pseudoconvex. Now pseudoconvexity is a local property in the sense that if any boundary point of an open set $\Omega$ has an open neighborhood $\omega$ such that $\Omega \cap \omega$ is pseudoconvex, then $\Omega$ is pseudoconvex; the analogous result holds for convexity. But it is well known that the property of lineal convexity is not a local property in this sense—for easy examples see Subsection 9.4.3. The purpose of this section is to investigate to what extent this is true for sets that are of a special form: the Hartogs domains.

Let us now give the main definition.

Definition 9.4.1 A set $A$ in $\mathbb{C}^n$ is said to be \textit{lineally concave} if it is a union of hyperplanes. It is called \textit{lineally convex} if its complement is lineally concave. $\square$

A lineally convex set whose boundary is sufficiently smooth satisfies a differential condition. Let $\rho$ be a defining function for $\Omega$ (see Definition 9.4.18), and let $H$ and $L$ denote, respectively, the Hessian and the Levi form at a boundary point $a$ of $\Omega$. Then the differential condition says that

$$|H(s)| \leq L(s) \text{ for all vectors } s \in T_{\mathbb{C}}(a),$$

(9.41)

where $T_{\mathbb{C}}(a)$ is the complex tangent space at the point $a$. See Subsection 9.4.5 for details. Every lineally convex domain of class $C^2$ satisfies the differential condition—for the converse, see Section 9.6. Here we shall prove that this is so in the special case of Hartogs domains, which we now proceed to define.

Definition 9.4.2 A \textit{Hartogs set} in $\mathbb{C}^n \times \mathbb{C}$ is a set which contains, along with a point $(z,t) \in \mathbb{C}^n \times \mathbb{C}$, also every point $(z,t')$ with $|t'| = |t|$. It is said to be a \textit{complete Hartogs set} if it contains, with $(z,t)$, also $(z,t')$ for all $t'$ with $|t'| \leq |t|$. $\square$
Here we shall study open and complete Hartogs sets; they are always defined by a strict inequality $|t| < R(z)$, thus
\[ \Omega = \{(z, t) \in \mathbb{C}^n \times \mathbb{C}; \ |t| < R(z)\}, \tag{9.42} \]
where $R$ is a function on $\mathbb{C}^n$ with values in $\mathbb{R}$.

Given $R$, we define a set $\omega$ in $\mathbb{C}^n$ by
\[ \omega = \{z \in \mathbb{C}^n; 0 < R(z) \leq +\infty\}. \]
We shall say that $\Omega$ is a Hartogs domain over $\omega$, or that $\omega$ is the base of $\Omega$, if (9.42) holds with $R(z)$ positive if and only if $z \in \omega$.

Most of our results will be concerned with the case $n = 1$, thus
\[ \Omega = \{(z, t) \in \mathbb{C} \times \mathbb{C}; \ |t| < R(z)\} = \{(z, t) \in \omega \times \mathbb{C}; \ |t| < R(z)\}. \tag{9.43} \]

The main result here is the following theorem.

**Theorem 9.4.3** Let $\Omega$ be a bounded complete Hartogs domain in $\mathbb{C}^2$ with boundary of class $C^2$. If $\Omega$ satisfies the differential condition (9.41) at all boundary points, then $\Omega$ is lineally convex.

Thus for complete Hartogs domains, the property of being lineally convex is a local property. Next we consider sets with $R$ of class $C^2$ in $\omega$ but which do not necessarily have a smooth boundary at points $(z, t)$ with $z \in \partial \omega$.

In this case we prove:

**Theorem 9.4.4** Let $\omega$ be a bounded open set in the complex plane $\mathbb{C}$. If the closure of $\omega$ is not a disk, then lineal convexity over $\omega$ is not a local condition: we can find a Hartogs domain $\Omega$ over $\omega$ and two open sets $\omega_0$ and $\omega_1$ such that the Hartogs domains $\Omega_j$ over $\omega_j$ are lineally convex, $j = 0, 1$, but their union $\Omega = \Omega_0 \cup \Omega_1$ is not. If on the other hand $\omega$ is a disk, and $\Omega$ is a Hartogs domain satisfying the differential condition (9.41) at all boundary points over $\omega$, then $\Omega$ is lineally convex.

**Corollary 9.4.5** Let $\omega$ be an open set in $\mathbb{C}$ which is equal to the interior of its closure, and let $\Omega$ be a Hartogs domain over $\omega$. Then the differential condition (9.41) imposed on all boundary points over $\omega$ is equivalent to lineal convexity if and only if $\omega$ is a disk.

### 9.4.2 Weak lineal convexity

There are several other notions related to lineal convexity:

**Definition 9.4.6** An open connected set is called weakly lineally convex if through any boundary point there passes a complex hyperplane which does not intersect the set. An open set is said to be locally weakly lineally convex if through every boundary point $a \in \partial \Omega$ there is a complex hyperplane $Y$ passing through $a$ such that $a$ does not belong to the closure of $Y \cap \Omega$. \(\square\)

\(\text{\textsuperscript{3}}\)There are actually two versions of this concept: see Definitions 9.5.7 and 9.5.8.
It is not difficult to prove that local weak lineal convexity implies pseudoconvexity.

For complete Hartogs sets it is very easy to see that weak lineal convexity implies lineal convexity:

**Lemma 9.4.7** A complete Hartogs domain which is weakly lineally convex and has a lineally convex base is lineally convex.

**Proof** Let \((z^0, t^0) \in \mathbb{C}^n \times \mathbb{C}\) be an arbitrary point in the complement of \(\Omega\), a Hartogs domain defined by (9.42). If \(R(z^0) > 0\), then the point \((z^0, R(z^0)t^0 / |t^0|)\) belongs to \(\partial \Omega\), and if \(\Omega\) is weakly lineally convex, there is a hyperplane passing through that point which does not cut \(\Omega\). Then the parallel plane through \((z^0, t^0)\) does not cut \(\Omega\) either. If \(R(z^0) \leq 0\), then \(z^0\) does not belong to the base, and a hyperplane with equation \(\zeta \cdot z = \zeta \cdot z^0\) will do, since the base is lineally convex. This proves the lemma.

### 9.4.3 The non-local character of lineal convexity

The domain
\[
V = \{ (z,t) \in \mathbb{C}^2; \quad |t| < |z| \}
\]  
(9.44)
is easily seen to be lineally convex. Indeed, if \((z_0, t_0) \notin V\) with \(t_0 \neq 0\), then the complex line \(\{ (z,t); \quad z_0t = t_0z \}\) passes through \((z_0, t_0)\) and does not cut \(V\); if on the other hand \(t_0 = 0\), we can for instance take the line \(\{0\} \times \mathbb{C}\). A simple example of a domain that is locally lineally convex but not lineally convex can be built up from this set as follows.

**Example 9.4.8** *(Kiselman 1996, Example 2.1.)*

![FIGURE 9.1](image-url)

**FIGURE 9.1**

An open connected Hartogs set in \(\mathbb{C}^2\) which is locally weakly lineally convex but not weakly lineally convex. Coordinates \((z, t) \in \mathbb{C}^2; (x, y, |t|) \in \mathbb{R}^3\).
Define first

\[ \Omega_+ = \{(z,t); \, |z| < 1 \, \text{and} \, |t| < |z - 2|\}; \]

\[ \Omega_- = \{(z,t); \, |z| < 1 \, \text{and} \, |t| < |z + 2|\}, \]

and then

\[ \Omega_0 = \Omega_+ \cap \Omega_-; \quad \Omega_0' = \{(z,t) \in \Omega_0; \, |t| < r\}, \]

where \( r \) is a constant with \( 2 < r < \sqrt{5} \). All these sets are lineally convex. The two points \((\pm 1, \sqrt{3})\) belong to the boundary of \( \Omega_0 \); in the three-dimensional space of the variables \((\text{Re} z, \text{Im} z, |t|)\), the set representing \( \Omega_0 \) has two peaks, which have been truncated in \( \Omega_0' \).

We now define \( \Omega' \) by glueing together \( \Omega_0 \) and \( \Omega_0' \): Define \( \Omega' \) as the subset of \( \Omega_0 \) such that \((z,t) \in \Omega_0' \) if \( \text{Im} z > 0 \); we truncate only one of the peaks of \( \Omega_0 \).

The point \((i-\varepsilon, r)\) for a small positive \( \varepsilon \) belongs to the boundary of \( \Omega' \) and the tangent plane at that point has the equation \( t = r \) and so must cut \( \Omega' \) at the point \((-i+\varepsilon, r)\). Therefore \( \Omega' \) is not lineally convex, but it agrees with the lineally convex sets \( \Omega_0 \) and \( \Omega_0' \) when \( \text{Im} z < \delta \) and \( \text{Im} z > -\delta \), respectively, for a small positive \( \delta \). The set has Lipschitz boundary; in particular it is equal to the interior of its closure. \( \square \)

**Proposition 9.4.9** Let \( \omega_0 \) and \( \omega_1 \) be two bounded open subsets in the complex plane such that none is contained in the closure of the other. Then there exists a Hartogs domain over \( \omega = \omega_0 \cup \omega_1 \) that is not lineally convex, but is such that the subsets \( \Omega_j \) over \( \omega_j \) are both lineally convex, \( j = 0, 1 \).

**Proof** Take two points \( a \in \omega_1 \setminus \varpi_0 \) and \( b \in \omega_0 \setminus \varpi_1 \), which exist by hypothesis. It is no restriction to assume that \( a = i, \ b = -i \). Then take \( c > 0 \) so large that \( \omega \) is contained in the disk of radius \( c-1 \) and with center at the origin. We then define as in Example 9.4.8,

\[ \Omega_0 = \left\{(z,t) \in \mathbb{C}^2; \, |t| < |z \pm c| \, \text{and} \, |t| < \left| z \pm i(1 + \sqrt{c^2 + 1}) \right| \right\}, \]

and

\[ \Omega_1 = \{(z,t) \in \Omega_0; \, |t| < r\}, \]

where \( r \) is a number slightly smaller than \( \sqrt{c^2 + 1} \) but so close to that number that the peak that we have truncated in \( \Omega_1 \) near \( i \in \mathbb{C} \) lies outside \( \omega_0 \), and the peak near \(-i\) lies outside \( \omega_1 \). This is possible since we have assumed that \( i \notin \varpi_0, \ -i \notin \varpi_1 \), and \( \Omega_0 \) and \( \Omega_1 \) differ only above small neighborhoods of \( i \) and \(-i \). These neighborhoods shrink to \( \{ i, -i \} \) as \( r \) increases to \( \sqrt{c^2 + 1} \). We now define \( \Omega \) to agree with \( \Omega_j \) over \( \omega_j, \ j = 0, 1 \). The conclusion is as in Example 9.4.8.

**9.4.4 Smooth vs. Lipschitz boundaries**

The lineally convex set \( \Omega_0 \) constructed in Example 9.4.8 has the remarkable property that it cannot be approximated by lineally convex sets with smooth
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boundary. Its boundary, which is Lipschitz, cannot in any reasonable way be rounded off if we want to preserve lineal convexity. This is why we shall continue this investigation to see whether smoothly bounded sets admit a passage from the local to the global.

Before doing so, however, we shall illustrate the difference between domains which can be approximated by smoothly bounded lineally convex domains and those that have only Lipschitz boundary.

Let \( \Omega \) be a complete Hartogs domain with \( R \) a function of class \( C^1 \), \( \omega \) being the open set where \( R > 0 \). Often it will be convenient to use not \( R \) but \( h = R^2 \) to define the set, thus

\[
\Omega = \{(z,t) \in \omega \times \mathbb{C}; \ |t| < R(z)\} = \{(z,t) \in \omega \times \mathbb{C}; \ |t|^2 < h(z)\}. \quad (9.45)
\]

The complex tangent plane at a boundary point \((z_0, t_0)\) with \( z_0 \in \omega \) has the equation

\[
t - t_0 = \alpha(z - z_0), \text{ where } \alpha = \frac{h_z(z_0)}{t_0} = \frac{2t_0R_z(z_0)}{R(z_0)}. \quad (9.46)
\]

The tangent plane intersects the plane \( t = 0 \) in the point

\[
b(z_0) = z_0 - \frac{h(z_0)}{h_z(z_0)} = z_0 - \frac{R(z_0)}{2R_z(z_0)}. \quad (9.47)
\]

If \( R_z(z_0) = 0 \), the tangent plane has the equation \( t = t_0 \), and in this case we define \( b(z_0) = \infty \), the infinite point on the Riemann sphere \( S^2 = \mathbb{C} \cup \{\infty\} \).

**Proposition 9.4.10** Let \( R \in C^1(\mathbb{C}) \) and define \( \Omega \) by (9.43). If \( \Omega \) is bounded and lineally convex, then \( b(z) \), defined by (9.47), does not belong to \( \omega \), so that \( b \) is a continuous mapping from \( \omega \) into \( S^2 \setminus \omega \). Its range contains \( S^2 \setminus \overline{\omega} \).

**Proof** Clearly \( b \) is continuous as a mapping into \( \mathbb{C} \) except where \( R_z = 0 \). Near such points, however, \( 1/b \) is continuous. The point \((b(z_0), 0)\) cannot belong to \( \Omega \) since \( \Omega \) is lineally convex; thus \( b(z_0) \notin \omega \). From every point \((z,0)\) outside the closure of \( \Omega \) we can draw a tangent to \( \Omega \); this shows that the range of \( b \) contains \( \mathbb{C} \setminus \overline{\omega} \); clearly it also contains \( \infty \).

**Corollary 9.4.11** If \( \Omega \) is as in Proposition 9.4.10, then \( \Omega \) is connected. The same is true if \( \Omega \) is the union of an increasing family of bounded lineally convex sets \( \Omega_j \) defined by functions \( R_j \in C^1(\mathbb{C}) \).

**Proof** Let \( \omega_1 \) be a component of \( \omega \) and let \( \Omega_1 \) be the set over \( \omega_1 \). Then the image of the boundary of \( \Omega_1 \) under \( b \) contains \( S^2 \setminus \overline{\omega_1} \). Since \( b(z_0) \notin \omega \) there can be no other component: we must have \( \omega_1 = \omega \). The statement about \( \bigcup \Omega_j \) is now immediate.

Corollary 9.4.11 should be compared with the following easy result for Lipschitz boundaries.
Proposition 9.4.12 Let $\omega$ be any open set in $\mathbb{C}$. Then there exists a Lipschitz continuous function $R \in C(\mathbb{C})$ such that $\omega$ is the set where $R$ is positive and the set $\Omega$ defined by $R$ is lineally convex.

Proof We define $R(z) = \inf_{a \in \omega} |z - a|$. The set $\Omega$ is lineally convex since it is an intersection of sets of the type $V$ defined in (9.44).

The set $M_{\sup R}$ where the function $R$ assumes its maximum can be rather arbitrary as shown be the next proposition.

Proposition 9.4.13 Given any closed set $M$ in the complex plane such that its complement is a union of open disks of radius $r$ there exists a Lipschitz continuous function $R$ such that $M_{\sup R} = M$ and the domain $\Omega$ defined by (9.45) with this $R$ is lineally convex.

Proof Define $R(z) = \min(r, \inf_{a \in A} |z - a|)$, where $A$ is the set of all centers of disks of radius $r$ in the complement of $M$.

But when $R$ is of class $C^1$, the set $M_{\sup R}$ is convex:

Theorem 9.4.14 Let $R: \mathbb{C}^n \to \mathbb{R}$ be a function of class $C^1$ or more generally a continuous function which is the limit of an increasing sequence of functions $R_j$ of class $C^1$ in the sets $\{z; R_j(z) > 0\}$. We assume that $R$ is positive only in a bounded subset of the complex plane. The functions $R_j$ define open sets $\Omega_j$, which we assume to be lineally convex. Then the set $M_{\sup R} = \{z; R(z) = \sup_w R(w)\}$ is convex. □

This could be proved here, but it is more easily done with the methods of Section 9.7: see Theorems 9.7.29 and 9.7.30.

If a set does not have a boundary of class $C^1$, we cannot give a meaning to the notion of tangent plane. However, if the set is the union of an increasing family of sets with smooth boundaries, it is possible to use instead their tangent planes and then pass to the limit. Such limits of tangent planes can serve as well, as explained in the following lemma.

Lemma 9.4.15 Let $\Omega$ be the union of an increasing family of open lineally convex sets $\Omega_j$ with boundaries of class $C^1$. Let $(j_k)$ be a sequence tending to $+\infty$, and let $Y_{j_k}$ be the complex tangent plane of $\partial \Omega_{j_k}$ at some point in the boundary of $\Omega_{j_k}$, $k \in \mathbb{N}$. Assume that the $Y_{j_k}$ converge to a hyperplane $Y$ in the topology of hyperplanes. Then $Y$ does not intersect $\Omega$.

Proof Suppose there is a point $z \in Y \cap \Omega$. Then also $z \in Y \cap \Omega_{j_k}$ for all large $k$. Since $\Omega_{j_k}$ is open, there is a ball $B_<(z, \varepsilon) \subset \Omega_{j_k}$ for large $k$, say for $k \geq k_0$. But then $Y_{j_k}$ intersects $B_<(z, \varepsilon)$ for all large $k$, say for $k \geq k_1$. Thus $Y_k \cap \Omega_{j_k}$ is non-empty for all $k \geq \max(k_0, k_1)$, contradicting the lineal convexity of $\Omega_{j_k}$.

To recognize such limits of tangent planes we shall use the concept in the following definition.
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Definition 9.4.16 Let $X$ be any subset of $\mathbb{C}^n$ and $a$ a point in the boundary $\partial X$. We shall say that a complex hyperplane $Y$ is an admissible tangent plane to $\partial X$ at $a$ if there exists an open set $A$ with boundary of class $C^1$ such that $A$ and $X$ are disjoint, $a$ belongs to the boundary of $A$, and $Y$ is the complex tangent plane to $A$ at $a$.

Proposition 9.4.17 Let $\Omega \subset \mathbb{C}^n$ be the union of an increasing family of lineally convex open sets $\Omega_j$ with boundaries of class $C^1$. Then any admissible tangent plane $Y$ to $\partial \Omega$ is the limit of a sequence of tangent planes $Y_j$ to $\partial \Omega_j$. Therefore, in view of Lemma 9.4.15, $Y$ cannot intersect $\Omega$.

Proof Let $a$ and $A$ be as in Definition 9.4.16. By a coordinate change we may suppose that $a = 0$, that the real tangent plane to $\partial A$ at the origin has the equation $y_n = 0$, and that $A$ is defined by an inequality $y_n > f(z_1, \ldots, z_{n-1}, x_n)$ near the origin for some function $f$ of class $C^1$, which consequently vanishes at the origin together with its gradient. Write $z' = (z_1, \ldots, z_{n-1}) \in \mathbb{C}^{n-1}$. We then know that all points in $\Omega$ satisfy $y_n < f(z', x_n)$. Define $g(z', x_n) = f(z', x_n) + \|z'||^2 + x_n^2$, and let $A_c$ be the set of all points such that $y_n > g(z', x_n) - c$. We let $c = c_j$ be the largest real number such that $A_c$ and $\Omega_j$ are disjoint. Now $0 \in \partial \Omega$ and $\Omega_j \not\supset \Omega$; therefore we can be sure that $c_j$ tends to zero as $j \to \infty$. There is a point $z^j$ which is common to the boundaries of $A_{c_j}$ and $\Omega_j$. Since $A$ and $\Omega_j$ are disjoint, we have $\|(z^j)'\|^2 + (x^j_n)^2 \leq c_j$. The real tangent plane to $\partial A_{c_j}$ at $z^j$ is identical to the real tangent plane to $\partial \Omega_j$ at that point. We can control its slope, for the gradient of $g$ is

$$\text{grad } g = \text{grad } f + \text{grad } (\|z'||^2 + x_n^2),$$

which is continuous and vanishes at the origin. Since $((z^j)', x_n^j)$ tends to the origin, this shows that the real tangent plane to $\partial A_{c_j}$ at $z^j$ must be close to the real hyperplane $y_n = 0$ if $j$ is large, and then the complex tangent plane to $\partial A_{c_j}$ at $z^j$ is close to the complex hyperplane $z_n = 0$. The last statement now follows from Lemma 9.4.15.

9.4.5 Differential conditions

Definition 9.4.18 Let $\Omega$ be an open set in $\mathbb{C}^n$ with boundary of class $C^1$. Then a function $\rho \in C^1(\mathbb{C}^n)$ is called a defining function for $\Omega$, if $d\rho \neq 0$ wherever $\rho = 0$ and if $\Omega = \{z \in \mathbb{C}^n; \rho(z) < 0\}$. (Here the differential $d$ is defined in (9.2).)

Definition 9.4.19 The complex tangent space at a point $a$ on the boundary of $\Omega$ is defined by

$$\sum_{j=1}^n \frac{\partial \rho}{\partial z_j}(a)s_j = 0.$$
We shall denote it by $T_C(a)$. The **real tangent space** is defined by

$$\text{Re} \sum_{j=1}^{n} \frac{\partial \rho}{\partial z_j}(a)s_j = 0$$

and will be denoted by $T_R(a)$. The **complex tangent plane** is then $a + T_C(a)$; it is contained in the **real tangent plane** $a + T_R(a)$.

To be able to characterize sets by infinitesimal conditions, we shall describe boundaries and their curvature using defining functions and the Hesse and Levi forms. We now give the needed definitions.

**Definition 9.4.20** The **complex Hessian** (or **complex Hesse form**) of a function $f$ of class $C^2$ is defined to be

$$H^C_f(z; t) = \sum \frac{\partial^2 f}{\partial z_j \partial z_k}(z)t_j t_k, \quad z \in \mathbb{C}^n, \quad t \in \mathbb{C}^n, \quad (9.48)$$
a quadratic form in the $t_j$.

**Definition 9.4.21** The **Levi form** of $f$ is the Hermitian form

$$L_f(z; t) = \sum \frac{\partial^2 f}{\partial z_j \partial \overline{z}_k}(z)t_j \overline{t}_k, \quad z \in \mathbb{C}^n, \quad t \in \mathbb{C}^n. \quad (9.49)$$

We say that $\Omega$ satisfies the **Levi condition at** $a \in \partial \Omega$ if

$$L_\rho(a; t) \geq 0 \quad \text{when} \quad t \in T_C(a), \quad (9.50)$$

where $\rho$ is a defining function for $\Omega$; and that $\Omega$ satisfies the **strong Levi condition at** $a$ if strict inequality holds in (9.50) for $t \neq 0$.

**Definition 9.4.22** The **real Hessian** of a function $f$ of real variables $x_1, \ldots, x_m$ is

$$H^R_f(x; s) = \sum \frac{\partial^2 f}{\partial x_j \partial x_k}(x)s_j s_k, \quad x \in \mathbb{R}^m, \quad s \in \mathbb{R}^m, \quad (9.51)$$
a quadratic form.

When a function of $n$ complex variables is given, its real Hessian in the $2n$ real variables ($\text{Re} \, z_1, \text{Im} \, z_1, \ldots, \text{Re} \, z_n, \text{Im} \, z_n$) can be expressed using its complex Hessian and its Levi form as

$$H^R_f(z; s) = 2(\text{Re} \, H^C_f(z; t) + L_f(z; t)),$$

for $z \in \mathbb{C}^n, \quad s \in \mathbb{R}^{2n}, \quad t \in \mathbb{C}^n, \quad t_j = s_{2j-1} + is_{2j}$.

Thus the characterization of convexity mentioned in the introduction is that $\text{Re} \, H^C_\rho(a; t) + L_\rho(a; t)$ be nonnegative for all $a \in \partial \Omega$ and all $t \in T_R(a)$. For a lineally convex set the same inequality holds for all $t \in T_C(a)$. It is then equivalent to $L(a; t) \geq |H(a; t)|$ for $a \in \partial \Omega$ and $t \in T_C(a)$. 

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Definition 9.4.23  We shall say that a set \( \Omega \) with boundary of class \( C^2 \) satisfies the Behnke–Peschl differential condition at a boundary point \( a \) of \( \Omega \) if

\[
|H^C_\rho(a; s)| \leq L_\rho(a; s) \quad \text{for all vectors } s \in T_C(a),
\]

where \( \rho \) is a defining function for \( \Omega \). We shall say that \( \Omega \) satisfies the strict Behnke–Peschl differential condition at \( a \) if there exists a positive number \( \varepsilon \) such that we have

\[
|H^C_\rho(a; s)| \leq L_\rho(a; s) - \varepsilon \|s\|_2^2 \quad \text{for all } s \in T_C(a).
\]

It is easy to prove that these conditions are invariant under complex affine mappings. They also do not depend on the choice of defining function. They were introduced for \( n = 2 \) by Behnke and Peschl (1935:169).

These conditions should be compared with the differential condition for convexity: \( |H_\rho(s)| \leq L_\rho(s) \) for all vectors \( s \) in the real tangent space \( T_R(a) \). This is a local condition, and it is well known that it is equivalent to convexity of \( \Omega \). The proof of this fact most conveniently goes via approximation of the set by sets satisfying the corresponding strong condition, i.e., \( |H_\rho(s)| \leq L_\rho(s) - \varepsilon \|s\|_2^2 \) for a positive \( \varepsilon \) and for all \( s \in T_R(a) \).

The following two lemmas are well known; cf. (Zinov'ev 1971) and (Hörmander 1994: Corollary 4.6.5). We include them for ease of reference.

Lemma 9.4.24  Let \( \Omega \) be an open subset of \( \mathbb{C}^n \) with boundary of class \( C^2 \). If \( \Omega \) is locally weakly lineally convex, then \( \Omega \) satisfies the Behnke–Peschl differential condition at every boundary point.

Proof  Let \( a \) be an arbitrary boundary point of a locally weakly lineally convex open set \( \Omega \). Then there exists a complex hyperplane through \( a \) that does not cut \( \Omega \) close to \( a \). This hyperplane cannot be anything but \( T_C(a) \) since the boundary is of class \( C^1 \). Therefore if we take an arbitrary vector \( s \in T_C(a) \) and consider the function \( \varphi(t) = \rho(a + ts) \) of a real variable \( t \), its second derivative must be non-negative at the origin. If we express the condition \( \varphi''(0) \geq 0 \) in terms of \( H \) and \( L \) we get \( \text{Re} \, H(s) + L(s) \geq 0 \), which, since \( H \) is quadratic and \( L \) sesquilinear, is equivalent to \( |H| \leq L \).

Lemma 9.4.25  Let \( \Omega \) be an open subset of \( \mathbb{C}^n \) with boundary of class \( C^2 \). If \( \Omega \) satisfies the strict Behnke–Peschl differential condition at every boundary point, then \( \Omega \) is locally weakly lineally convex.

Proof  With \( \varphi \) as in the proof of the previous lemma we must have \( \varphi''(0) > 0 \) if \( \Omega \) satisfies the strict Behnke–Peschl differential condition. This implies that \( T_C(a) \) cannot cut \( \Omega \) close to \( a \).

It is known that if \( \Omega \) is a connected open set with boundary of class \( C^1 \) which is locally weakly lineally convex, then \( \Omega \) is weakly lineally convex; see, e.g., (Hörmander 1994: Proposition 4.6.4). We shall come back to this result in Subsection 9.4.7.
9.4.6 Differential conditions for Hartogs domains

In this subsection we shall see what the differential conditions look like in the case of a complete Hartogs domain in \( \mathbb{C}^2 \). Let \( \Omega \) be a complete Hartogs domain in \( \mathbb{C}^2 \) defined by (9.45). If \( h \) is of class \( C^1 \), we can choose as its defining function
\[
\rho(z,t) = t\bar{t} - h(z), \quad (z,t) \in \mathbb{C} \times \mathbb{C}.
\]
It must satisfy \( d'\rho \neq 0 \) when \( \rho = 0 \), which means that \( d'\rho = \bar{t}dt - h_z dz \neq 0 \) when \( |t|^2 = h(z) \). Since the first term of \( d'\rho \) is \( \bar{t}dt \), which is non-zero everywhere except in the plane \( t = 0 \), the only condition is that \( h_z \neq 0 \) when \( h = 0 \), i.e., that \( h \) itself shall be a defining function in \( \mathbb{C} \). It defines a subset \( \omega \) of the complex plane over which \( \Omega \) is situated.

**Lemma 9.4.26** Let \( h \) be a defining function of an open set \( \omega \) in \( \mathbb{C} \) of class \( C^k \), \( k \geq 1 \). Then the complete Hartogs domain in \( \mathbb{C}^2 \) defined by (9.45) has boundary of class \( C^k \). When \( k \geq 2 \), it satisfies the Behnke–Peschl differential condition at every boundary point if and only if \( h \) satisfies the condition
\[
\frac{|h_z|^2}{h} \geq h_z z + |h_{zz}| \quad \text{wherever } h > 0.
\]
(9.54)

Furthermore \( \Omega \) satisfies the strict Behnke–Peschl differential condition if and only if there is strict inequality in (9.54).

**Proof** Let us look at the Hessian and Levi forms of \( \rho(z,t) = |t|^2 - h(z) \). They are, respectively,
\[
H(s) = -h_{zz}s_1^2 \quad \text{and} \quad L(s) = -h_z \bar{z}|s_1|^2 + |s_2|^2, \quad s = (s_1, s_2) \in \mathbb{C}^2.
\]
The differential condition \( |H| \leq L \) takes the form
\[
|h_z||s_1|^2 \leq -h_z \bar{z}|s_1|^2 + |s_2|^2 \quad \text{for all } s \in T_{\mathbb{C}}(a).
\]
The tangent plane is defined by \( -h_z s_1 + \bar{t} s_2 = 0 \). When \( t \neq 0 \) we use this equation to eliminate \( s_2 \); the condition takes the form (9.54). Near \( t = 0 \) we eliminate instead \( s_1 \) and get
\[
(h_z \bar{z} + |h_{zz}|) \frac{h}{|h_z|^2} \leq 1.
\]
This inequality is satisfied, even strictly, at all boundary points sufficiently close to \( t = 0 \), provided \( h_z \neq 0 \) near \( h = 0 \). Therefore, if \( h \) is a defining function for \( \omega \), then \( \rho \) is a defining function for \( \Omega \) and condition (9.54) implies the Behnke–Peschl differential condition at all boundary points of \( \Omega \), including those where \( t = 0 \). Conversely, if \( \rho \) is a defining function for \( \Omega \), then \( h \) is a defining function for \( \omega \), and the Behnke–Peschl differential condition for \( \rho \) implies the condition (9.54) for \( h \).
Remark 9.4.27 We can also express the Behnke–Peschl differential condition (9.54) in terms of the radius $R = \sqrt{h}$. It becomes

$$|R_z|^2 \geq |R_z^2 + RR_{zz}| + RR_{z\bar{z}},$$

(9.55)

which is less convenient to work with than (9.54). If $h$ is concave, then $h_{z\bar{z}} \leq 0$, so that (9.54) holds. More generally, if $R$ is concave, then $R_{z\bar{z}} \leq 0$, which implies that (9.55) holds. It is also possible to express the Behnke–Peschl differential condition in terms of the function $f = -\log R$. It then takes the form

$$|f_{zz} - 2f_z^2| \leq f_{z\bar{z}}.$$  (9.56)

We note that $(z,t) \mapsto |t|^2 - h(z)$ is convex if and only if $h$ is concave, and that $(z,t) \mapsto \log \|z\|^2 + f(z)$ is plurisubharmonic if and only if $f$ is.

□

9.4.7 Approximating bounded lineally convex Hartogs domains by smoothly bounded ones

Theorem 9.4.28 Let

$$\Omega = \{(z,t) \in \mathbb{C}^2; |t|^2 < h(z)\}$$

(9.57)

be a bounded complete Hartogs domain in $\mathbb{C}^2$ with boundary of class $C^2$. Suppose $\Omega$ satisfies the Behnke–Peschl differential condition at all boundary points. Then $\Omega$ can be approximated from the inside by Hartogs domains

$$\Omega_\varepsilon = \{(z,t); |t| < R_\varepsilon(z)\}$$

which satisfy the strict Behnke–Peschl differential condition all boundary points $(z,t)$ except those where $h_z(z) = 0$. In fact, we can take $h_\varepsilon = h - \varepsilon$ with $\varepsilon$ positive and small enough.

Proof This is an instance where it is more convenient to use $h$ rather than $R$. The Behnke–Peschl differential condition (9.54) contains the value of $h$ only at one place, and $h_\varepsilon = h - \varepsilon$ has the same derivatives as $h$, so we can write

$$\frac{|h_z|^2}{h - \varepsilon} > \frac{|h_z|^2}{h} \geq h_{z\bar{z}} + |h_{zz}|$$

except when $h_z = 0$. Thus the boundary of $\Omega_\varepsilon$ satisfies the strict Behnke–Peschl differential condition except at the points where $h_z = 0$. So far the argument is valid for all positive $\varepsilon$. We need to check that $h_\varepsilon$ is a defining function; otherwise we cannot apply Lemma 9.4.26. But the gradient of $h_\varepsilon$ is the same as that of $h$, which is non-zero when $h = 0$, hence also when $h_z = 0$, provided $\varepsilon$ is small enough. Thus $h_\varepsilon$ is indeed a defining function for all small $\varepsilon$, proving the theorem.
We shall now see that the approximating sets $\Omega_\varepsilon$ that we constructed in Theorem 9.4.28 are, in fact, linearly convex. Let us agree to say that a complex plane with the equation $z = \text{constant}$ is \textit{vertical} and a plane with the equation $t = \text{constant}$ is \textit{horizontal}.

**Proposition 9.4.29** Let $\Omega$ be a bounded complete Hartogs domain in $\mathbb{C}^2$ with boundary of class $C^2$ satisfying the strict Behnke–Peschl differential condition except possibly at the points where the tangent plane is horizontal. Then $\Omega$ is linearly convex.

We shall need the following three lemmas.

**Lemma 9.4.30** Let $\Omega$ be as in Proposition 9.4.29 and let $L$ be a complex line in $\mathbb{C}^2$ which is not horizontal. Then $L \cap \Omega$ consists of a finite number of open sets bounded by $C^2$ curves obtained as transversal intersections of $L$ and $\partial \Omega$ (and $L \cap \partial \Omega$ consists of these curves plus a finite number of isolated points).

**Proof** Take an arbitrary boundary point $a$ and let $L$ be a complex line through $a$ which is not horizontal. If $L$ is the tangent plane, $L = a + T_C(a)$, then the proof of Lemma 9.4.25 shows that $L$ intersects $\Omega$ near $a$ only in the point $a$. If, on the other hand, $L$ is not the tangent plane, then $L \cap (a + T_C(a)) \neq L$, so $\partial \Omega$ cuts $L$ transversally, and $\partial \Omega \cap L$ is a $C^2$ curve in $L$ near $a$. Thus $L \cap \partial \Omega$ consists of a number of $C^2$ curves plus isolated points—by compactness there can only be finitely many curves and points.

**Lemma 9.4.31** Let $\Omega$ and $L$ satisfy the hypotheses of the previous lemma. Then $\Omega \cap L$ is connected, and $\Omega \cap (a + T_C(a))$ is empty for all $a \in \partial \Omega$.

**Proof** We shall follow the proof of Proposition 4.6.4 in (Hörmander 1994)—we only have to be careful to avoid horizontal planes. Let $(z_j, t_j)$, $j = 0, 1$, be two points in $L \cap \Omega$. We have to prove that they belong to the same component of $L \cap \Omega$. Suppose first that both $t_0$ and $t_1$ are non-zero. Since $\Omega$ is connected, there is a curve $\gamma$ which goes from $\gamma(0) = (z_0, t_0)$ to $\gamma(1) = (z_1, t_1)$. We can actually do this in such a way that the complex line $L_s$ that contains $\gamma(0)$ and $\gamma(s)$, $0 < s \leq 1$, is never horizontal. Indeed, we first go from $(z_0, t_0)$ to $(z_0, 0)$ along a curve in the plane $z = z_0$ avoiding $(z_0, t_1)$; then along a curve in the plane $t = 0$ from $(z_0, 0)$ to $(z_1, 0)$; and then finally from $(z_1, 0)$ to $(z_1, t_1)$ along a curve in the plane $z = z_1$ avoiding $(z_1, t_0)$. (We know that $t_0 \neq t_1$.)

Thus none of the lines $L_s$ is horizontal, and we can apply Lemma 9.4.30 to them. Consider the set $C$ of all $s \in [0, 1]$ such that $\gamma(0)$ and $\gamma(s)$ belong to the same component of $L_s \cap \Omega$. Then certainly $C$ contains all sufficiently small numbers, for $\gamma(0)$ and $\gamma(s)$ are then in the line $z = z_0$, whose intersection with $\Omega$ is a disk. The set $C$ is open as a subset of $[0, 1]$ in view of Lemma 9.4.30, but so is its complement with respect to $[0, 1]$. Since it is non-empty, it must contain 1, i.e., $(z_0, t_0)$ and $(z_1, t_1)$ belong to the same component of $L \cap \Omega$. If one of $t_0, t_1$ is zero, we choose a point with non-zero second coordinate in the neighborhood and argue as above.
Consider now a tangent plane \( L = a + T_{C}(a) \) and planes \( L_{\varepsilon} = a_{\varepsilon} + T_{C}(a) \) parallel to it, where we write \( a_{\varepsilon} = (z_{0}, (1 - \varepsilon)t_{0}) \) if \( a = (z_{0}, t_{0}) \). We already know from Lemma 9.4.25 that \( L \) cannot intersect \( \Omega \) close to \( a \). However, it cannot cut \( \Omega \) at all, for if it did, then a parallel plane \( L_{\varepsilon} \) for some small positive \( \varepsilon \) would intersect \( \Omega \) in a component close to \( a \) and another nonempty set at some distance from \( a \), thus in a disconnected set. This proves Lemma 9.4.31.

**Lemma 9.4.32** Let \( \Omega \) be as in Proposition 9.4.29 and let \( a \in \partial \Omega \) be such that the tangent plane is horizontal. Then \( \Omega \cap (a + T_{C}(a)) \) is empty; in other words \( R \) has a global maximum at \( a \). Consequently any horizontal plane \( L \) intersects \( \Omega \) in finitely many open sets bounded by \( C^{2} \) curves obtained as transversal intersections of \( L \) by \( \partial \Omega \).

**Proof** Let \((z_{0}, t_{0})\) be a boundary point such that the tangent plane is horizontal, i.e., \( R_{z}(z_{0}) = 0 \). Suppose the tangent plane cuts \( \Omega \) in some point \((z_{1}, t_{1})\). We must then have \( t_{1} = t_{0} \). Since \( \Omega \) and its base \( \omega \) are connected, we can find a curve \( \gamma \) in \( \omega \) connecting \( z_{0} \) to \( z_{1} \), say \( \gamma(s) = z_{s}, s \in [0, 1] \).

Consider now the tangent planes at the points \((z_{s}, R(z_{s}))\); we denote them by \( L_{s} = (z_{s}, R(z_{s})) + T_{C}(z_{s}, R(z_{s})) \). It is no restriction to assume \( t_{0} > 0 \), so that \( R(z_{0}) = t_{0} \). We know that \( L_{0} \) is horizontal, but certainly not all the \( L_{s} \) can be horizontal, since \( R(z_{1}) > |t_{1}| = |t_{0}| = R(z_{0}) \). Let \( s_{0} \) be the infimum of all \( s \) such that \( L_{s} \) is not horizontal; we must have \( 0 \leq s_{0} < 1 \). The planes \( L_{s} \) with \( 0 \leq s \leq s_{0} \) are identical and all intersect \( \Omega \) in the point \((z_{1}, t_{1})\). It is now clear that there exists a tangent plane \( L_{s} \) with \( s \) just a little bit larger than \( s_{0} \) which is not horizontal and still cuts \( \Omega \). This contradicts Lemma 9.4.31.

**Proof of Proposition 9.4.29.** We know from Lemma 9.4.31 that a tangent plane which is not horizontal does not intersect \( \Omega \); we obtain the same conclusion from Lemma 9.4.32 for a horizontal tangent plane. Thus \( \Omega \) is weakly lineally convex. Lemma 9.4.7 shows that this implies lineal convexity.

We can now finally state the main result of this section:

**Theorem 9.4.33** Let \( \Omega \) be a bounded complete Hartogs domain in \( C^{2} \) with boundary of class \( C^{2} \). If \( \Omega \) satisfies the Behnke–Peschl differential condition (9.52) at all boundary points, then \( \Omega \) is lineally convex.

**Proof** Using Theorem 9.4.28 we construct open sets \( \Omega_{\varepsilon} \), which tend to \( \Omega \). Also, if \( R(z_{0}) > 0 \), the tangent plane of \( \partial \Omega_{\varepsilon} \) at \((z_{0}, \sqrt{R(z_{0})^{2} - \varepsilon})\) tends to that of \( \partial \Omega \) at \((z_{0}, R(z_{0}))\). The sets \( \Omega_{\varepsilon} \) are lineally convex by Proposition 9.4.29. Then also their limit \( \Omega \) is lineally convex. Indeed, if a tangent plane to \( \partial \Omega \) intersected \( \Omega \), then it would cut also \( \Omega_{\varepsilon} \) for all sufficiently small \( \varepsilon \), and then also for \( \varepsilon \) small enough the corresponding tangent plane to \( \partial \Omega_{\varepsilon} \) would cut \( \Omega_{\varepsilon} \). This is a contradiction.
9.4.8 The non-local character of lineal convexity, revisited

Having settled the question of lineal convexity of smoothly bounded Hartogs domains we now turn to sets of the form

\[ \Omega = \{ (z, t) \in \omega \times \mathbb{C}; \ |t| < R(z) \} = \{ (z, t) \in \omega \times \mathbb{C}; \ |t|^2 < h(z) \}, \]  

(9.58)

where \( \omega \) is a given open set in \( \mathbb{C} \) and \( h \) is a \( C^2 \) function in the closure of \( \omega \) satisfying \( h > 0 \) and the Behnke–Peschl differential condition (9.54). Its boundary is smooth enough over points in \( \omega \), but is only Lipschitz at points over \( \partial \omega \). It turns out that when \( \omega \) is a disk, then the Behnke–Peschl differential condition implies lineal convexity: we shall study this question in Subsection 9.4.9. On the other hand, if \( \omega \) is a set such that \( \omega \) is not a disk, then the Behnke–Peschl differential condition does not imply lineal convexity. This is the topic of the present subsection.

The property of being a disk is invariant under Möbius mappings, and disks are the only sets that remain convex under all Möbius mappings. This is a kind of explanation for the phenomenon we encounter here, and it is therefore natural to study how the Behnke–Peschl differential condition (9.52) behaves under Möbius mappings. This is explained in the next lemma.

**Lemma 9.4.34** Let \( \Omega \) be a Hartogs domain in \( \mathbb{C}^2 \) defined by \( |t| < R(z) \), let \( a, b, c, d \) be four complex numbers with \( ad - bc \neq 0 \), and let \( \Omega_1 \) be the Hartogs domain defined by \( |t| < R_1(z) = |c + dz|R((a + bz)/(c + dz)) \). Then \( \Omega \) and \( \Omega_1 \) are lineally convex simultaneously. The two functions \( h \) and \( h_1(z) = |c + dz|^2h((a + bz)/(c + dz)) \) satisfy the Behnke–Peschl differential condition (9.54) simultaneously.

**Proof** Take constants \( \alpha, \beta \) and \( c \) of which not both of \( \alpha \) and \( \beta \) are zero, and consider the mapping

\[ (\mathbb{C} \setminus \{0\}) \times \mathbb{C} \times \mathbb{C} \ni (z_0, z_1, t) \mapsto (z_1/z_0, t/z_0) \in \mathbb{C}^2. \]

Under it the pull-back of the hyperplane of equation \( c + \alpha z + \beta t = 0 \) is the hyperplane of equation \( cz_0 + \alpha z_1 + \beta t = 0 \). It follows that the pull-back of a lineally convex set in \( \mathbb{C}^2 \) is a complex homogeneous lineally convex set in \( \mathbb{C}^3 \).

Now any linear mapping of the form

\[ \mathbb{C}^3 \ni (z_0, z_1, t) \mapsto (cz_0 + dz_1, az_0 + bz_1, t) \in \mathbb{C}^3 \]

with \( ad - bc \neq 0 \) preserves lineal convexity, and mappings

\[ \mathbb{C}^3 \ni (z_0, z_1, t) \mapsto \left( \frac{a z_0 + b z_1}{c z_0 + d z_1}, \frac{t}{c z_0 + d z_1} \right) \in \mathbb{C}^3 \]

preserve lineally convex sets which are complex homogeneous. If we transport this back to \( \mathbb{C}^2 \) we get a mapping of the form

\[ (z, t) \mapsto \left( \frac{a + bz}{c + dz}, \frac{t}{c + dz} \right). \]
This proves that $\Omega$ and $\Omega_1$ as defined in the statement of the lemma are lineally convex at the same time. The statement about the differential condition for $h$ and $h_1$ can be verified directly, perhaps most easily if we check it for the special mappings $z \mapsto c + dz$ and $z \mapsto 1/z$, which together generate all Möbius mappings.

**Lemma 9.4.35** Let $K$ be a compact subset of $\mathbb{C}$ with connected complement. Assume that $K$ is not a disk. Then there exists a closed disk $D_1$ containing $K$ such that $K \cap \partial D_1$ has at least two components.

**Proof** Let $D_0$ be the closed disk of minimal radius that contains $K$. By hypothesis $K \neq D_0$ and $\mathbb{C} \setminus K$ is connected, so there exists a point $a_0 \in \partial D_0 \setminus K$. Let $H$ be an open half plane that contains $K$ but is such that $a_0 \notin H$. Now consider the closed disk $D_1$ of minimal radius among those that contain $K$ and have $\partial H$ as a tangent. We claim that there are four points $a,b,c,d \in \partial D_1$ which are in that order along the circumference and with $a,c \notin K$, $b,d \in K$. This will show that $b$ and $d$ belong to different components of $K \cap \partial D_1$. To find these points we argue as follows. Let $a$ be the point of $\partial D_1$ at which $\partial H$ is tangent; thus $a \in \partial D_1$ and $a \notin K$. Next, $D_1 \notin D_0$, so there is a point $c \in \partial D_1 \setminus D_0$. Thus $c \notin K$. Finally we claim that there are two points $b,d \in \partial D_1 \cap K$ on either side of the segment $[a,c]$. This is so because if one of the arcs from $a$ to $c$ were disjoint from $K$, then it can easily be seen that $D_1$ would not be minimal among the disks that contain $K$ and are tangent to $\partial H$. This completes the proof.

**Theorem 9.4.36** Let $\omega$ be a bounded connected open subset of $\mathbb{C}$ such that the complement $S^2 \setminus \omega$ of its closure with respect to the Riemann sphere $S^2 = \mathbb{C} \cup \{\infty\}$ has at least one component which is not a disk. Then there exists a Hartogs domain defined by a smooth function and with base $\omega$ such that it is not lineally convex, although $\omega = \omega_0 \cup \omega_1$ and the Hartogs domains over $\omega_0$ and $\omega_1$ are both lineally convex. In particular the function defining $\Omega$ satisfies the Behnke–Peschl differential condition (9.54).

**Proof** Let $K$ be the complement of a component of $S^2 \setminus \omega$ which is not a disk; thus $K$ contains $\omega$. Moreover the complement of $K$ is connected and $\partial K \subset \partial \omega$. We may assume that $K$ is compact: if not we use a Möbius mapping to reduce ourselves to that case. Let $a,b,c,d \in \partial D_1$ be the four points whose existence is guaranteed by Lemma 9.4.35; recall that $b,d \in K$ and $a,c \notin K$. Now take a new closed disk $D_2$ which does not contain $a,b$, or $d$, but contains $c$ in its interior, and is so close to $D_1$ that $b$ and $d$ belong to different components of $K \setminus D_2$. This is possible because $a$ does not belong to $K$. Now we map $D_2$ onto the closed right half plane, taking $a$ to 0 and some point outside $K$ and near $c$ to infinity. We are thus reduced to a situation where $K$ is still compact in $\mathbb{C}$, whereas $\partial D_2$ is the imaginary axis, with $a = 0$ and $\text{Im} b$ and $\text{Im} d$ of different signs, say for definiteness $\text{Im} b < 0$ and $\text{Im} d > 0$. Moreover we can take $D_2$ so close to $D_1$ that the points in $K$ which are not in $D_2$ are never
real. Then we can define a function \( R \) as follows. First take a smooth concave function \( \psi \) of a real variable such that \( \psi(s) = 1 \) when \( s \geq 0 \) and \( \psi(s) < 1 \) for \( s < 0 \), but still so that \( \psi(\Re z) > 0 \) for all points \( z \in \overline{\omega} \). Then define

\[
R(z) = \begin{cases} 
\psi(\Re z) & \text{when } z \in \omega, \ \Re z < 0, \ \Im z < 0; \\
1 & \text{at other points in } \omega.
\end{cases}
\]

This function is continuous, even identically one, in a neighborhood of the intersection of \( \omega \) and the real axis.

The tangent plane at a point \((z_0, t_0) \in \partial \Omega \) with \( z_0 \in \omega \) has the equation (9.46). In particular, we may take \( t_0 = R(z_0) \) and get

\[
t = R(z_0) + 2R_z(z_0)(z - z_0).
\]

In the present case \( R \) is locally a function of \( \Re z \), say \( R(z) = k(x) \), so that \( R_z = k_x/2 \) is real. Thus the tangent plane is

\[
t = R(z_0) + k_x(x_0)(z - z_0) = R(z_0) + k_x(x_0)(x - x_0) + ik_x(x_0)(y - y_0),
\]

and, writing \( z = z_0 + z_1 \), we obtain

\[
|t|^2 = R(z_0)^2 + 2k_x(x_0)R(z_0)x_1 + k_x(x_0)^2 x_1^2 + k_x(x_0)^2 y_1^2.
\]

When \( x_1 < 0 \) and \( k_x(x_0) \) is positive and small,

\[
|t|^2 \approx R(z_0)^2 + 2k_x(x_0)R(z_0)x_1 < R(z_0)^2.
\]

Since \( \omega \) is connected and has the point \( b \) on its boundary, we can choose \( z_0 \) such that \( y_0 < 0 \) and \( x_0 < 0 \) with \( k_x(x_0) \) arbitrarily small, so small that indeed \( |t| < R(z_0) \). Then we choose \( z = z_0 + z_1 \in \omega \) with \( \Im z > 0 \). Thus \( R(z) = 1 \), so the tangent plane at \((z_0, R(z_0))\) cuts \( \Omega \) in a point above \( z \). This proves that \( \Omega \) is not lineally convex. However, if we look at the parts of \( \omega \) where \( \Im z > -\varepsilon \) and \( \Im z < \varepsilon \) respectively, then \( R \) is the restriction of a globally concave function in each of them and therefore defines a lineally convex set.

**Theorem 9.4.37** Let \( \omega \) be a bounded open set in \( \mathbb{C} \) such that \( S^2 \setminus \overline{\omega} \) is not connected. Then there is a function \( h \in C^\infty(\overline{\omega}) \), \( h > 0 \), which satisfies the Behnke–Peschl differential condition (9.54) but is such that the Hartogs domain it defines over \( \omega \) is not lineally convex.

**Proof** If one of the components of \( S^2 \setminus \overline{\omega} \) is not a disk, we already know the result by Theorem 9.4.36. The case when all components of \( S^2 \setminus \overline{\omega} \) are disks remains to be considered. This means that \( \overline{\omega} \) is a disk from which countably many disks (at least one) have been removed. Any one of these holes can be moved by a Möbius transformation so that it becomes concentric with the outer circumference of \( \overline{\omega} \); in other words \( \overline{\omega} \) is an annulus \( r_0 \leq |z| \leq r_1 \) from which possibly a number of disks have been removed. It is clearly enough to
consider the case of the annulus, for the possible presence of other holes will not destroy our conclusion.

So assume \( \Omega \) is the annulus \( r_0 \leq |z| \leq r_1 \) and define \( R_0(z) = 1 - ax^2 - by^2 \), where \( 0 < a < b \) and \( b \) is so small that \( R_0 > 0 \) in \( \Omega \). Next define \( \varphi \) to be a concave \( C^\infty \) function of one real variable such that \( \varphi(s) = s \) for all \( s \leq 1 - br_0^2 + \epsilon \) and \( \varphi(s) = c \) when \( s \geq 1 - ar_0^2 - \epsilon \) for some positive \( \epsilon \) and a suitable constant \( c \); by necessity we must have \( c < 1 - ar_0^2 \). Define \( R_1(z) = \varphi(R_0(z)) \). We observe that \( R_0 = R_1 \) in a neighborhood of the intersection of the imaginary axis and \( \Omega \). Both \( R_0 \) and \( R_1 \) are concave in \( \mathbb{C} \), so the corresponding Hartogs domains over \( |z| < r_1 \) are convex and therefore lineally convex. It follows that the Hartogs domains over \( \omega \) are lineally convex. Now define \( R \) to agree with \( R_0 \) in the right half plane and with \( R_1 \) in the left half plane. Note that \( R(z) = R_1(z) = c \) at points \( z \in \omega \) close to \( -r_0 \), so that the tangent plane at a boundary point over such a point has the equation \( t = t_0 \) with \( |t_0| = c < 1 - ar_0^2 \). But over a point \( z \) in \( \omega \) close to \( r_0 \) we have \( R(z) = R_0(z) > c \), so the tangent plane \( t = t_0 \) cuts \( \Omega \). This proves that \( \Omega \) cannot be lineally convex.

### 9.4.9 Hartogs domains over a disk

The Behnke–Peschl differential condition over a disk remains to be studied. We shall see that it is then equivalent to lineal convexity.

We shall write \( D(c, r) \) for the open disk in the complex plane with center \( c \) and radius \( r \), and just \( D \) for the open unit disk \( D(0, 1) \).

**Proposition 9.4.38** Let \( h \in C^2(D) \), \( h > 0 \), be a real-valued function which satisfies the Behnke–Peschl differential condition

\[
\frac{|h_z|^2}{h} \geq h_{zz} + |h_{zz}|, \quad |z| < 1. \tag{9.59}
\]

Let \( \varphi \in C^2(\mathbb{R}) \) be real-valued, decreasing and satisfy \( \varphi \leq 1 \) everywhere and \( \varphi'' < 0 \) wherever \( \varphi < 1 \). Assume that there are constants \( a \) and \( A \) such that

\[
\text{Re} \left[ \frac{2zh_z(z)}{h(z)} \right] \leq a < 1 \tag{9.60}
\]

and

\[
\left| \frac{2zh_z(z)}{h(z)} \right| \leq A < +\infty \tag{9.61}
\]

wherever \( 0 < \varphi(z \bar{z}) < 1 \). Then \( g(z) = \varphi(z \bar{z})h(z) \) satisfies the differential condition wherever \( \varphi(z \bar{z}) > 0 \) and \( |z| < 1 \), provided \( \varphi'/\varphi'' \) is small enough, more precisely if either \( A \leq 1 \) or else

\[
\frac{\varphi'(s)}{s\varphi''(s)} \leq \frac{2(1 - a)}{A^2 - 1} \quad \text{when } s \text{ is such that } 0 < \varphi(s) < 1.
\]
Proof With \( g(z) = \varphi(z)h(z) \) we have
\[
g_z = \varphi' \cdot \overline{z}h + \varphi h_z, \\
g_{zz} = \varphi'' \cdot \overline{z}^2 h + 2 \varphi' \overline{z} h_z + \varphi h_{zz}, \\
g_{z\overline{z}} = \varphi'' \cdot |z|^2 h + \varphi' \overline{h} + 2 \varphi \Re z h_z + \varphi h_{z\overline{z}}.
\]
Thus what we have to prove is, writing \( r \) for \(|z|\),
\[
\frac{|\varphi' \overline{z} + \varphi h_z|^2}{\varphi h} \geq r^2 \varphi'' h + \varphi' h + 2 \varphi' \Re z h_z + \varphi h_{z\overline{z}} + |\varphi'' \overline{z}^2 h + 2 \varphi' \overline{z} h_z + \varphi h_{z\overline{z}}|.
\]
We expand the left-hand side and find that the term \( 2\varphi' \Re z h_z \) appears on both sides. We shall therefore prove
\[
\frac{r^2 \varphi'' h}{\varphi} + \frac{\varphi h_z^2}{h} \geq r^2 \varphi'' h + \varphi' h + \varphi h_{z\overline{z}} + |\varphi'' \overline{z}^2 h + 2 \varphi' \overline{z} h_z + \varphi h_{z\overline{z}}|.
\]
This formula follows from \(|h_z|^2/h \geq h_{z\overline{z}} + |h_{zz}|\), which holds by hypothesis, and
\[
\frac{r^2 \varphi'' h}{\varphi} \geq r^2 \varphi'' h + \varphi' h + |\varphi'' \overline{z}^2 h + 2 \varphi' \overline{z} h_z + \varphi h_{z\overline{z}}|,
\]
which we shall prove now. We divide both sides of this inequality by the positive quantity \(-r^2 \varphi'' h\) (if \( \varphi'' \) is zero there is nothing to prove), and find the equivalent inequality
\[
-\frac{\varphi'^2}{\varphi''} \geq -1 - \frac{\varphi'}{r^2 \varphi''} + \left| -\frac{\pi^2}{r^2} - 2 \frac{\varphi' \overline{z} h_z}{r^2 \varphi'' h} \right| = -1 - \frac{\varphi'}{r^2 \varphi''} + \left| 1 + \frac{\varphi'}{r^2 \varphi''} \frac{2zh_z}{h} \right|.
\]
Since \(-\frac{\varphi'^2}{\varphi''}\) is positive, it suffices to prove that
\[
1 + t \geq |1 + tw| \quad \text{when} \quad t = \frac{\varphi'(r^2)}{r^2 \varphi''(r^2)} \quad \text{and} \quad w = \frac{2zh_z(z)}{h(z)}.
\]
This inequality, in turn, follows from
\[
(1 + t)^2 \geq |1 + tw|^2 = 1 + 2\Re w + t^2|w|^2,
\]
which holds as soon as \(2 + t \geq 2\Re w + t|w|^2\). By hypothesis \(\Re w \leq a < 1\) and \(|w| \leq A\), so (9.62) follows as soon as either \(A \leq 1\) or else \(A > 1\) and \(t \leq 2(1 - a)/(A^2 - 1)\). This proves the proposition.

Example 9.4.39 As an example of the function \( \varphi \) in Proposition 9.4.38 we let \( s_0 \) be an arbitrary number such that \(0 < s_0 < 1\) and take a smooth function \( \varphi \) satisfying \( \varphi(s) = 1 \) for \( s \leq s_0 \) and whose derivative is \( \varphi'(s) = -C \exp(-1/(s-s_0)) \) for \( s > s_0 \). Then we determine \( C \) to make \( \varphi(1) = 0 \); this means that we choose \( C \) to satisfy
\[
C \int_{s_0}^{1} e^{-1/(s-s_0)}ds = 1.
\]
Lineally Convex Hartogs Domains

We note that \( \phi'(s)/s \phi''(s) = (s-s_0)^2/s \), which varies between 0 and \((1-s_0)^2\). Thus if \(1-s_0\) is small enough, we can conclude that the new function \(\phi(z)h(z)\) satisfies the Behnke–Peschl differential condition (9.59) over the open unit disk and it agrees with \(h\) when \(|z| \leq \sqrt{s_0}\).

\[\square\]

We need to study condition (9.60) more closely. In fact it has a simple geometric meaning.

**Definition 9.4.40** Let a complete Hartogs domain

\[\Omega = \{(z, t) \in \omega \times \mathbb{C}; \ |t|^2 < h(z)\}\]

be defined over a bounded domain \(\omega\) in \(\mathbb{C}\) by a function \(h \in C^2(\omega), \ h > 0\). Denote by \((b(z), 0)\) the point at which the tangent at a point \((z, t) \in \partial \Omega\) with \(z \in \omega\) intersects the plane \(t = 0\) (put \(b(z) = \infty\) if there is no such point in \(\mathbb{C}\)).

We shall say that \(\Omega\) satisfies the tangent condition if

\[\inf_{z \in \omega} d(b(z), \omega) > 0, \quad (9.63)\]

where \(d\) denotes the distance from a point to a set.

If \(\Omega\) is defined by a function \(h \geq c > 0\) and is lineally convex, then it must satisfy the tangent condition, but not only that—we can deduce important quantitative information from its lineal convexity:

**Lemma 9.4.41** Let \(R \in C^1(\omega)\) be such that the set \(\Omega\) defined by (9.58) is lineally convex. Then

\[\inf_{z \in \omega} d(b(z), \omega) \geq \inf_{z \in \omega} \frac{R}{2\sup_{\omega} |R_z|} \geq \inf_{z \in \omega} \frac{h}{\sup_{\omega} |h_z|}. \quad (9.64)\]

If \(R \geq c > 0\) in \(\omega\), then \(\Omega\) satisfies the tangent condition (9.63).

**Proof** The tangent plane at a point \((z_0, t_0) \in \partial \Omega\) with \(z_0 \in \omega\) is given by equation (9.46), and \(b(z)\) is given by equation (9.47). The equation for the tangent can also be written as \(t = \alpha(z - b(z_0))\). If \(\Omega\) is lineally convex, then this tangent cannot intersect \(\Omega\), so we must have \(|t| \geq R(z)\) whenever \(z, z_0 \in \omega\).

Thus

\[|t| = |\alpha(z - b(z_0))| \geq R(z)\]

for all \(z, z_0 \in \omega\);

inserting the value of \(\alpha = 2|R_z(z_0)| = |h_z(z_0)|/\sqrt{h(z_0)}\) we obtain

\[|z - b(z_0)| \geq \frac{R(z)}{2|R_z(z_0)|} = \frac{\sqrt{h(z_0)|h_z(z_0)|}}{\sqrt{h(z_0)}}\]

We now let \(z, z_0\) vary in \(\omega\) to get the desired conclusion.

The idea is to prove that the tangent condition is not only necessary as in Lemma 9.4.41, but also sufficient if \(\omega\) is a disk, which we shall do in Proposition 9.4.42. We then proceed to prove that \(\Omega\) does satisfy the tangent condition under our hypotheses if \(\omega\) is a disk.
Proposition 9.4.42 Assume that $h \in C^2(D)$, $h > 0$, satisfies the Behnke–Peschl differential condition (9.59) and that $\Omega$ satisfies the tangent condition. Let $\varphi$ be the function constructed in Example 9.4.39. Then $\varphi(z\overline{z})h(z)$ satisfies the differential condition if $s_0$ is sufficiently close to 1. Therefore, by Theorem 9.4.33, the open set $\{(z, t) \in D \times C; |t|^2 < \varphi(z\overline{z})h(z)\}$, which has a $C^2$ boundary, is lineally convex; as a consequence also its limit as $s_0$ tends to 1, viz. $\Omega$ itself, is lineally convex.

Proof Using formula (9.47) for $b(z)$, the relation between the inequality (9.60) used in the proof of Proposition 9.4.38 and the tangent condition is easy to establish. We observe that $|b(z)| = |z - h(z)/h(z)| > |z|$ if and only if $\text{Re} 2zh_z(z)/h(z) < 1$. Thus if $\Omega$ satisfies the tangent condition, then $h$ satisfies (9.60) for some $a < 1$ and all $z$ in some sufficiently narrow annulus $\sqrt{s_0} \leq |z| \leq 1$.

Define

$$A = \sup_{|z| \leq 1} \left| \frac{2zh_z(z)}{h(z)} \right| \quad \text{and} \quad a(s_0) = \sup_{\sqrt{s_0} \leq |z| \leq 1} \text{Re} \left[ \frac{2zh_z(z)}{h(z)} \right].$$

If $A \leq 1$ we are done; otherwise we can choose $s_0 < 1$ so close to 1 that $(1 - s_0)^2 \leq 2(1 - a(s_0))/(A^2 - 1)$. Proposition 9.4.38 can be applied and shows that $\varphi(z\overline{z})h(z)$ satisfies the differential condition.

We shall now prove that it can never happen that $\text{Re} 2zh_z(z)/h(z) \geq 1$ for any $z$ with $|z| \leq 1$.

Proposition 9.4.43 If $h \in C^2(D)$, $h > 0$, satisfies the Behnke–Peschl differential condition (9.59), then $\Omega$ satisfies the tangent condition (9.63).

Proof Let us define

$$b_0(r) = \inf_{|z| \leq r} |b(z)|, \quad 0 < r \leq 1.$$

This is a decreasing function and it is continuous where it is finite. The tangent condition for $\Omega_r = \{(z, t) \in D(0, r) \times C; |t|^2 < h(z)\}$ means precisely that $b_0(r) > r$. It is clear that the condition is satisfied for a very small $r$. Indeed, $b(0) = -h(0)/h_z(0)$ is either $\infty$ or a non-zero complex number; in view of the continuity, $|b(z)| > r$ if $|z| \leq r$ and $r$ is small enough.

If the tangent condition is satisfied for a particular $\Omega_r$, then by Proposition 9.4.42 the set $\Omega_r$ is lineally convex, so Lemma 9.4.41 can be applied and shows that $b_0(r) > r + \varepsilon$, where

$$\varepsilon = \frac{\inf_{|z| \leq 1} R}{2 \sup_{|z| \leq 1} |R_z|} > 0.$$

Here we could remark that it would be enough to require that $b(z) \notin \mathbb{D}$ only for all $z \in \partial \omega$, supposing that $h \in C^2(\overline{\omega})$. The stronger condition used in Definition 9.4.40 is however easier to handle in the proof of Proposition 9.4.43.
We know that \( b_0(r) > r \) for small values of \( r \), and we have just seen that if \( b_0(r) > r \), then also \( b_0(r) \geq r + \varepsilon \), for a positive \( \varepsilon \) that does not depend on \( r \). Therefore that function cannot assume any value in the interval \( ]r, r + \varepsilon[ \): it must satisfy \( b_0(r) > r \) all the way up to and including \( r = 1 \). This means that \( \Omega \) satisfies the tangent condition.

**Theorem 9.4.44** Let \( h \in C^2(D) \), \( h > 0 \), satisfy the Behnke–Peschl differential condition \( (9.59) \). Then the open set \( \Omega = \{(z, t) \in D \times \mathbb{C}; \ |t|^2 < h(z)\} \) is lineally convex.

**Proof** If \( h \in C^2(\overline{D}) \) with \( h > 0 \) in \( \overline{D} \) we see from Proposition 9.4.43 that \( \Omega \) satisfies the tangent condition, so that Proposition 9.4.42 can be applied. In the general case with \( h \in C^2(D) \), \( h > 0 \), we apply this result to a smaller disk \( rD, r < 1 \), to conclude that the domain over \( rD \) is lineally convex. Then we let \( r \to 1 \).
9.5Weak Lineal Convexity

We start this section with a general presentation of weak lineal convexity. We then discuss local variants of this property.

A locally weakly lineally convex open set with boundary of class $C^1$ is also (globally) weakly lineally convex provided that it is bounded. But, as shown by Yuri˘ı Zelinski˘ı, this is not true for unbounded domains. The purpose here is to construct explicit examples, Hartogs domains, showing this. Their boundary can have regularity $C^{1,1}$ or $C^\infty$.

Obstructions to constructing smoothly bounded domains with certain homogeneity properties will be discussed.

9.5.1 Introduction

After the main definitions about variants of lineal convexity, we shall approach the comparison global vs. local. In my paper (1998) I proved that a differential condition that I called the Behnke–Peschl differential condition implies that a bounded and connected open subset of $\mathbb{C}^n$ with boundary of class $C^2$ is weakly lineally convex. The proof relied on a result by Yužakov and Krivokolesko (1971a, 1971b), proved also in (Hörmander 1994: Proposition 4.6.4).

Yuri˘ı Zelinskij (2002a, 2002b) published an example of an unbounded set that is locally lineally convex but not lineally convex. His example is not very explicit. We shall construct here an explicit example—actually a Hartogs domain, which has the advantage of being easily visualized in three real dimensions. We construct domains with boundary of class $C^{1,1}$ and a certain homogeneity property (Example 9.5.13), and show that this cannot be done with a boundary of class $C^2$ (Proposition 9.5.18). However, the boundary can be of class $C^\infty$ if the homogeneity requirement is dropped (Example 9.5.14).

9.5.2 Lineal convexity

The property of being lineally convex was defined in Definition 9.4.1 on page 279. To wit:

**Definition 9.5.1** A subset of $\mathbb{C}^n$ is said to be lineally convex if its complement is a union of complex affine hyperplanes. □

To every set $A$ there exists a smallest lineally convex subset $\mu(A)$ that contains $A$. Clearly the mapping $\mu: \mathcal{P}(\mathbb{C}^n) \to \mathcal{P}(\mathbb{C}^n)$, where $\mathcal{P}(\mathbb{C}^n)$ denotes the family of all subsets of $\mathbb{C}^n$ (the power set), is increasing and idempotent, in other words an ethnomorphism (morphological filter). It is also larger than the identity, so that $\mu$ is a cleistomorphism (closure operator) in the ordered set $\mathcal{P}(\mathbb{C}^n)$. 
This kind of complex convexity was introduced by Heinrich Behnke (1898–1997) and Ernst Ferdinand Peschl (1906–1986). I learnt about it from André Martineau (1930–1972) when I was in Nice during the academic year October 1967 through September 1968. See Martineau’s papers (1966, 1967, 1968), also in (Œuvres de André Martineau 1977).

Are there lineally convex sets which are not convex? This is obvious in one complex variable, and from there we can easily construct, by taking Cartesian products, lineally convex sets in any dimension that are not convex. But these sets do not have smooth boundaries. Hörmander (1994:293, Remark 3) constructs open connected sets in \( \mathbb{C}^n \) with boundary of class \( C^2 \) as perturbations of a convex set. These sets are lineally convex and close to a convex set in the \( C^2 \) topology, and therefore starshaped with respect to some point if the perturbation is small. Also the symmetrized bidisk

\[
\{(z_1 + z_2, z_1z_2) \in \mathbb{C}^2; |z_1|, |z_2| < 1\},
\]

studied by Agler & Young (2004) and Pflug & Zwonek (2012), is not convex—not even biholomorphic to a convex domain (Nikolov et al. (2008)—but it is starshaped with respect to the origin (Agler & Young 2004: Theorem 2.3). So we may ask:

**Question 9.5.2** Does there exist a lineally convex set in \( \mathbb{C}^n, n \geq 2 \), with smooth boundary that is not starshaped with respect to any point? □

We shall return to this question in Subsection 9.5.10.

### 9.5.3 Weak lineal convexity

**Definition 9.5.3** An open subset \( \Omega \) of \( \mathbb{C}^n \) is said to be **weakly lineally convex** if there passes, through every point on the boundary of \( \Omega \), a complex affine hyperplane which does not cut \( \Omega \). □

It is clear that every lineally convex open set is weakly lineally convex. The converse does not hold. This is not difficult to see if we allow sets that are not connected:

**Example 9.5.4** Given a number \( c \) with \( 0 < c < 1 \), define an open set \( \Omega_c \) in \( \mathbb{C}^2 \) as the union of the set

\[
\{z = (x_1 + iy_1, x_2 + iy_2) \in \mathbb{C}^2; c < |x_1| < 1, |y_1| < 1, |x_2| < c, |y_2| < c\}
\]

with the two sets obtained by permuting \( x_1, x_2 \) and \( y_2 \). Thus \( \Omega_c \) consists of six boxes. It is easy to see that it is weakly lineally convex, but there are many points in its complement such that every complex line passing through that point hits \( \Omega_c \).

Any complex line intersects the real hyperplane defined by \( y_1 = 0 \) in the empty set or in a real line or in a real two-dimensional plane, and the three-dimensional set \( \{z; y_1 = 0\} \cap \Omega_c \) is easy to visualize. □
It is less easy to construct a connected set with these properties, but this has been done by Yužakov & Krivokolesko (1971b:325, Example 2). See also an example due to Hörmander in the book by Andersson, Passare & Sigurdsson (2004:20–21, Example 2.1.7).

However, the boundary of the constructed set is not of class $C^1$, and this is essential. Indeed, Yužakov & Krivokolesko (1971b:323, Theorem 1) proved that a connected bounded open set with “smooth” boundary is locally weakly lineally convex in the sense of Definition 9.5.8 below if and only if it is lineally convex. It is then even $C$-convex (1971b:324, Assertion). See also Corollary 4.6.9 in (Hörmander 1994), which states that a connected bounded open set with boundary of class $C^1$ is locally weakly lineally convex if and only if it is $C$-convex (and every $C$-convex open set is lineally convex).

There cannot be any cleistomorphism connected with the notion of weak lineal convexity for the simple reason that the property is defined only for open sets. We might therefore want to define weak lineal convexity for arbitrary sets. We may ask:

**Question 9.5.5** Is there a reasonable definition of weak lineal convexity for all sets which keeps the definition for open sets and is such that there is a cleistomorphism associating to any $A \subset \mathbb{C}^n$ the smallest set that contains $A$ and is weakly lineally convex? □

**Question 9.5.6** The operation $L \mapsto L \cap \Omega$ associating to a complex line $L$ its intersection with an open set $\Omega$ has continuity properties that seem to be highly relevant for weak lineal convexity. Here the family of complex lines can arguably have only one topology, but for the family of sets $L \cap \Omega$ there is a choice of several topologies, especially if $\Omega$ is unbounded.

Can an interesting theory be built starting from this remark?

### 9.5.4 Local weak lineal convexity

**Definition 9.5.7** We shall say that an open set $\Omega \subset \mathbb{C}^n$ is **locally weakly lineally convex** if for every point $p$ there exists a neighborhood $V$ of $p$ such that $\Omega \cap V$ is weakly lineally convex. □

Obviously, a weakly lineally convex open set has this property, but the converse does not hold, which is obvious for sets that are not connected: Take the union of two open balls whose closures are disjoint. Also for connected sets the converse does not hold as we showed in Example 9.4.8. In that example it is essential that the boundary is not smooth.

Zelinskij (1993:118, Example 13.1) constructs an open set which is locally weakly lineally convex but not weakly lineally convex. The set is not equal to the interior of its closure.
Definition 9.5.8 Let us say that an open set $\Omega$ is locally weakly lineally convex in the sense of Yužakov and Krivokolesko (1971b:323) if for every boundary point $p$ there exists a complex hyperplane $Y$ passing through $p$ and a neighborhood $V$ of $p$ such that $Y$ does not meet $V \cap \Omega$. □

Zelinskij (1993:118, Definition 13.1) uses this definition and calls the property локальная линейная выпуклость (lokal'naia linejnaja vypuklost’). As we shall see, this property is strictly weaker than the local weak lineal convexity defined above in Definition 9.5.7.

Hörmander (1994:Proposition 4.6.4) and Andersson, Passare & Sigurdsson (2004: Proposition 2.5.8) use this property only for open sets with boundary of class $C^1$. Then the hyperplane $Y$ is unique.

For all open sets, local weak lineal convexity obviously implies local weak lineal convexity in the sense of Yužakov and Krivokolesko. In the other direction, Hörmander’s Proposition 4.6.4 shows that for bounded open sets with boundary of class $C^1$, local weak lineal convexity in the sense of Yužakov and Krivokolesko implies local weak lineal convexity (even weak lineal convexity if the set is connected).

Nikolov (2012:Proposition 3.7.1) and Nikolov et al. (2013:Proposition 3.3) have a local result in the same direction: If $\Omega$ has a boundary of class $C^k$, $2 \leq k \leq \infty$, and $\Omega \cap B_{<}(p,r)$, where $p$ is a given point, is locally weakly lineal convex in the sense of Yužakov and Krivokolesko at all points near $p$, then there exists a $C$-convex open set $\omega$ (hence lineally convex) with boundary of class $C^k$ such that $\omega \cap B_{<}(p,r') = \Omega \cap B_{<}(p,r')$ for some positive $r'$.

However, in general, the two properties are not equivalent:

Example 9.5.9 The bounded connected open subset $\Omega'$ of $C^2$, taking $r = 2$, which was defined in Example 9.4.8 on page 281, has Lipschitz boundary and is locally weakly lineally convex in the sense of Yužakov and Krivokolesko but not locally weakly lineally convex. While $\Omega'$ is locally weakly lineally convex for $2 < r < \sqrt{5}$, the set $\Omega^2$ is not locally weakly lineally convex: The point $(0,2)$ does not have a neighborhood with the desired property. But it does satisfy the property of Yužakov and Krivokolesko. □

9.5.5 Approximation by smooth sets

Let $A_j \subset C^{n_j}$ be two lineally convex sets in $C^{n_j}$, $j = 1,2$. Then it is easy to see that their Cartesian product $A_1 \times A_2 \subset C^{n_1+n_2}$ is lineally convex. In particular, if $n_1 = n_2 = 1$, then every Cartesian product in $C^2$ is lineally convex. However, these sets cannot always be approximated by lineally convex sets with smooth boundaries.

If $\Omega_j$, $j = 1,2$, are convex open sets, then $\Omega = \Omega_1 \times \Omega_2$ is convex and can be approximated from within by convex open set $\Omega_{[\varepsilon]}$ with $C^\infty$ boundaries, $\Omega_{[\varepsilon]} \nearrow \Omega$ as $\varepsilon \searrow 0$. 
But if we let $\Omega_1$ be an annulus and $\Omega_2$ a disk, e.g.,

$$\Omega = \Omega_1 \times \Omega_2 = \{ z \in \mathbb{C}^2; 1 < |z_1| < 3, |z_2| < 1 \},$$

then it cannot be approximated by smooth weakly convex sets from the inside as we shall see in the next proposition and its corollary.

**Proposition 9.5.10** Let $\omega$ be a nonempty bounded open subset of $\mathbb{R}^2$ with boundary of class $C^1$. Suppose that $\inf_{x \in \omega} |x_1| > 0$. Define a Reinhardt domain $\Omega$ as

$$\Omega = \{ z \in \mathbb{C}^2; (|z_1|, |z_2|) \in \omega \}.$$  

Then $\Omega$ is not locally weakly lineally convex.

**Proof.** Take a point $q = (q_1, q_2) \in \Omega$ with $q_1 < 0$, $q_2 \geq 0$. Denote by $\Omega^+$ the set of all $x \in \Omega$ such that $x_1 > 0$ and $x_2 > 0$, and by $L_\alpha$ the complex line of equation $z_2 - q_2 = \alpha(z_1 - q_1)$, $\alpha > 0$. For $\alpha = 0$, the line cuts $\Omega^+$ in $(-q_1, q_2)$; for large $\alpha$ it does not cut $\Omega^+$. Now choose the smallest $\alpha$ such that $L_\alpha$ does not cut $\Omega^+$. Then $L_\alpha$ contains at least one point $p \in \partial \Omega^+$, and $L_\alpha$ is the tangent plane of $\partial \Omega$ at $p$. Since this line meets $\Omega$ in $q$, $\Omega$ is not weakly lineally convex. But we can say more: It is not even locally weakly lineally convex. To see this, first note that $\alpha = (p_2 - q_2)/(p_1 - q_1) > 0$. Then there are points $z \in \Omega$ belonging to the tangent at $p$ arbitrarily close to $p$. Indeed, since $\alpha$ is positive, a point $z$ satisfying

$$|z_1| > p_1 \text{ and } |z_2| < p_2$$

belongs to $\Omega$ if it is close enough to $p$. In terms of $z_1$ this means that

$$|z_1| > p_1 \text{ and } |q_2 + \alpha(z_1 - q_1)| < p_2;$$

in other words that $z_1 \notin D_{\leq}(0, p_1)$ and that $z_1 \in D_{<}(c_1, r_1)$, the open disk with center at $c_1 = q_1 - q_2/\alpha$ and radius $r_1 = p_2/\alpha = p_1 - c_1$. Since $r_1 = p_1 - c_1$, there are points $z_1 \in D_{<}(c_1, r_1) \setminus D_{\leq}(0, p_1)$ which are arbitrarily close to $p_1$.

**Corollary 9.5.11** A Reinhardt domain

$$\Omega = \{ z \in \mathbb{C}^2; r_1 < |z_1| < R_1, |z_2| < R_2 \}$$

with $r_1 > 0$ is lineally convex but cannot be approximated by lineally convex domains with boundary of class $C^1$.

**Proof.** If a domain $\Omega_{\epsilon}$ approximates $\Omega$ from the inside in the sense that

$$\Omega_{\epsilon} \subset \Omega \subset \Omega_{\epsilon} + B_{\leq}(0, \epsilon),$$

then there is also a Reinhardt domain with this property: We may construct such a set by averaging over all rotations.

We can now apply the proposition to $\Omega_{\epsilon}$. □
9.5.6 The Behnke–Peschl and Levi conditions

We refer to Subsection 9.4.5 for the definitions of the real and complex Hessian, the Levi form as well as the Levi condition and the strong Levi condition. Moreover, we defined there the Behnke–Peschl differential condition and the strict Behnke–Peschl differential condition.

The Behnke–Peschl differential condition says that the restriction of the real Hessian to the complex tangent space at any boundary point shall be positive semidefinite; for the strong case, positive definite.

Because of the different homogeneity of $H^C$ and $L$, the inequality $\text{Re}H^C + L \geq 0$ is equivalent to $L \geq |H^C|$. The inequality $L \geq |H^C| \geq 0$ shows that the Behnke–Peschl differential condition implies the Levi condition.

In my paper (1998) I proved that a bounded connected open set with boundary of class $C^2$ is weakly lineally convex if it satisfies the Behnke–Peschl differential condition.

That this condition is necessary for weak lineal convexity was known since Behnke and Peschl (1935); the sufficiency was unknown at the time.

9.5.7 Yužakov and Krivokolesko: Passage from local to global

Let us quote the part of Proposition 4.6.4 in (Hörmander 1994) which is important for us:

**Proposition 9.5.12** Let $\Omega \subset \mathbb{C}^n$ be a bounded connected open set with boundary of class $C^1$ and assume that $\Omega$ is locally weakly lineally convex in the sense of Yužakov and Krivokolesko. Then $\Omega$ is weakly lineally convex. □

The result was proved by Yužakov & Krivokolesko (1971a, 1971b) under the condition that the boundary is “smooth.”

9.5.8 A new example

We shall construct explicit Hartogs domains here with the properties mentioned in Zelinskij’s example. We start with the simplest.

**Example 9.5.13** Define a function $\varphi^\circ : \mathbb{C} \to \mathbb{R}$ by

$$
\varphi^\circ(z_1) = \begin{cases} 
-x_1^2 - y_1^2, & x_1 \leq 0 \text{ or } y_1 \leq 0; \\
-x_1^2 + y_1^2, & 0 \leq y_1 \leq x_1; \\
x_1^2 - y_1^2, & 0 \leq x_1 \leq y_1.
\end{cases}
$$

Then $\Omega_{\varphi^\circ} = \{z \in \mathbb{C}^2; 1 + \varphi^\circ(z_1) + |z_2|^2 < 0\}$ has boundary of class $C^{1,1}$ and is locally weakly lineally convex but not weakly lineally convex. □
FIGURE 9.2
The set $\Omega_{\varphi^0} \cap \{ z \in \mathbb{C}^2; \; z_2 = 0 \}$.

The properties of the set in this example will be discussed now and the properties will be seen to hold from Proposition 9.5.15.

The tangent plane at a boundary point $p = (p_1, p_2)$ with $\text{Re} \; p_1 > 0$, $\text{Im} \; p_1 > 0$, and $(\text{Re} \; p_1)^2 > (\text{Im} \; p_1)^2 + 1$, has the equation $-p_1(z_1 - p_1) + p_2(z_2 - p_2) = 0$ and it passes through the point $q = (p_1 - |p_2|^2/p_1, 0)$. Choosing $p = (3 + i, \sqrt{7})$ we get $q = (\frac{3}{10} + \frac{13}{10}i, 0) \in \Omega_{\varphi^0}$, proving that $\Omega_{\varphi^0}$ is not lineally convex.

We note that the tangent plane at a boundary point $p$ with $\text{Re} \; p_1 \leq 0$ or $\text{Im} \; p_1 \leq 0$ is contained in the complement of $\Omega_{\varphi^0}$; in particular, it hits the plane $z_2 = 0$ at the point $q = (p_1/|p_1|^2, 0) \notin \Omega_{\varphi^0}$. We also note that the part of $\Omega_{\varphi^0}$ where $0 < x_1 < y_1$ is convex, so any tangent plane of this part does not intersect it. Similarly, the part where $0 < y_1 < x_1$ is convex. Therefore $\Omega^0_{\varphi}$ is the union of two lineally convex sets, taking the subsets where $x_1 < \max(y_1, 0)$, and $y_1 < \max(x_1, 0)$, respectively.

When $x_1 < 0$ or $y_1 < 0$ we get $\varphi^0_{z_1}(z_1) = -z_1$, $\varphi^0_{z_1\bar{z}_1}(z_1) = 0$, $\varphi^0_{z_2\bar{z}_1}(z_1) = -1$; when $0 < y_1 \leq x_1$ we have $\varphi^0_{z_2}(z_1) = -z_1$, $\varphi^0_{z_2\bar{z}_1}(z_1) = -1$ and $\varphi^0_{z_2\bar{z}_1}(z_1) = 0$; when $0 < x_1 \leq y_1$ we have $\varphi^0_{z_1}(z_1) = z_1$, $\varphi^0_{z_1\bar{z}_1}(z_1) = 1$ and $\varphi^0_{z_1\bar{z}_1}(z_1) = 0$. In all three cases $|\varphi_{z_1\bar{z}_1}| - \varphi_{z_1\bar{z}_1} = 1$. An application of Proposition 9.5.15 below now gives the result, except that it does not give anything at the exceptional points, where the function is not of class $C^\infty$, i.e., those with $y_1 = 0$, $x_1 > 0$ or $x_1 = 0$, $y_1 > 0$. However, we have already seen that at these points, the tangent plane does not cut $\Omega_{\varphi^0}$.

The boundary of $\Omega_{\varphi^0}$ is of class $C^2$ at the points where $y_1 = 0$, $x_1 > 0$ or $x_1 = 0$, $y_1 > 0$. The passage from $-x_1^2 - y_1^2$ for $y_1 \leq 0$ to $-x_1^2 + y_1^2$ for $0 \leq y_1 \leq x_1$ cannot be made analytically.
Weak Lineal Convexity

The function $\varphi^o$ is not of class $C^{1,1}$ at the points where $x_1 = y_1$, $x_1 > 0$, but this is of no consequence, since these points do not belong to the closure of the set it defines.

We note that the function $\varphi^o$ in the example is homogeneous of degree two:

$$\varphi^o(z_1) = \varphi^o(|z_1|e^{it}) = |z_1|^2 \psi(t), \quad z_1 \in \mathbb{C}, \ t \in \mathbb{R}.$$ 

It is therefore natural to ask if there is a $C^\infty$ homogeneous function $\varphi$ with the same properties. More precisely, we may ask for functions $\varphi : \mathbb{C} \to \mathbb{R}$ which yield a locally weakly lineally convex domain that is not weakly lineally convex in four different cases.

1.1. Is there a $C^\infty$ function $\varphi$ with these properties?

1.2. Is there a homogeneous $C^\infty$ function $\varphi$ with these properties?

2.1. Is there an analytic function $\varphi$ with these properties?

2.2. Is there a homogeneous analytic function $\varphi$ with these properties?

As we shall see, the answer to the first question is in the affirmative (Example 9.5.14). But the answer to Question 1.2 is in the negative (Proposition 9.5.18).

**Example 9.5.14** Now define $\varphi^* : \mathbb{C} \to \mathbb{R}$ by

$$\varphi^*(z_1) = \begin{cases} -x_1^2 + \chi(y_1), & x_1 \geq y_1; \\ -y_1^2 + \chi(x_1), & x_1 \leq y_1, \end{cases}$$

where $\chi \in C^\infty(\mathbb{R})$ is a function of one real variable such that $\chi'$ is convex and which satisfies

$$\chi(y_1) = \begin{cases} -y_1^2 + \rho, & y_1 \leq -\frac{1}{2}; \\ y_1^2 + \sigma, & y_1 \geq \frac{1}{2}. \end{cases}$$

The convexity of $\chi'$ implies that $2|y_1| \leq |\chi'(y_1)| \leq \max(2|y_1|, 1)$ with equality to the left for $|y_1| \geq \frac{1}{2}$. This implies that we must have $\frac{1}{2} < \chi(\frac{1}{2}) - \chi(-\frac{1}{2}) < 1$, and we can actually choose $\chi$ so that $\chi(\frac{1}{2}) - \chi(-\frac{1}{2})$ is any given number in that interval.

For definiteness we now choose $\rho = -\frac{1}{2}$, $\sigma = 0$, $\chi(\frac{1}{2}) - \chi(-\frac{1}{2}) = \frac{1}{2}$, which implies that $\varphi^o - \frac{1}{4} \leq \varphi^* \leq \varphi^o$, that $\Omega_{\varphi^*}$ contains $\Omega_{\varphi^o}$, and that the set of points $z \in \Omega_{\varphi^*}$ with $\text{Re} \ z_1 \geq \frac{1}{2}$ and $\text{Im} \ z_1 \geq \frac{1}{2}$ is unchanged compared to $\Omega_{\varphi^o}$.

We choose $\chi$ as a suitable third primitive of

$$\chi''(y_1) = C \exp(1/(y_1 - c) - 1/(y_1 + c)), \quad -c < y_1 < c,$$

for a number $c$, $0 < c \leq \frac{1}{2}$, and a positive constant $C$, taking $\chi''(y_1)$ equal to zero when $|y_1| \geq c$. This implies that $\chi'$ is even and that $\chi(0) = \frac{1}{2} \rho + \frac{1}{2} \sigma = -\frac{1}{8}$.

Then

$$\Omega_{\varphi^*} = \{ z \in \mathbb{C}^2; \ 1 + \varphi^*(z_1) + |z_2|^2 < 0 \}.$$
has boundary of class $C^\infty$ and is locally weakly lineally convex but not lineally convex, since, just as for $\Omega_{\varphi^*}$, the tangent plane at the boundary point $p = (3 + i, \sqrt{7})$ passes through $q = (\frac{9}{11}, \frac{17}{11}, 0) \in \Omega_{\varphi^*}$. □

The properties mentioned in these two examples will follow from the next proposition and its corollary.

**Proposition 9.5.15** Let $\varphi : \mathbb{C} \to \mathbb{R}$ be a function of class $C^k$, $k = 2, 3, \ldots, \infty, \omega$ ($C^\omega$ denoting the family of all real analytic functions), and define an open set in $\mathbb{C}^2$ as

$$\Omega_{\varphi} = \{ z \in \mathbb{C}^2; 1 + \varphi(z_1) + |z_2|^2 < 0 \}.$$

We assume that

$$\varphi_{z_1} \neq 0 \text{ wherever } \varphi = -1,$$

and that

$$(\varphi - 1)(|\varphi_{z_1}| - \varphi_{z_1}) \leq |\varphi_{z_1}|^2 \text{ in the set where } -\varphi - 1 \geq 0. \tag{9.66}$$

Then $\Omega_{\varphi}$ has boundary of class $C^k$ and satisfies the Behnke–Peschl differential condition at every boundary point. If the inequality is strict at a certain point, we get the strict Behnke–Peschl differential condition at that point.

**Proof** In Lemma 9.4.26 on page 288 I described the domain by an inequality of the form $|z_2|^2 < h(z_1)$ and found that the Behnke–Peschl differential condition takes the form $h(h_{z_1} + |h_{z_1}|) \leq |h_{z_1}|^2$, which, with $h(z_1) = -\varphi(z_1) - 1$, yields (9.66).

**Corollary 9.5.16** Let $\varphi$ have the form $\varphi(z_1) = -x_1^2 + \chi(y_1)$ for $x_1 \geq y_1$ and $\varphi(z_1) = -y_1^2 + \chi(x_1)$ for $y_1 \geq x_1$. We assume that $\chi \in C^k(\mathbb{R})$, $k \geq 2$, with $-2 \leq \chi''$ and such that $\chi(y_1) > -1$ when $\chi'(y_1) = 0$. Then the conclusion of Proposition 9.5.15 holds under the assumption

$$\frac{1}{4} \chi'(y_1)^2 + \chi(y_1) + 1 \geq 0, \quad y_1 \in \mathbb{R}. \tag{9.67}$$

**Proof** The condition (9.65) is satisfied, since the gradient of $\varphi$ in this case vanishes only when $x_1 = 0$ and $\chi'(y_1) = 0$. Then $1 + \varphi(z_1) + |z_2|^2 = 1 + \chi(y_1) + |z_2|^2 > 0$, so $z = (iy_1, z_2)$ cannot be a boundary point of $\Omega$.

Condition (9.66) reduces to

$$(x_1^2 - \chi(y_1) - 1)\left(\left|\frac{1}{2} - \frac{1}{4} \chi''(y_1)\right| + \frac{1}{2} - \frac{1}{4} \chi''(y_1)\right) \leq |x_1 - \frac{1}{2} i \chi'(y_1)|^2 = x_1^2 + \frac{1}{4} \chi'(y_1)^2,$$

provided $x_1^2 - \chi(y_1) - 1 \geq 0$. If $-2 \leq \chi''$, we have

$$\left|\frac{1}{2} - \frac{1}{4} \chi''(y_1)\right| + \frac{1}{2} - \frac{1}{4} \chi''(y_1) = \frac{1}{2} + \frac{1}{4} \chi''(y_1) + \frac{1}{2} = \frac{1}{2} \chi''(y_1) = 1,$$

which gives (9.67). We then see that in this case the inequality holds also if $x_1^2 - \chi(y_1) - 1 < 0$.

In Example 9.5.14, the defining function $1 - x_1^2 + \chi(y_1) + |z_2|^2$ has nonvanishing gradient everywhere since $\chi' > 0$ everywhere. Smoothness follows.
Weak Lineal Convexity

The function $\varphi^*$ is not of class $C^\infty$ in the set where $x_1 = y_1$, $x_1 > 0$, but again this is unimportant since these points do not belong to the closure of $\Omega_{\varphi^*}$. An application of Corollary 9.5.16 now gives the result. In fact, with the choice of $\rho = -\frac{1}{2}$, $\sigma = 0$, we need only note that $\chi(y_1) \geq -y_1^2 - \frac{1}{4}$ everywhere, and that $\chi'(y_1) \geq 2|y_1|$, so that

$$\frac{1}{4}\chi'(y_1)^2 + \chi(y_1) + 1 \geq \frac{3}{4} > 0, \quad x_1 \geq y_1,$$

thus with strict inequality in (9.67) and (9.66); similarly for $x_1 \leq y_1$.

**Remark 9.5.17** It might be of interest to understand where the proof of Hörmander’s Proposition 4.6.4 quoted above breaks down in the unbounded case. An important step in the proof is to see that, if we have a continuous family $(L_t)_{t \in [0,1]}$ of complex lines, the set $T$ of parameter values $t$ such that $L_t \cap \Omega$ is connected is both open and closed. Thus, if $0 \in T$, then also $1 \in T$. We shall see that closedness is no longer true for the sets in Examples 9.5.13 and 9.5.14.

Define complex lines

$$L_t = \{z \in \mathbb{C}^2; \quad z_2 = t(z_1 - 1 - i)\}, \quad t \in [0,1],$$

which all pass through $(1+i,0) \notin \Omega_{\varphi^*}$. Then $L_t \cap \Omega_{\varphi^*}$ is connected for $0 \leq t < 1$ while $L_1 \cap \Omega_{\varphi^*}$ is not.

We shall first see that $L_1 \cap \Omega_{\varphi^*}$ is disconnected. If $z \in L_1 \cap \Omega_{\varphi^*}$, and $x_1 \leq 0$ or $y_1 \leq 0$, then

$$f(z) = 1 + \varphi^*(z_1) + |z_2|^2 \geq 1 - |z_1|^2 + \rho + |z_1 - 1 - i|^2 = 3 + \rho - 2(x_1 + y_1).$$

Since we have chosen $\rho = -\frac{1}{4}$, the quantity $f(z)$ can be negative only if $x_1 + y_1 > 0$, which implies that $z_1$ satisfies either $x_1 > |y_1|$ or $y_1 > |x_1|$. Therefore the real hyperplane of equation $x_1 = y_1$ divides $L_1 \cap \Omega_{\varphi^*}$ into two sets, which are nonempty since $(2,1-i)$ and $(2i,1-i)$ both belong to $L_1$, the first with $y_1 < x_1$, the second with $y_1 > x_1$, and that both belong to $\Omega_{\varphi^*}$ in view of the fact that $\chi(0) < 0$.

Next we shall see that $L_t \cap \Omega_{\varphi^*}$ is connected when $0 \leq t < 1$. Given $t$ such that $0 \leq t < 1$, we obtain for $z \in L_t \cap \Omega_{\varphi^*}$ with $x_1, y_1 \leq 0$,

$$1 + \varphi^*(z_1) + |z_2|^2 \leq 1 - (1-t^2)|z_1|^2 - 2t^2(x_1 + y_1) + 2t.$$  

This yields the estimate

$$1 + \varphi^*(z_1) + |z_2|^2 \leq 3 + 4|z_1| - (1-t^2)|z_1|^2,$$

which is negative when $|z_1| = R_t$ for a large enough number $R_t$, which depends on $t$. Obviously $R_t$ tends to $+\infty$ as $t \to 1$, which explains that $L_1 \cap \Omega_{\varphi^*}$ is disconnected. Let $\Gamma_t$ be the arc in $L_t$ with $|z_1| = R_t$ and $x_1 \leq 0$ or $y_1 \leq 0$, thus contained in $\Omega_{\varphi^*}$.
An arbitrary point \( a \in L_t \cap \Omega_\varphi \) can be joined to a point in \( \Gamma_t \) by a straight-line segment contained in \( \Omega_\varphi \), and therefore also contained in \( L_t \cap \Omega_\varphi \). If \( \text{Re} \, a_1 \leq 0 \) or \( \text{Im} \, a_1 < 0 \) this follows from the fact that the set of points in \( \Omega_\varphi \) with argument of \( z_1 \) equal to that of \( a_1 \) is convex; otherwise from the fact that the points in \( \Omega_\varphi \) with \( 0 \leq \text{Im} \, z_1 < \text{Re} \, z_1 \) is convex, as is the set of points with \( 0 \leq \text{Re} \, z_1 < \text{Im} \, z_1 \).

\( \square \)

### 9.5.9 An impossibility result

**Proposition 9.5.18** Let \( \Omega_\varphi = \{ z \in \mathbb{C}^2; 1 + \varphi(z_1) + |z_2|^2 < 0 \} \), where \( \varphi \) is positively homogeneous of degree two and of class \( C^2 \) where it is negative. Then either \( \varphi \) is constant and \( \Omega_\varphi \) is linearly convex; or \( \varphi \) is not constant and \( \Omega_\varphi \) is not connected.

The set \( \Omega_\varphi \) in Example 9.5.13 has the properties mentioned here except that its boundary is not of class \( C^2 \): We have a striking contrast between the regularity classes \( C^{1,1} \) and \( C^2 \).

**Proof.** For functions \( \varphi : \mathbb{C} \to \mathbb{R} \) which are positively homogeneous of degree two, i.e., of the form \( \varphi(z_1) = |z_1|^2 \psi(t) \), \( z_1 = |z_1|e^{it} \in \mathbb{C} \), \( t \in \mathbb{R} \), condition (9.66) on \( \varphi \) takes the form

\[
(-r^2 \psi - 1) \left[ -\psi + \sqrt{(\frac{1}{2} \psi')^2 + (\frac{1}{2} \psi'')^2 - \frac{1}{4} \psi''} \right] \leq r^2 (\psi^2 + \frac{1}{2} \psi'^2),
\]

to hold in the set where \( -r^2 \psi - 1 \geq 0 \); equivalently

\[
4\psi + (-r^2 \psi - 1) \left[ \sqrt{\psi'^2 + 4\psi'^2 - \psi''} \right] \leq r^2 \psi'^2.
\]

From this we obtain, if we divide by \( r^2 \) and let \( r \) tend to \( +\infty \),

\[
(-\psi) \left[ \sqrt{\psi'^2 + 4\psi'^2 - \psi''} \right] \leq \psi'^2.
\]

(9.68)

But this condition is also sufficient, which follows on multiplication by \( r^2 \) and adding the trivial inequality \( 4\psi - \left[ \sqrt{\psi'^2 + 4\psi'^2 - \psi''} \right] \leq 0 \).

To get rid of the square root in (9.68) we rewrite it as

\[
\psi'^2 \left( \psi'^2 + 2(-\psi)\psi'' - 4\psi^2 \right) \geq 0,
\]

where the left-hand side is of degree four.

We now introduce a function \( g \) by defining \( g(t) \) as the positive square root of \( -\psi(t) \) if \( \psi(t) \) is negative and as \( 0 \) at all other points. Thus \( g \) is of class \( C^2 \) where it is positive, and \( \psi = -g^2 \) there. The points where \( g = 0 \), equivalently \( \psi > 0 \), are not of interest, since for them \( 1 + |z_1|^2 \psi(t) + |z_2|^2 \geq 1 > 0 \), implying that \( z \) does not belong to the closure of \( \Omega_\varphi \).
We get an inequality of degree eight but which is easy to analyze:

\[ g^5 g'' (g + g'') \leq 0. \]  

(9.69)

Thus, for each \( t \) such that \( g(t) > 0 \), either \( g'(t) = 0 \) or \( g(t) + g''(t) \leq 0 \). If \( g' \) is zero everywhere, i.e., if \( g \) is constant, it is known that \( \Omega_\varphi \) is lineally convex, in particular weakly lineally convex. Wherever \( g \) is positive and \( g' \) is nonzero we get \( g + g'' \leq 0 \). This implies that any local maximum of \( g \) is isolated and that there can only be one point where the maximum is attained.\(^5\) Hence, unless \( g' \) vanishes everywhere, \( g + g'' \leq 0 \) everywhere. We define \( h = g + g'' \leq 0 \) and obtain for any \( a \in \mathbb{R} \):

\[ g(t) = g(a) \cos(t - a) + g(a) \int_a^t \sin(t - s) h(s) ds, \quad t \in \mathbb{R}. \]

The function \( g \) attains its maximum at some point which we may call \( a \), and the formula then shows that \( g(t) \leq g(a) \cos(t - a) \) for all \( t \) with \( a \leq t \leq a + \pi/2 \). In particular, \( g \) must have a zero \( t_0 \) in the interval \([a, a + \pi/2]\). By symmetry, \( g \) has a zero \( t_1 \) also in the interval \([a - \pi/2, a]\), hence at least two zeros in a period. This means that \( \Omega \) is not connected, since the union of the rays \( \text{arg } z_1 = t_0 \) and \( \text{arg } z_1 = t_1 \) divides the \( z_1 \)-plane.

\[ \square \]

9.5.10 A set which is not starshaped

A subset \( A \) of a vector space is said to be \textit{starshaped with respect to a point} \( a \in A \) if the segment \([a, b]\) is contained in \( A \) as soon as \( b \) belongs to \( A \).

In answer to Question 9.5.2 we mention a modification of the set \( \Omega_{\varphi^2} \) which is not starshaped.

**Example 9.5.19** Define \( \varphi^2 : \mathbb{C} \to \mathbb{R} \) by

\[
\varphi^2(z_1) = \begin{cases} 
-x_1^2 - y_1^2, & x_1 + y_1 \leq 0; \\
-\frac{1}{2}(x_1 - y_1)^2, & x_1 + y_1 > 0.
\end{cases}
\]

Then \( \Omega_{\varphi^2} = \{ z \in \mathbb{C}^2 ; \ 1 + \varphi^2(z_1) + |z_2|^2 < 0 \} \) has boundary of class \( C^{1, 1} \) and is lineally convex, but it is not starshaped with respect to any point. \[ \square \]

This set can conceivably be modified to have a boundary of class \( C^\infty \) like in Example 9.5.14. However, it is unbounded.

**Question 9.5.20** Does there exist a bounded set with boundary of class \( C^2 \) which is lineally convex but not starshaped?

\(^5\)Defining a function \( g : \mathbb{R} \to \mathbb{R} \) of period \( 2\pi \) by \( g(t) = |\cos 2t| \) when \( 0 \leq t \leq \pi \) and \( g(t) = 1 \) when \( \pi < t < 2\pi \), we get a function which satisfies inequality (9.69) in \( \mathbb{R} \times \pi \mathbb{Z} \), but not the conclusions we have drawn from it. This function is not of class \( C^2 \) (but of class \( C^{1, 1} \)).
Question 9.5.21 What about a Hartogs domain with these properties?

The set $\Omega$ defined in Corollary 9.5.11 is bounded, lineally convex and not starshaped, but it has only a Lipschitz boundary. It cannot be approximated by a lineally convex set with smooth boundary.
9.6 A Differential Inequality Characterizing Weak Lineal Convexity

Abstract of this section

Behnke and Peschl introduced in 1935 the notion of *Planarkonvexität*, now called weak lineal convexity. They showed that, for domains with smooth boundary, it implies that a differential inequality is satisfied at every boundary point. We shall prove the converse here.

9.6.1 Introduction

In an article published in the *Mathematische Annalen* in 1935, Heinrich Behnke (1898–1979) and Ernst Peschl (1906–1986) introduced a notion of convexity called *Planarkonvexität*, nowadays known as *weak lineal convexity*. They showed that for domains in the space of two complex variables with boundary of class $C^2$, this property implies that a differential inequality is satisfied at every boundary point. Here we shall prove that, conversely, the differential inequality is sufficient for weak lineal convexity.

Behnke and Peschl (1935:170) proved that for sets with smooth boundary, weak lineal convexity is a local property (see Theorem 9.6.2 below).

Both usual convexity and pseudoconvexity can be characterized infinitesimally. The simplest example of such a result is that a $C^2$ function of one real variable is convex if and only if its second derivative is nonnegative. More generally, a domain in $\mathbb{R}^n$ with boundary of class $C^2$ is convex if and only if the Hessian of a defining function is positive semidefinite in the tangent space at every boundary point. Similarly, an open set in $\mathbb{C}^n$ with boundary of class $C^2$ is pseudoconvex if and only if the Levi form of a defining function is positive semidefinite in the complex tangent space at every boundary point (the Levi condition).

In analogy with these two classical results, we shall prove in the present section that a bounded connected open subset of $\mathbb{C}^n$ with boundary of class $C^2$ is weakly lineally convex if and only if the real Hessian of a defining function is positive semidefinite in the complex tangent space at every boundary point (the Behnke–Peschl condition).

It is easy to see that semidefiniteness is necessary. It is also known—indeed, this is the *Hauptsatz* of Behnke and Peschl (1935)—that the corresponding strong condition, i.e., that the real Hessian be positive definite, is sufficient. Thus what we have proved is that semidefiniteness is sufficient.

In the case of convexity and pseudoconvexity, the best way to deal with semidefiniteness is to approximate the domain by domains which satisfy the corresponding stronger condition of definiteness. This is not how we approach the problem here, at least not directly. The idea of proof of the main result
here is instead to construct Hartogs domains which share a tangent plane with the given domain.

**Question 9.6.1** Can a weakly lineally convex domain with smooth boundary be approximated from the inside by domains satisfying the strong Behnke–Peschl condition? (For Hartogs domains this is known; see Theorem 9.4.33. □

### 9.6.2 The main result

To be able to characterize sets by infinitesimal conditions, we shall describe boundaries and their curvature using defining functions and the Hesse and Levi forms. We refer to the definitions already given in Subsection 9.4.5.

As noted in the introduction, lineal convexity is not a local condition. Simple examples of sets which are locally lineally convex but not weakly lineally convex can be found in Section 9.4. However, weak lineal convexity is a local condition for sets with smooth boundary. The precise result is as follows.

**Theorem 9.6.2** Let $\Omega$ be a bounded connected open set in $\mathbb{C}^n$ with boundary of class $C^1$. Assume that for every boundary point $a$, the closure of the intersection of $\Omega$ with the complex tangent plane at $a$ does not contain $a$. Then $\Omega$ is weakly lineally convex. □

For sets in $\mathbb{C}^2$ or $\mathbb{P}^2$ with boundary of class $C^2$, this was proved by Behnke and Peschl (1935:170). For a proof under the hypotheses stated here, see (Hörmander 1994: Proposition 5.6.4.) See also (Andersson, Passare & Sigurdsson 2004: Proposition 2.5.8). We shall need this result in our proof.

We recall two lemmas from Section 9.4: Lemmas 9.4.24 and 9.4.25, both due to Behnke and Peschl (1935: Theorems 7 and 8); local weak lineal convexity is called Planarkonvexität im kleinen by them. Cf. also (Zinovi’ev 1971), (Hörmander 1994: Corollary 5.6.5).

Combining Lemma 9.4.25 and Theorem 9.6.2 we can deduce that the strict Behnke–Peschl differential condition (9.53) at all boundary points is sufficient for weak lineal convexity. This is the Hauptsatz of Behnke and Peschl (1935:170) (for sets in $\mathbb{C}^2$ or $\mathbb{P}^2$). We now state our main result, that in fact also the weaker condition (9.52) is sufficient:

**Theorem 9.6.3** Let $\Omega$ be a bounded connected open set in $\mathbb{C}^n$ with boundary of class $C^2$. Then $\Omega$ is weakly lineally convex if and only if $\Omega$ satisfies the Behnke–Peschl differential condition condition (9.52) at every boundary point.

If $\Omega$ is locally weakly lineally convex, has a $C^1$ boundary, and in addition is bounded, then $\Omega$ is also $C$-convex and lineally convex. This follows from (Andersson, Passare & Sigurdsson 2004: Proposition 2.5.8), who consider sets in projective space. I do not know how their result can be applied to unbounded domains in $\mathbb{C}^n$ with smooth boundary; such domains are not necessarily smoothly bounded in $\mathbb{P}^n$. 
9.6.3 Results for Hartogs domains

Lineal convexity for Hartogs sets is easier to handle than in the general case. For some results, see Section 9.4, in particular Theorems 9.4.33 and 9.4.44.

Proposition 9.6.4 Let $\Omega$ be an open set in $\mathbb{C}^n$ and define

\[ \Omega_H = \{ z \in \mathbb{C}^n; (z_1, \ldots, z_{n-1}, \lambda z_n) \in \Omega \text{ for all } \lambda \in \mathbb{C} \text{ with } |\lambda| \leq 1 \}. \quad (9.70) \]

This is the largest complete Hartogs set contained in $\Omega$. If $\Omega$ is lineally convex, then $\Omega_H$ is lineally convex; similarly for weak lineal convexity. If $\partial \Omega$ is of class $C^2$ except perhaps where $z_n = 0$, then so is the boundary of $\Omega_H$ at all points $z$ with $z_n \neq 0$ and satisfying the condition

\[ 2M|z_n| < |\rho_{z_n}(z)|, \quad (9.71) \]

where $M$ is a bound for the second derivatives $\rho_{z_n z_n}$ and $\rho_{z_n z_n}$. If, in addition, $\Omega$ satisfies the Behnke–Peschl differential condition (9.52) at all boundary points with $z_n \neq 0$, then so does $\Omega_H$ at all boundary points with $z_n \neq 0$ satisfying (9.71).

Proof If $\Omega$ is lineally convex, then also $\Omega_H$, as an intersection of lineally convex sets, has this property:

\[ \Omega_H = \bigcap_{|\lambda| \leq 1} \Omega_{\lambda}, \quad \text{where} \quad \Omega_{\lambda} = \{ z \in \mathbb{C}^n; (z_1, \ldots, z_{n-1}, \lambda z_n) \in \Omega \}. \]

Assume now that $\Omega$ is only weakly lineally convex, and let a point $a$ on the boundary of $\Omega_H$ be given. Then for some $\lambda$ with $|\lambda| = 1$, $a$ is on the boundary of $\Omega_{\lambda}$ defined above, and a hyperplane through $a$ which does not intersect $\Omega_{\lambda}$ does not intersect $\Omega_H$ either. (The argument is valid for all $a$; if $a_n = 0$ we even have $a \in \partial \Omega_{\lambda}$ for all $\lambda$.)

If $\rho$ defines $\Omega$, then

\[ \rho_H(z) = \sup_{\theta \in \mathbb{R}} \rho(z_1, \ldots, z_{n-1}, e^{i\theta} z_n) \quad (9.72) \]

defines $\Omega_H$ in a neighborhood of its closure. Define

\[ \varphi(z_1, \ldots, z_n, \theta) = \rho(z_1, \ldots, z_{n-1}, e^{i\theta} z_n), \quad (z, \theta) \in \mathbb{C}^n \times \mathbb{R}. \]

We can calculate

\[ \varphi_\theta = -2\text{Im} (\rho_{z_n} e^{i\theta} z_n); \]
\[ \varphi_{\theta\theta} = -2\text{Re} (\rho_{z_n} e^{i\theta} z_n) - 2\text{Re} (\rho_{z_n z_n} e^{2i\theta} z_n^2) + 2\rho_{z_n z_n} z_n^2. \]

The value of $\theta$ which defines the supremum in (9.72) solves the equation $\varphi''_\theta = 0$, and the implicit function theorem can be applied if $\varphi''_{\theta\theta} \neq 0$ there. This condition is fulfilled if

\[ |\text{Re} (\rho_{z_n} e^{i\theta} z_n)| > 2M|z_n|^2, \quad (9.73) \]
where $M$ is a bound for the second derivatives of $\rho$ as defined in the statement of the proposition. However, when $\varphi' = 0$, the expression $\rho_n e^{i\lambda}z_n$ is real, so that (9.73) simplifies to (9.71). The implicit function theorem then says that the boundary of $\Omega_H$ is as smooth as that of $\Omega$ where the condition is satisfied.

Now assume that $\Omega$ satisfies the Behnke–Peschl condition at a boundary point $a$ of $\Omega_H$ with $a_n \neq 0$. Then $a$ is on the boundary of some $\Omega_\lambda$, $|\lambda| = 1$, as already noted above. Consider the functions

$$\varphi_\lambda(s) = \rho_\lambda(a + st), \quad \varphi_H(s) = \rho_H(a + st), \quad s \in \mathbb{R}, \ t \in T_C(a),$$

where $\rho_\lambda(z) = \rho(z_1, \ldots, z_{n-1}, \lambda z_n)$, the defining function for $\Omega_\lambda$ obtained by rotating $\rho$ in the last coordinate.

The Behnke–Peschl condition holds for $\Omega_\lambda$, which means that $(\varphi_\lambda)''(0) \geq 0$. Now $\varphi_H \geq \varphi_\lambda$ and both functions vanish at the origin, which implies $(\varphi_H)''(0) \geq (\varphi_\lambda)''(0)$. Thus the condition holds for $\Omega_H$. This completes the proof.

In an application of this proposition in the next subsection we shall let $\Omega$ be defined near an arbitrarily given point by an inequality $y_n < f(z', x_n)$ for some real-valued function $f$ of $n - 1$ complex variables and one real variable. Then $\rho(z) = y_n - f(z', x_n)$ is a defining function for $\Omega$ near the given point. (Here $x_n = \text{Re } z_n$, $y_n = \text{Im } z_n$, and $z' = (z_1, \ldots, z_{n-1})$.) We see that $\rho_n = -\frac{1}{2}(f_n + i)$, so that $|\rho_n| > \frac{1}{2}$. Moreover

$$\rho_{n} z_n = \rho_{n} \overline{z}_n = -\frac{1}{4} f_n x_n.$$

This implies that a sufficient condition for (9.71) to hold is

$$C|z_n| < 1, \quad (9.74)$$

where $C$ is a bound for $f_n x_n$.

**Remark 9.6.5** Condition (9.71) has a simple geometric meaning. With the defining function $\rho(z) = y_n - f(z', x_n)$ it says that the intersection of the boundary of $\Omega$ with the subspace $z' = \text{constant}$ has smaller curvature than the intersection of the boundary of $\Omega_H$ with the same subspace where the two boundaries meet. For simplicity we shall use the stronger condition (9.74) instead.

### 9.6.4 Proof of the main result

We shall now prove Theorem 9.6.3. In view of Theorem 9.6.2 it is enough to prove that the complex tangent plane $a + T_C(a)$ does not cut $\Omega$ near $a$. We shall assume that $a + T_C(a)$ cuts $\Omega$ in a point $b$ and then show that this leads to a contradiction if $b$ is close to $a$.

First of all we may assume that $n = 2$ by looking at the two-dimensional affine complex subspace that contains $a$, $b$ and a third point on the normal to
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∂Ω through a. We may also assume that the coordinate system is chosen so that \( a = 0 \) and the real tangent plane \( a + T_R(a) \) has the equation \( \text{Im} \; z_2 = 0 \). We recall that both weak lineal convexity and the Behnke–Peschl condition (9.52) are invariant under complex affine mappings. The complex tangent plane at \( a \) then has the equation \( z_2 = 0 \), so that \( b_2 = 0 \). We shall consider a neighborhood \( W \) of \( a \) such that three conditions are satisfied. Let

\[
W = \{ z \in \mathbb{C}^2; \; |z_1| < R_1, |z_2| < R_2 \}
\]

and let \( V \) be its intersection with \( \mathbb{C} \times \mathbb{R} \):

\[
V = \{ (z_1, x_2) \in \mathbb{C} \times \mathbb{R}; \; |z_1| < R_1, |x_2| < R_2 \}.
\]

The three conditions are:

(A) First of all the set \( \Omega \) shall be defined in \( W \) by an inequality \( \text{Im} \; z_2 < f(z_1, \text{Re} \; z_2) \) for some function \( f \) which is of class \( C^2 \) in a neighborhood of the closure of \( V \).

(B) Next we shall assume that condition (9.74) is satisfied for all \( z \in W \) with some margin:

\[
R_2 \sup_V |f_{x_2 x_2}| < \frac{2}{3} < 1.
\]

(This is to allow a change of coordinates later.)

(C) Third, \( R_1 \) shall be so small that \( MR_1 + C(1 + M^2)R^2_1 < \frac{1}{4}R_2 \), where \( M = \frac{1}{2}CR_2 \) and \( C \) is defined below.

To satisfy these conditions we have to specify the numbers \( R_1, R_2 \) and \( C \). We first choose \( R_1 \) and \( R_2 \) so that (A) and (B) hold, and then define a constant \( C \) as follows. Since \( f \) is a function of class \( C^2 \) defined in a neighborhood of the closure of \( V \) and with vanishing derivatives of order up to one at the origin, there exists a constant \( C \) such that

\[
|f(z_1, x_2)| \leq C(|z_1|^2 + x_2^2),
\]

\[
|f_{x_2}(z_1, x_2)| \leq C(|z_1| + |x_2|), \quad \text{and}
\]

\[
|f_{x_2 x_2}(z_1, x_2)| \leq C
\]

for all \((z_1, x_2) \in V\). We finally shrink \( R_1 \) if necessary to make (C) hold.

With the choice of coordinate system we have made, the normal at \( a \) is the \( y_2 \)-axis. Let \( c \) be a point on that axis with \( \text{Im} \; c_2 < 0 \); it is convenient to take \( c = -\frac{1}{4}iR_2 \). Thus \( c = (0, c_2) \) and \( |c| = -\text{Im} \; c_2 = \frac{1}{4}R_2 \). The circle in the plane \( z_1 = 0 \) with center at \( c \) and radius \( |c| \) passes through \( a \) and is tangent to the \( x_2 \)-axis at that point.

We shall prove that \( f(b) \leq 0 \) (hence that \( b \notin \Omega \)) for all \( b \) with \( |b_1| < R_1 \). Assume the contrary: \( f(b) > 0 \). Consider the plane \( z_1 = b_1 \) and the graph of \( f \) restricted to that plane. Draw the normal to the graph of \( f(b_1, \cdot) \) through the
point \( z_2 = if(b_1,0) \) in the \( z_2 \)-plane. This normal intersects the line \( y_2 = \text{Im} \, c_2 \) at a point, which we call \( p_2 \). Define \( p_1 = b_1 \) and \( p = (p_1, p_2) \in \mathbb{C}^2 \). The slope of the normal is determined by the slope of the graph at \( z_1 = b_1, \, x_2 = 0 \), i.e., by \( f_{x_2}(b_1,0) \). This derivative can however be controlled: we know that \( f_{x_2}(b_1,0) \) is not more than \( C|b_1| \) in modulus. The distance between \( p \) and \( c \) is

\[
|p - c| = |f_{x_2}(b_1,0)||c| + f(b_1,0) \leq C|b_1|(\frac{1}{4}R_2 + C|b_1|^2) \leq \frac{1}{2}CR_2|b_1|
\]

where the last estimate is a consequence of (C). Thus \( |p - c| \leq M|b_1| \) with \( M = \frac{1}{2}CR_2 \).

We have constructed a disk \( D_0 \) in the plane \( z_1 = 0 \) with center at \( c_2 \) and with \( z_2 = 0 \) on its boundary, and now let \( D_1 \) be the disk in the plane \( z_1 = b_1 \) with center at \( p_2 \) and \( if(b_1,0) \) on its boundary (and therefore containing \( z_2 = 0 \)):

\[
D_0 = \{ z \in \mathbb{C}^2; \, z_1 = 0, \, |z_2 - c_2| < |c| \}; \\
D_1 = \{ z \in \mathbb{C}^2; \, z_1 = b_1, \, |z_2 - p_2| < |if(b_1,0) - p_2| \}.
\]

Both disks are moreover contained in \( \Omega \cap W \). For \( D_0 \) this is obvious from the construction; for \( D_1 \) this can be seen as follows. The center of \( D_1 \) is \( p_2 \) and its radius \( r_1 \) is \( |if(b_1) - p_2| \). The disk is contained in \( W \) if \( |p_2| + r_1 \leq R_2 \). This inequality follows from the estimates we already have:

\[
|p_2| + r_1 \leq 2|p_2| + C|b_1|^2 \leq 2|c_2| + 2|p_2 - c_2| + C|b_1|^2 \leq \frac{1}{2}R_2 + 2MR_1 + CR_1^2 \leq R_2,
\]

where the last inequality follows from (C). Thus \( D_1 \subset W \). That \( D_1 \subset \Omega \) now follows from (B); cf. Remark 9.6.5.

If we construct a Hartogs domain by rotating \( \Omega \) around an axis which passes through \( c \) and \( p \), then this Hartogs domain will have \( a \) on its boundary and contain \( b \). This is precisely what we shall do.

We introduce new coordinates \( (w_1, w_2) \) so that the \( w_1 \)-axis, i.e., the plane \( w_2 = 0 \), passes through \( c \) and \( p \). The \( w_2 \)-axis need not be changed. This means that the new coordinates shall be defined as

\[
w_1 = z_1, \quad w_2 = z_2 - c_2 - (p_2 - c_2)z_1/b_1.
\]

Indeed \( z = c \) gives \( w = 0 \) and \( z = p \) yields \( w = b = (b_1,0) \). We now define \( \Omega_H \) in the \( w \)-coordinates. The tangent plane with equation \( z_2 = 0 \) has the equation \( w_2 = (p_2 - c_2)z_1/b_1 \) and is also the tangent plane to \( \partial \Omega_H \) at the point \( w = (0,-c_2) \). It intersects \( \Omega_H \) at the point \( z = b \), i.e., \( w = (b_1,-p_2) \). That this point is an element of \( \Omega_H \) follows from the construction of \( D_1 \).

We shall now apply Theorem 9.4.33 to \( \Omega_H \) over the disk \( |w_1| < R_1 \) in the \( w_1 \)-plane. To be able to do so we have to check that there is a point of \( \Omega_H \) over every point \( w_1 \) with \( |w_1| < R_1 \), or equivalently that \( (w_1,0) \in \Omega_H \) for all \( w \) with \( |w_1| < R_1 \).

In the new coordinate system, the inequality defining \( \Omega \) becomes

\[
\text{Im} \, w_2 < -\text{Im} \, c_2 - \text{Im} \, (p_2 - c_2)w_1/b_1 + f(w_1, \text{Re} \, w_2 + \text{Re} \, (p_2 - c_2)w_1/b_1).
\]
Denote the right-hand side by \( g(w_1, \text{Re} w_2) \). In particular

\[
g(w_1, 0) = -\text{Im} c_2 - \text{Im} (p_2 - c_2)w_1/b_1 + f(w_1, \text{Re} (p_2 - c_2)w_1/b_1).
\]

Recalling the estimate \( |p_2 - c_2| \leq M|b_1| \) above, we get

\[
g(w_1, 0) \geq \frac{1}{4}R_2 - M|w_1| - C(1 + M^2)|w_1|^2 \geq \frac{1}{4}R_2 - MR_1 - C(1 + M^2)R_1^2 > 0,
\]

the last inequality coming from (C). This ensures that every point \((w_1, 0)\) with \(|w_1| < R_1\) lies in \(\Omega\) and therefore also in \(\Omega_H\).

We know that \(\Omega_H\) satisfies the Behnke–Peschl differential condition at all boundary points if the condition in the \(w\)-coordinates corresponding to (9.74) is valid. Note that \(|w_2 - z_2 + c_2| \leq M|z_1|\) independently of the choice of \(b \in W\), from which we deduce

\[
|w_2| \leq |z_2| + \frac{1}{4}R_2 + MR_1 \leq \frac{3}{2}R_2.
\]

The second derivative of \(g\) with respect to \(\text{Re} w_2\) is the same as the second derivative of \(f\) with respect to \(x_2 = \text{Re} z_2\), so from (B) we can conclude that the condition (4.5) is satisfied also in the \(w\)-coordinates for all points \(w \in \partial\Omega_H\) with \(|w_1| < R_1\).

It now follows from Theorem 9.4.44 that \(\Omega_H\) is lineally convex, which contradicts the fact that the tangent plane at the point \(w = -c\) intersects \(\Omega_H\) in \(w = (b_1, -p_2)\). This completes the proof.
9.7 Generalized Convexity

Abstract of this section
Inspired by mathematical morphology we study generalized convexity and prove that certain subsets of Hartogs domains are convex in a generalized sense.

9.7.1 Introduction to this section
By the Hahn–Banach theorem, an open convex set in $\mathbb{R}^m$ is an intersection of open half-spaces; its complement a union of closed half-spaces. What if we replace the latter by balls? We shall study here a kind of generalized convexity where a set is called concave if it is a union of closed balls; its complement thus being an intersection of complements of closed balls. This will be done in particular for Hartogs domains which are lineally convex.

Lineal convexity is a kind of complex convexity intermediate between usual convexity and pseudoconvexity. More precisely, if $A$ is a convex set in $\mathbb{C}^n$ which is either open or closed, then $A$ is lineally convex (this is true also in the real category), and if $\Omega$ is a lineally convex open set in $\mathbb{C}^n$, then $\Omega$ is pseudoconvex.

As mentioned on page 263, there are several different notions of convexity related to lineal convexity.

The main results are presented in Subsections 9.7.8 and 9.7.10. It is shown there that certain subsets of Hartogs domains have convexity properties originating in mathematical morphology. We also study external tangent planes of sets that do not necessarily have a smooth boundary.

9.7.2 Hyperplanes, tangent planes, and multifunctions
Hyperplanes are affine subspaces with real or complex codimension 1, and they will play an important role in the sequel.

To any real hyperplane $Y$ in $\mathbb{C}^n$ and every point $a \in Y$ there is a unique complex hyperplane $Y_{[a]}$ that contains $a$ and is contained in $Y$. In fact

$$Y_{[a]} = Y \cap (i(Y - a) + a).$$

We note that $Y_{[a]}$ depends continuously on $(Y, a)$ for the natural topology on hyperplanes and points.

Conversely, every complex hyperplane $Z$ in $\mathbb{C}^n$ is contained in a real hyperplane, but there are now several choices. If a complex hyperplane $Z$ is given and is defined by the equation $\beta \cdot (z - a) = 0$, then for any complex number $\theta$ with $|\theta| = 1$ the real hyperplane $Z^{[\theta]}$ defined by $\text{Re}(\beta \cdot (z - a)) = 0$ contains $Z$. The real hyperplane $Z^{[\theta]}$ does not depend on the choice of $a \in Z$ and satisfies $(Z^{[\theta]})_{[b]} = Z$ for every $b \in Z$.
If a real hyperplane $Y$ and a point $a \in Y$ are given, then $(Y_{[a]})^{[0]} = Y$ for two values of $\theta$ with $|\theta| = 1$. Explicitly, if $Y$ is given by the equation $\Re \beta \cdot (z - a) = 0$, then $Y_{[a]}$ is given by $\beta \cdot (z - a) = 0$ and $(Y_{[a]})^{[0]}$ by $\Re \theta(\beta \cdot (z - a)) = 0$; the two choices $\theta = \pm 1$ give us $Y$ back.

For the definition of the real and complex tangent spaces to an open subset $\Omega$ of $\mathbb{C}^n$ with boundary of class $C^1$, as well as the real and complex tangent planes, we refer to Definition 9.4.19.

Clearly $T_{\Omega,\mathbb{C}}(b) = T_{\Omega,\mathbb{R}}(b)^{[0]}$; for the tangent planes,

$$b + T_{\Omega,\mathbb{C}}(b) = (b + T_{\Omega,\mathbb{R}}(b))^{[0]}.$$

**Definition 9.7.1** If $A$ is a subset of $\mathbb{C}^n$, we shall denote by $\Gamma_A(a)$ the set of all complex hyperplanes $Z$ which pass through the origin and are such that $a + Z$ does not intersect $A$. $\square$

**Definition 9.7.2** A mapping $F: X \rightarrow \mathcal{P}(Y)$ will be called a multifunction from $X$ into $Y$ and will be written $F: X \rightrightarrows Y$. (This means that the value, image, or fiber $F(x)$ of $F$ at a point $x$ is a subset of $Y$, possibly empty.) The **graph** of a multifunction $F$, denoted by $\text{graph}(F)$, is the set $\{(x, y) \in X \times Y; y \in F(x)\}$.

If $X$ and $Y$ are topological spaces, we can equip $X \times Y$ with the Cartesian product topology. In all cases considered here, $X$ is a $T_1$ space—equivalently, all singleton sets are closed. If so, for $\text{graph}(F)$ to be a closed subset of $X \times Y$, it is necessary but not sufficient that the fiber

$$F(a) = \{(a) \times Y \cap \text{graph}(F)$$

be a closed subset of $Y$ for every $a \in X$.

Thus $\Gamma_A$ is a multifunction $\Gamma_A: \mathbb{C}^n \rightrightarrows \text{Gr}_{n-1}(\mathbb{C}^n) = M_{n,n-1}(\mathbb{C})$ with values in the Grassmann manifold of all complex hyperplanes in $\mathbb{C}^n$ passing through the origin. If $\Omega$ is open, $\Gamma_{\Omega}(a)$ is closed for every $a \in \mathbb{C}^n$. See also Proposition 9.7.15.

Lineal convexity of a set $A$ means that $\Gamma_A(a)$ is nonempty for every $a \in \mathbb{C}^n \setminus A$; weak lineal convexity of an open set $\Omega$ that $\Gamma_{\Omega}(b)$ is nonempty for every $b \in \partial \Omega$.

Let us agree to say that a topological space is **connected** if the only sets which are both open and closed are the empty set and the whole space (not necessarily distinct). A subset of a topological space is said to be **connected** if it is connected as a topological subspace.

Zelinski˘ı (1981) has proved that a bounded lineally convex open set $\Omega$ is $C$-convex if and only if $\Gamma_{\Omega}(b)$ is connected for every boundary point $b$. See also (Andersson, Passare & Sigurdsson 2004:46, Theorem 2.5.2) for the corresponding result on subsets of projective space.

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6 We follow here Bourbaki (1961 I: §11:1) in that the empty space is defined to be connected. Adrien Douady (personal communication, 2000 June 26) argued for the empty space not to be connected. The difference is important in Definition 5, where $C$-convexity is defined.
9.7.3 Accessibility

We refer to Subsection 9.3.7 for the set-theoretical operations $\delta_S$, $\varepsilon_S$, $\kappa_S$ and $\alpha_S$ with respect to a structuring element $S$. It is convenient to express closedness and openness for some of these operators in terms of accessibility:

**Definition 9.7.3** If $A$ is a subset of $\mathbb{R}^m$ or $\mathbb{C}^n$ and $b$ a point in this space, we shall say that $b$ is $S$-**accessible from the outside** if $b$ belongs to the closure of $\alpha_S(\mathbb{L}A)$. In particular, we shall speak about accessibility from the outside by balls of radius $r$ if $S$ is equal to $B_\varepsilon(0,r)$ or $B_\varepsilon(0,r)$.

**Remark 9.7.4** If $b$ is $S$-accessible from the outside of a certain class, then there is also a set $T$ of the same class such that $\mathbb{A} \cap T = \{ b \}$. Indeed, if $S$ satisfies

$$\{ x; f(x) < 0 \} \subset S \subset \{ x; f(x) \leq 0 \},$$

then $T$ can be taken as the set of all $x$ such that $f(x) + \| x - b \|^2 \leq 0$.

We shall consider regularity classes $C^{k,\beta}$, where $k \in \mathbb{N}$ and $0 \leq \beta \leq 1$, meaning that the functions considered are of class $C^k$ and all derivatives of order $k$ are Hölder continuous of order $\beta$, with the understanding that $C^{k,0} = C^k$.

**Definition 9.7.5** If $b \in \partial A$ is accessible from the outside by a structuring element $S$ having boundary of class $C^{k,\beta}$ with $k \geq 1$, then we shall say that the unique tangent plane to $S$ at $b$ is an **external tangent space** of $A$ at $b$.

The set of all external tangent spaces at a point $b$, a subset of the Grassmann manifold $\text{Gr}_{m-1}(\mathbb{R}^m) = M_{m,m-1}(\mathbb{R})$ of all real hyperplanes passing through the origin, will be denoted by $\Theta_{A,R}^{k,\beta}(b)$, and the corresponding multifunction $\partial A \ni \text{Gr}_{m-1}(\mathbb{R}^m)$ by $\Theta_{A,R}^{k,\beta}$.

If $\Omega$ is an open subset of $\mathbb{C}^n$, we shall denote by $\Theta_{\Omega,R}^{k,\beta}(b)$ the set of all complex hyperplanes through the origin contained in planes in $\Theta_{\Omega,R}^{k,\beta}(b)$; we call them **complex external tangent spaces**. It is the set of all complex hyperplanes $Z = Y[0]$, $Y \in \Theta_{\Omega,R}^{k,\beta}(b)$.

When the class is clear from the context or is unimportant, we shall omit the superscripts $k,\beta$.

It is easy to see that $\Theta_{\Omega,R}^{1,1} = \Theta_{\Omega,R}^2 = \Theta_{\Omega,R}^\infty$.

The relation between $\Gamma_\Omega(b)$ and $\Theta_{\Omega,C}(b)$, $b \in \partial \Omega$, seems to be of interest.

**Definition 9.7.6** Let us say that $\Omega$ is **tangentially lineally convex at** $b \in \partial \Omega$ if no complex external tangent plane of class $C^2$ at $b$ meets $\Omega$, i.e., if $\Theta_{\Omega,C}(b) \subset \Gamma_\Omega(b)$.

**Proposition 9.7.7** Let $b \in A \subset \mathbb{R}^m$ be accessible from the outside by balls of radius $r > 0$. Then $\Theta_{A,R}^{k,\beta}(b)$ is connected.
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Proof Take $b = 0$ and assume that $\overline{A} \cap \overline{U}_j = \{0\}$, $j = 0, 1$, where $U_j$ is the set of all points $x$ such that $f_j(x) < 0$, and $f_j$ is a function of a given regularity and with nonvanishing gradient wherever it is zero. This is justified by Remark 9.7.4. We now form $f_s = (1 - s)f_0 + sf_1$, $0 \leq s \leq 1$, and claim that the set where $f_s$ is negative defines an open set $U_s$ which serves to prove that all gradients

$$(\text{grad } f_s)(0) = (1 - s)(\text{grad } f_0)(0) + s(\text{grad } f_1)(0)$$

can occur, implying that there is a curve connecting the hyperplane defined by $f_0$ to that defined by $f_1$. We note that the gradient of $f_s$ is nonzero at the origin except in the case when $(\text{grad } f_1)(0)$ is a negative multiple of $(\text{grad } f_0)(0)$. In that case, however, the hyperplanes defined by the two gradients are the same, so there is nothing to prove. We modify $f_s$ outside a neighborhood of the origin if necessary to make sure that it satisfies the requirement that its gradient be nonzero everywhere where the function itself vanishes.

If $x \in \overline{A} \setminus \{0\}$, then $x \notin \overline{U}_j$, $j = 0, 1$, so that $f_j(x) > 0$, $j = 0, 1$. This implies that $f_s(x) > 0$, so that $x \notin \overline{U}_s$. Thus we have proved that $\overline{A} \cap \overline{U}_s \subset \{0\}$; obviously $\overline{A} \cap \overline{U}_s \supset \{0\}$. In conclusion, we have proved that the tangent plane of $U_s$ at $b = 0$ belongs to $\Theta_{\mathcal{A}, \mathcal{R}}(0)$ for all $s$ with $0 \leq s \leq 1$.

Example 9.7.8 Let us define a cleistomorphism

$$\kappa_r: \mathcal{P}(C^n) \to \mathcal{P}(C^n) \text{ or } \kappa_r: \mathcal{P}(\mathcal{R}^m) \to \mathcal{P}(\mathcal{R}^m)$$

as the cleistomorphism with structuring element $U = \mathcal{L}B_{C}(0, r)$ for some positive radius $r$. It follows that $\kappa_r(A)$ is closed for any set $A$, perhaps most easily seen by observing that its complement, denoted by $\alpha_r(\mathcal{L}A)$, is the union of all open balls $B_{C}(x, r)$ that are contained in $\mathcal{L}A$.

Thus $\kappa_r(A)$ is the smallest invariance set containing $A$ whose boundary points are all accessible by balls of radius $r$, and we see that the boundary points of a closed set $F$ are accessible by such balls if and only if $\kappa_r(F) = F$.

To treat open sets, we define $\lambda_r(A)$ as the interior of $\kappa_r(A)$. In view of Proposition 9.3.8 the operation $A \mapsto \lambda_r(A) = (\kappa_r(A))^\circ$ is an ethnomorphism. If we restrict it to open sets, it is larger than the identity, i.e., $\lambda_r(\Omega) \supset \Omega$ for all open sets $\Omega$. So accessibility for open sets is defined by the fixed points of $\lambda_r$.

The infimum of all the $\kappa_r$, $r > 0$, is just the topological closure. □

9.7.4 Concavity and convexity with respect to a structuring element or a family of structuring elements

Just as it is sometimes easier to look at lineally concave sets rather than lineally convex sets, it can be more convenient to define accessibility from the inside than from the outside. We shall do this in terms of concavity and convexity with respect to a structuring element, treating both properties in parallel:
Definition 9.7.9 Given a subset $S$ (called structuring element) of an abelian group $G$, we shall say that a subset $A$ of $G$ is $S$-concave if it is a union of translates $x + S$ with $x$ in some subset $X$ of $G$. We shall say that it is $S$-convex if its complement is $S$-concave.

We define the $S$-kernel of a set $A$, denoted by $\alpha_S(A)$, as the union of all translates $x + S$ contained in $A$. We define the $S$-hull of a set $B$, denoted by $\kappa_S(B)$, as the complement of the $S$-kernel of $\mathcal{C}B$.

Obviously $\mathcal{C}(\alpha_S(A)) = \kappa_S(\mathcal{C}A).

The anisotropomorphism $\alpha_S$

$$\alpha_S(A) = \bigcup_{x \in G} (x + S; x + S \subset A), \quad A \in \mathcal{P}(G),$$

defined as the $S$-kernel of $A$, and $\kappa_S(B) = \mathcal{C}\alpha_S(\mathcal{C}B)$, defined as the $S$-hull of $B$.

Thus \{S\}-concavity is the same as $S$-concavity.

Classical examples are when we take $\mathcal{I}$ as the family $\mathcal{H}$ of all open half-spaces in $\mathbb{R}^m$, defined by an inequality $\xi \cdot x > c$, or the family $\mathcal{G}$ of all closed half-spaces in $\mathbb{R}^m$, defined by an inequality $\xi \cdot x \geq c$, with $\xi \in \mathbb{R}^m \setminus \{0\}$, $c \in \mathbb{R}$. We can also consider the set of all real or complex hyperplanes, or intersections of complex hyperplanes with balls.

Example 9.7.11 The set $A = [0, 1]^2 \cup \{(0, 0)\} \subset \mathbb{R}^2$ (an open square with a vertex added) is convex, but is not an intersection of open half planes, nor of closed half planes; in other words, it is not even convex in the sense of Fenchel (1952)—see the discussion about this class of sets in Subsection 9.2.4, page 257. We obtain

$$A^o = [0, 1]^2 \not\subset A \not\subset \kappa_{\mathcal{H}}(A) = [0, 1]^2 \not\subset \kappa_{\mathcal{G}}(A) = [0, 1]^2 = \mathcal{A},$$
indicating that more general half planes are needed. □

In view of the above example we now define more general half-spaces, called here **refined half-spaces**, by which we mean convex sets $Y$ such that

$$\{x \in \mathbb{R}^m; \xi \cdot x < c\} \subset Y \subset \{x \in \mathbb{R}^m; \xi \cdot x \leq c\}$$

for some $\xi \in \mathbb{R}^m \setminus \{0\}$ and $c \in \mathbb{R}$. Let us denote by $\mathcal{Y}$ the family of all such sets $Y$.

Obviously $\kappa_{\mathcal{Y}}(A)$ is always a closed set. In view of the Hahn–Banach theorem it is equal to the closed convex hull of $A$. The mapping $\kappa_{\mathcal{Y}}$ takes an open set to its convex hull (which is open) and a compact set to its convex hull (which is closed).

This is convexity viewed from the outside. We can also work with convexity from the inside: We defined in Definition 9.2.7 on page 256 the convex hull of a set $A \subset \mathbb{R}^m$. It can easily be proved that $\text{cvxh} = \kappa_{\mathcal{Y}}$, showing that the refined half-spaces serve also for convex sets which are not evenly convex—see Section 9.2, page 257, for these sets.

The operation $\text{cvxh}$ maps any set to its convex hull, which need not be closed even if $A$ is closed. The composition $\text{clos} \circ \text{cvxh}$ takes any set to its closed convex hull. (The composition $\text{cvxh} \circ \text{clos}$ is not idempotent if $m \geq 2$.)

**Definition 9.7.12** We shall say that an open subset of $\mathbb{R}^m$ or $\mathbb{C}^n$ is **$r$-concave** if it is a union of open balls of radius $r$. A closed subset is called **$r$-concave** if it is a union of closed balls of radius $r$. A set is called **$r$-convex** if its complement is $r$-concave.

This definition agrees for open sets in $\mathbb{C}$ with that of Sergey Favorov and Leonid Golinskii (2015:3). They defined the $r$-convex hull of a set $E \subset \mathbb{C}$, denoted by $\text{conv}_r(E)$, as the set

$$\text{conv}_r(E) = \bigcap \{\mathbb{C}D_<(z,r); E \subset \mathbb{C}D_<(z,r)\}, \quad E \subset \mathbb{C}, \quad r > 0.$$ 

Thus $\mathbb{C}\text{conv}_r(E)$ is a union of open disks. They call a set $r$-convex if $\text{conv}_r(E) = E$. Such a set is always closed. The generalization to $\mathbb{R}^m$ or $\mathbb{C}^n$ is obvious, and we see that $\text{conv}_r(E)$ is exactly the set $\kappa_{B_<}(0,r)(E)$ with the notation from Definition 9.7.9. When $r$ tends to $+\infty$, we get the closed convex hull $\text{cvxh}(E)$ as a limiting case.\(^7\)

\(^7\)The notion of $r$-convex closed sets is used by these authors as an hypothesis in results on Blaschke-type conditions for the Riesz measure of a subharmonic function, thus in a context quite different from the one studied here. Since I worked on generalized convexity during the period 1996–2001 see for example Proposition 4.9 in my paper (1996) and then again since 2014, and with quite different problems, our respective studies are independent.
9.7.5 Lineal convexity viewed from mathematical morphology

Lineal concavity is an example of $\mathcal{S}$-concavity, taking $\mathcal{S}$ equal to the family $\mathcal{Z}$ of all complex hyperplanes in $\mathbb{C}^n$ containing the origin. Weak lineal convexity means that $\kappa_{\mathcal{Z}}(\Omega)$ does not meet the boundary of $\Omega$.

There are also local variants of these definitions: we take $\mathcal{S} = \mathcal{Z}_r$ as the family of all intersections $Z \cap B_{\leq}(0, r)$, where $Z$ is a complex hyperplane passing through the origin. The corresponding $\mathcal{Z}_r$-convexity, for some positive $r$, can be called uniform local lineal convexity.

Let us take again the family $\mathcal{S}$ of structuring elements in Definition 9.7.10 as the set $\mathcal{Z} \subset P(\mathcal{P}(\mathbb{C}^n))$ of all complex affine hyperplanes in $\mathbb{C}^n$. We define a dilation $\psi: \mathcal{P}(\mathcal{Z}) \to \mathcal{P}(\mathbb{C}^n)$ by

$$
\psi(\mathcal{B}) = \bigcup_{Z \in \mathcal{B}} Z, \quad \mathcal{B} \in \mathcal{P}(\mathcal{Z}).
$$

(9.75)

Its lower inverse $\psi_{[-1]}: \mathcal{P}(\mathbb{C}^n) \to \mathcal{P}(\mathcal{Z})$ is defined by

$$
\psi_{[-1]}(A) = \bigcup_{\mathcal{B} \in \mathcal{Z}} (\mathcal{B}; \psi(\mathcal{B}) \subset A) = \{ Z \in \mathcal{Z}; Z \subset A \}, \quad A \in \mathcal{P}(\mathbb{C}^n).
$$

(9.76)

We note that $\varepsilon = \psi_{[-1]}$ is an erosion—as the lower inverse of a dilation, but also easily seen directly. There is a relation between $\Gamma_A$ and $\varepsilon$:

$$
\Gamma_A(b) = \{ Z \in \varepsilon(\mathcal{C}A); b \in Z \}.
$$

The upper inverse $\varepsilon^{[-1]}: \mathcal{P}(\mathcal{Z}) \to \mathcal{P}(\mathbb{C}^n)$ of $\varepsilon$ is a dilation defined by

$$
\varepsilon^{[-1]}(\mathcal{B}) = \bigcap_{A \in \mathcal{P}(\mathbb{C}^n)} (A; \varepsilon(A) \supset \mathcal{B}) = \bigcup_{Z \in \mathcal{B}} Z = \psi(\mathcal{B}), \quad \mathcal{B} \in \mathcal{P}(\mathcal{Z}).
$$

(9.77)

By composition we obtain an anoiktomorphism $\alpha_{\mathcal{Z}}: \mathcal{P}(\mathbb{C}^n) \to \mathcal{P}(\mathbb{C}^n)$:

$$
\alpha_{\mathcal{Z}}(A) = (\varepsilon^{[-1]} \circ \varepsilon)(A) = (\psi \circ \psi_{[-1]})(A) = \bigcup (Z; Z \subset A), \quad A \in \mathcal{P}(\mathbb{C}^n),
$$

the union of all complex affine hyperplanes contained in $A$. We can also form

$$
\kappa_{\mathcal{Z}}(\mathcal{B}) = (\varepsilon \circ \varepsilon^{[-1]})(\mathcal{B}) = (\psi_{[-1]} \circ \psi)(\mathcal{B}), \quad \mathcal{B} \in \mathcal{P}(\mathcal{Z}).
$$

We have $\alpha_{\mathcal{Z}}(A) = A$ (equivalently $\alpha_{\mathcal{Z}}(A) \supset A$) if and only if $A$ is lineally concave, which happens if and only if $\mathcal{C}A$ is lineally convex. If $\Omega$ is open, it is lineally convex if and only if $\alpha_{\mathcal{Z}}(\mathcal{C}\Omega) \supset \mathcal{C}\Omega$, and weakly lineally convex if and only if $\alpha_{\mathcal{Z}}(\mathcal{C}\Omega) \supset \partial\Omega$. 

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9.7.6 Exterior accessibility of Hartogs domains

We shall now study Hartogs domains in \( \mathbb{C}^n \times \mathbb{C} \), where we write coordinates as \((z, t) \in \mathbb{C}^n \times \mathbb{C}\).

To define complete Hartogs sets, we may use either the function \( R \), the function \( h = R^2 \), or the function \( f = - \log R \). An open complete Hartogs set is then defined equivalently by \(|t| < R(z); |t|^2 < h(z); |t| < e^{-f} \), and we are free to choose whichever is convenient for a specific calculation. We note that if \( f \) is plurisubharmonic, then \( \Omega \), defined by \(|t| + f(z) < 0 \), is pseudoconvex.

Complex hyperplanes in \( \mathbb{C}^n \times \mathbb{C} \) are of three kinds:

1. A hyperplane can be given by an equation \( \beta \cdot (z - z^0) = 0 \) for some \( \beta \in \mathbb{C}^n \setminus \{0\} \) and some point \( z^0 \in \mathbb{C}^n \) (we shall call it a \textit{vertical hyperplane}).
2. It can have the equation \( t = c \) for some complex constant \( c \) (we shall call it a \textit{horizontal hyperplane}).
3. Finally it can have the equation \( t = \beta \cdot (z - z^0) \), where \( \beta \) is nonzero. Such a hyperplane intersects the hyperplane \( t = 0 \) in a hyperplane in \( \mathbb{C}^n \) containing \( z^0 \).

The projection \( \mathbb{C}^n \times \mathbb{C} \ni (z, t) \mapsto (z, |t|) \in \mathbb{C}^n \times \mathbb{R} \) can be used to visualize the set. Equivalently, we can look at the intersection of \( \Omega \) with the set \( \{(z, t); z \in \mathbb{C}^n, t > 0\} \). A hyperplane is then represented in \( \mathbb{C}^n \times \mathbb{R} \) by either

1. a vertical plane;
2. a horizontal plane \(|t| = |c|\); or
3. a cone \(|t| = |\beta \cdot (z - z^0)|\) with vertices at all the points \( z \) satisfying \( \beta \cdot (z - z^0) = 0 \); when \( n = 1 \) just the unique point \( z^0 \).

If \( b = (z^0, t^0) \) is a boundary point with \( t^0 = 0 \), then there is a complex line of equation \( z = z^0 \) in the complement of \( \Omega \), and there may or may not exist a hyperplane in \( \Gamma_{\Omega}(b) \) if the set \( \omega \) in \( \mathbb{C}^n \) where \( R \) is positive is linearly convex, there is such a hyperplane. If on the other hand \( b = (z^0, t^0) \) is a boundary point satisfying \(|t^0| = R(z^0) > 0 \), then a hyperplane \( Z \in \Gamma_{\Omega}(b) \) is given by an equation \( t/t^0 = \beta \cdot z \); the parallel hyperplane \( b + Z \) passing through \( b \) has the equation \( t/t^0 = 1 + \beta \cdot (z - z^0) \). It may happen that all real hyperplanes containing \( b + Z \) cuts \( \Omega \), but if this is not the case, the only real hyperplane containing \( b + Z \) and not cutting \( \Omega \) is that of equation \( \text{Re}(t/t^0) = 1 + \text{Re} \beta \cdot (z - z^0) \).

**Theorem 9.7.13** Let a function \( R: \mathbb{C}^n \to [-\infty, +\infty] \) be given and consider the complete Hartogs set \( \Omega \) defined as in Definition 9.4.2. Assume that \( \Omega \) is open and weakly linearly convex. Then \( R \) is continuous at every point where it is finite and positive, and all boundary points of \( \Omega \) satisfying \((z^0, t^0)\) with \(|t^0| = R(z^0) > 0 \) are accessible from the outside of class \( C^2 \). In fact, every complex hyperplane which passes through a boundary point \((z^0, t^0)\) with \(|t^0| = R(z^0) > 0 \) and does not meet \( \Omega \) is contained in a real external tangent plane.
In particular $\Gamma_\Omega(b) \subset \Theta_{\Omega,C}(b)$ for all points $b = (z^0, t^0)$ with $|t^0| = R(z^0) > 0$ ($\Theta_{\Omega,C}(b)$ is defined in Definition 9.7.5).

**Proof** Any point $(z^0, t^0)$ with $|t^0| = R(z^0) > 0$ belongs to the boundary of $\Omega$, so there exists by hypothesis a vector $\beta \in \mathbb{C}^n$ such that the complex hyperplane defined by $t/t^0 = 1 + \beta \cdot (z - z^0)$ lies entirely in the complement of $\Omega$. We shall prove that there is a real external tangent plane of class $C^2$ containing it.

That the complex hyperplane does not meet $\Omega$ means that

$$\frac{R(z)}{|t^0|} \leq |1 + \beta \cdot (z - z^0)|, \quad z \in \mathbb{C}^n.$$  

Now

$$|1 + z| \leq \frac{1}{2} + \frac{1}{2}|1 + z|^2 = 1 + \text{Re} z + \frac{1}{2}|z|^2, \quad z \in \mathbb{C},$$

with equality if and only if $|1 + z| = 1$. It follows that for any $\gamma > \frac{1}{2}$,

$$|1 + z| \leq 1 + \text{Re} z + \gamma|z|^2, \quad z \in \mathbb{C},$$

with equality only when $z = 0$. Hence

$$|1 + \beta \cdot (z - z^0)| \leq 1 + \text{Re} \beta \cdot (z - z^0) + \gamma|\beta\cdot (z - z^0)|^2$$

$$\leq 1 + \text{Re} \beta \cdot (z - z^0) + \gamma\|\beta\|^2 \cdot \|z - z^0\|^2,$$

with equality between the first and last expression only when $z = z^0$ or $\beta = 0$. Therefore, if we choose $c > \frac{1}{4}\|\beta\|^2$,

$$R(z)/|t^0| \leq 1 + \text{Re} \beta \cdot (z - z^0) + c\|z - z^0\|^2, \quad z \in \mathbb{C}^n,$$

with equality only when $z = z^0$.

So the set

$$U = \{(z, t); \text{ Re } (t/t^0) > 1 + \text{Re} \beta \cdot (z - z^0) + c\|z - z^0\|^2\},$$

taking $c > \frac{1}{2}\|\beta\|^2$, is a set with smooth boundary and the real hyperplane defined by $\text{Re } t/t^0 = 1 + \text{Re} \beta \cdot (z - z^0)$ is an external tangent plane of class $C^2$ of $\Omega$ at $(z^0, t^0)$.

From what we just proved it follows in particular that $R$ is upper semicontinuous where positive. On the other hand, $\Omega$ is open by hypothesis, which, as we noted, implies that the restriction $R|_\omega$ is lower semicontinuous.

### 9.7.7 Unions of increasing sequences of domains

If an increasing family $(V_j)_{j \in \mathbb{N}}$ of open sets in $\mathbb{R}^m$ is given with union $V$ and if $b \in \partial V$, let us denote by $\limsup \Theta_{V_j, \mathbb{R}}(b)$, understood as $(\limsup \Theta_{V_j, \mathbb{R}})(b)$,
all limits of real hyperplanes \( Y_j \in \Theta_{V_j,R}(b^{(j)}) \) at points \( b^{(j)} \in \partial V_j \) such that \( b^{(j)} \to b \) as \( j \to \infty \). Here \( \Theta_{V,R}(b) \) is defined in Definition 9.7.5. We shall use a similar notation for the complex hyperplanes: \( \limsup \Theta_{\Omega_j,C}(b) \) when \( \Omega_j \) increases to \( \Omega \), and also \( \limsup \Gamma_{\Omega_j}(b) \).

**Proposition 9.7.14** Let \( (V_j)_{j \in \mathbb{N}} \) be an increasing family of open subsets of \( \mathbb{R}^m \). Define \( \Theta_{V,R} \) as in Definition 9.7.5 using as structuring element a set \( S \) with boundary of class \( C^{k,s} \) with \( k \geq 1 \). Then \( \Theta_{V,R}(b) \subset \limsup \Theta_{V_j,R}(b) \) for all points \( b \in \partial V \). A similar result holds for the complex external tangent planes \( \Theta_{\Omega,C}(b) \) of an open subset \( \Omega \) of \( \mathbb{C}^n \). Here the inclusion can be strict. The limit superior is always nonempty.

**Proof** Take \( b = 0 \) and let \( U \) be an open set with boundary of the class in question such that \( V \cap \overline{U} = \{0\} \), defined as the set of all points \( x \) where \( \varphi(x) \) is negative, \( \varphi \) being of the right class and with nonvanishing gradient where it is zero. Let \( \varphi_s \), \( s > 0 \), be the function

\[
\varphi_s(x) = \varphi(x) - s + \|x\|^2, \quad x \in \mathbb{R}^m,
\]

and let \( U_s \) be the set where \( \varphi_s \) is negative. We note that when \( x \in V \), then \( x \notin U_s \), so that \( \varphi_s(x) \geq 0 \). If \( x \in V \cap U_s \), then \( \varphi(x) \geq 0 \) while \( \varphi_s(x) < 0 \). So \( \|x\|^2 < s - \varphi(x) \leq s \). Since \( \varphi \) is of class \( C^1 \), its gradient at any point in \( V \cap U_s \) is close to its gradient at the origin. For every large enough \( j \) there is a smallest \( s_j \) such that \( U_{s_j} \) and \( V_j \) have a common boundary point \( b^{(j)} \). Necessarily, then, \( \|b^{(j)}\|^2 \leq s_j \). For large \( j \), \( s_j \) is small, so small that the external tangent plane of \( U_{s_j} \) at \( b^{(j)} \) is as close as we like to the tangent plane of \( U \) at the origin. This shows that any hyperplane in \( \Theta_{V,R}(0) \) can be approximated by hyperplanes in \( \Theta_{V_j,R}(b^{(j)}) \).

**Proposition 9.7.15** Let \( (\Omega_j)_{j \in \mathbb{N}} \) be an increasing family of lineally convex open subsets of \( \mathbb{C}^n \) and denote their union by \( \Omega \). Then \( \limsup \Gamma_{\Omega_j}(b) = \Gamma_{\Omega}(b) \). In particular the graph of \( \Gamma_{\Omega} \) is closed.

**Proof** If \( Z \notin \Gamma_{\Omega}(b) \), then \( b + Z \) intersects \( \Omega \). Take a compact ball \( K \) in \( \Omega \) that contains a point of \( b + Z \) in its interior. Then for all sufficiently large \( j \), \( \Omega_j \) contains \( K \). All hyperplanes which are close enough to \( b + Z \) intersect \( K \) and hence also \( \Omega_j \) for these \( j \). Therefore, if \( Z_j \) tends to \( Z \) and \( b^{(j)} \in \partial \Omega_j \) tends to \( b \), then \( b^{(j)} + Z_j \) intersects \( \Omega_j \) for large \( j \). This means that hyperplanes \( b^{(j)} + Z_j \) with \( Z_j \in \Gamma_{\Omega}(b^{(j)}) \) cannot approach \( b + Z \). So we have \( \limsup \Gamma_{\Omega_j}(b) \subset \Gamma_{\Omega}(b) \).

The opposite inclusion is trivially true.

**Lemma 9.7.16** If \( A \) is a closed set in \( \mathbb{R}^m \) and \( b \in \partial A \), then \( \overline{\Theta_A^{(R)}(b)} \), where we use a Euclidean ball as structuring element, is nonempty.

**Proof** Given \( b \in \partial A \) and a positive number \( s \), take \( c \notin A \) with \( \|c - b\|^2 < s \). Take then \( r > 0 \) maximal so that \( B_{<}(c,r) \) does not cut \( A \). Clearly \( r \leq s \). On the boundary of this ball, there must exist a point \( p \in A \). Then \( \|p - b\|^2 < s \).
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Let \( \Omega \) be an open subset of \( \mathbb{C}^n \) such that it is equal to the interior of its closure. If \( \Omega \) is tangentially lineally convex at all points \( b \) in some open subset \( B \) of \( \partial \Omega \) (see Definition 9.7.6), then \( \Theta_{\Omega,C}^1(b) \subset \overline{\Theta_{\Omega,C}^2(b)} \subset \Gamma_{\Omega}(b) \), and \( \Gamma_{\Omega}(b) \) is nonempty for all \( b \in B \). In particular, tangential lineal convexity at all points \( b \in \partial \Omega \) implies weak lineal convexity.

**Proof** We apply Lemma 9.7.16 to \( A = \overline{\Omega} \). Then the interior of \( A \) is equal to \( \Omega \). Moreover, \( \text{graph}(\Gamma_{\Omega}) \) is closed; see Proposition 9.7.15.

If \( \Omega \) is lineally convex, \( \Gamma_{\Omega}(b) \) is not necessarily connected, not even when \( \Omega \) is a Hartogs set, as is shown by the example below as well as by Example 9.4.8 in Section 9.4, page 281.

**Example 9.7.18** Let \( \Omega \) be the Cartesian product of an annulus and a disk,

\[
\Omega = \{(z_1, z_2) \in \mathbb{C}^2; \ 1 < |z_1| < 2, \ |z_2| < 1\},
\]

a lineally convex set. We define complex hyperplanes \( Z_\beta \) passing through 0 by the equations \( \beta z_2 = (1 - \beta)z_2, \ z \in \mathbb{C}^2, \ \beta \in [0, 1] \). Use a ball \( B_\varepsilon(0, r) \) with \( 0 < r < 1 \) as structuring element. Then \( \Theta_{\Omega,C}^1(b) \), where \( b = (1, 1) \), consists of all the \( Z_\beta, \ \beta \in [0, 1] \), whereas \( \Gamma_{\Omega}(b) \) consists of \( Z_0 \) and \( Z_{\frac{1}{2}} \) for \( \frac{1}{2} \leq \beta \leq 1 \). Thus \( \Gamma_{\Omega}(b) \) does not contain \( \Theta_{\Omega,C}^2(b) \). We also note that \( \Gamma_{\Omega}(b) \) is not connected; it has two components, \( \{Z_0\} \) and \( \{Z_{\frac{1}{2}}; \ \frac{1}{2} \leq \beta \leq 1\} \).

If \( 0 < \beta < \frac{1}{2} \), then there are points \( z = (-1 - s, z_2) \in Z_\beta \cap \Omega \) far from \( b = (1, 1) \) (take \( s > 0, \ s = -1 - z_1 < 2/\beta - 4 \) as well as points in \( Z_{\frac{1}{2}} \cap \Omega \) arbitrarily close to \( b \).

The set \( \Omega \) is lineally convex, but if we approximate it from the inside by a set with boundary of class \( C^1 \) containing all points in \( \Omega \) with distance to \( \partial \Omega \) at least equal to \( \varepsilon > 0 \), then we get a set which is \( \mathcal{Z}_1 \)-convex but not \( \mathcal{Z}_r \)-convex for \( r \geq 1 + \varepsilon > 1 \). (For \( \mathcal{F} \)-convexity, see Definition 9.7.10; for \( \mathcal{Z}_r \)-convexity, see the beginning of Subsection 9.7.5.)

### 9.7.8 Convexity properties of superlevel sets

**Definition 9.7.19** Given any function \( f \) on a set \( X \) and with values in the set \( \mathbb{R} \) of extended real numbers and an element \( c \) of \( [-\infty, +\infty] \), we define its (non-strict) **superlevel set** as \( \{x \in X; \ f(x) \geq c\} \). Analogously we define its (non-strict) **sublevel set** as \( \{x \in X; \ f(x) \leq c\} \).

Given a complete Hartogs set with radius function \( R \), we shall denote by \( M_c \) the superlevel set \( \{z \in \mathbb{C}^n; \ R(z) \geq c\} \).
Generalized Convexity

Example 9.7.20 Consider the lineally convex Hartogs set \( \Omega \subset \mathbb{C} \times \mathbb{C} \) defined by the radius function

\[ R(z) = \min(|z - 2|, |z + 2|), \quad |z| < 1; \quad R(z) = 0, \quad |z| \leq 1. \]

Then \((0,2)\) belongs to the boundary of \( \Omega \) and \( \Gamma_{\Omega}((0,2)) \) consists of precisely two elements, the hyperplanes defined by \( t = -z \) and \( t = z \), respectively; thus it is not connected. (This shows that \( \Omega \) is not \( \mathbb{C} \)-convex in view of Zelinskii’s criterion mentioned near the end of Subsection 9.7.2.) However, the union of all the \( \Gamma_{\Omega}(b) \) with \( b \in \Omega \) is connected. We note that \( \Gamma_{\Omega}((1, \sqrt{5})) \) is connected and contains \( \Gamma_{\Omega}((0,2)) \). See Example 9.4.8 in Section 9.4.

The boundary points of \( \Omega \) are accessible from the outside by balls of a not too large radius, and \( \Theta_{\Omega, C}((0,2)) \) consists of all hyperplanes \( t = \lambda z, \) with \( \lambda \in [-1,1] \). We also note that the intersection of \( \Omega \) with the complex line \( t = c \) has two components if \( 2 \leq |c| < \sqrt{5} \).

Also, for \( a = s + i(1 - s/2) \) with a small positive number \( s \), the superlevel set \( M_{R(a)} \) is \( B_{<}(0, r) \)-convex for \( r \) slightly smaller than \( \sqrt{5} \), whereas for \( s = 0, a = i \), the superlevel set \( M_{R(i)} \), now equal to \( \{1,-1\} \), is \( B_{<}(0, r) \)-convex for any \( r \) but not convex. (This is a warning that \( r \)-convexity is not so meaningful for sets that are not regular open or regular closed.)

For simplicity we shall assume below that \( n = 1 \).

Theorem 9.7.21 Let \( \Omega \subset \mathbb{C} \times \mathbb{C} \) be a complete lineally convex Hartogs domain defined as in (9.42) with \( n = 1 \). Assume that a point \( a \in \omega \) is such that \( R(a) < \sup R \). Then there exists an \( r > 0 \) such that if \( a \) belongs to the erosion \( \varepsilon_{D_{<}(0, r)}(\omega) \), then \( a \in \alpha_{D_{<}(0, r)}(\mathcal{C}M_{R(a)}) \). In other words, since \( a \) belongs to the open set \( \omega \), the distance \( r \) to \( \mathcal{L}\omega \) is positive, and \( a \) is exterior accessible in \( M_{R(a)} \) by disks of radius \( r \).

**Proof** There is a complex hyperplane (thus a complex line in the present situation) in the complement of \( \Omega \) which passes through \((a, R(a))\). It cannot be vertical since \( a \in \omega \) and it cannot be horizontal since \( R(a) < \sup R \), so it must have an equation of the form

\[ \frac{t}{R(a)} = 1 + \beta(z - a) = \beta(z - a_{\beta}), \]

where \( \beta \neq 0 \) and \( a_{\beta} = a - 1/\beta \) is the point where the line hits the line \( t = 0 \).

This implies that the cone in \( \mathbb{C} \times \mathbb{R} \) defined by \(|t|/R(a) \geq |\beta(z - a_{\beta})|\) does not meet any point \((z, |t|)\) in \( \Omega \), in particular that \( a \) belongs to the disk \( D_{<}(a_{\beta}, s) \) with center at \( a_{\beta} \) and radius \( s = |a - a_{\beta}| = 1/|\beta| \). As noted, this disk does not meet \( \omega \), so \( a \in \alpha_{D_{<}(0, s)}(\mathcal{C}M_{R(a)}) \). We note finally that \( s = |a - a_{\beta}| \geq d(a, \mathcal{L}\omega) = r \).

There is no uniformity here: \( r \) depends on \( a \). But if \( \Omega \) is bounded and we restrict attention to points \( a \) in a compact subset of \( \omega \) and with \( R(a) \geq c > 0 \), we can choose a fixed \( r > 0 \). Thus \( M_{R(a)} \) is \( D_{<}(0, r) \)-convex.
There may be several lines of the form \( t = \beta (z - a) \) as mentioned in the proof. Then among all the possible values of \( \beta \in \mathbb{C} \) we can take the infimum of their absolute values, and any limit of these numbers must also define a line in the complement of \( \Omega \), since the complement is closed. This gives the largest possible value to \( r = 1/|\beta| \).

In Example 9.7.20 we see that, for a real such that \( 0 < a < 1 \),
\[
    r = 2 - a > d(a, \mathbb{C}_\omega) = 1 - a,
\]
implying that the number \( r \) obtained in the proof can be smaller than it is in an actual situation.

**Remark 9.7.22** In the other direction, if a closed \( r \)-convex set \( M \) in \( \mathbb{C} \) is given, then there exists a lineally convex open set in \( \mathbb{C}^n \times \mathbb{C} \) with radius function \( R \) such that \( M \sup R = M \); see Proposition 9.4.12 on page 284.

**Corollary 9.7.23** If \( \Omega \) is lineally convex and bounded, and its boundary is of class \( \mathcal{C}^1 \) at the set where \( R > 0 \), then a point \( a \in \omega \) belongs to \( \alpha_{D_<(0,r)}(\mathbb{C}M_{R(a)}) \) if
\[
    r \leq \frac{1}{\| (\text{grad } R)(a) \|_2}.
\]
This is the case for all points \( z \) with \( R(z) = R(a) \) if
\[
    r \leq \frac{1}{\sup_{z \in \omega} (\| (\text{grad } R)(z) \|_2; R(z) = R(a))}.
\]

We see that \( r \nearrow +\infty \) when \( R(a) \nearrow \sup R \), meaning that the superlevel set becomes more and more convex. We shall make this precise in Theorem 9.7.29.

**Proof** In this situation there is only one line in the complement of \( \Omega \) passing through \((a, R(a))\), and the absolute value of the coefficient \( \beta \) is \( \| (\text{grad } R)(a) \|_2 = 2|R_a| \). The radius \( r \) depends on \( a \) and may vary, but among all the points \( z \) with \( R(z) = R(a) \) its lower bound is positive.

We now consider a situation with two levels, \( R(a) \) and \( R(a) + s \geq R(a) \).

**Theorem 9.7.24** Let \( \Omega \subset \mathbb{C} \times \mathbb{C} \) be a lineally convex Hartogs domain defined as in Definition 9.4.2 with \( n = 1 \) and take a point \( a \in \omega \subset \mathbb{C} \) with \( R(a) < \sup R \). Then there exists a number \( r > 0 \) such that for all \( s \geq 0 \),
\[
    d(a, M_{R(a)+s}) \geq \frac{sr}{R(a)},
\]
where the inequality means that any point \( w \) with \( R(w) = R(a) + s \) is outside the disk \( D_<(a_\beta, r_1) \) with \( r_1 = r + sr/R(a) \).

It follows that \( w \) is accessible with disks of radius \( r_1 \) in the complement of the superlevel set \( M_{R(a)+s} \).
Proof As in the proof of Theorem 9.7.21, we see that the cone defined by 
\[|t|/R(a) \geq |\beta(z - a\beta)|,\]
where \(\beta\) is the coefficient in the equation of the line in the complement of \(\Omega\) passing through \((a, R(a))\), viz. \(t/R(a) = 1 + \beta(z - a)\), does not contain any point of the form \((z, \beta|z|)\) in \(\Omega\). We take \(r = 1/|\beta|\). In particular the disk \(D < (a\beta, r)\) with \(r_1 = r + sr/R(a)\) for any \(w\) with \(R(w) = R(a) + s\) does not meet \(M_{R(a)+s}\).

Since also \((w, R(w))\) admits a line \(t/R(w) = 1 + \gamma(z - w)\) in the complement of \(\Omega\), we must have \(|\gamma| \leq |\beta|\), so the corresponding radius \(r_2 = 1/|\gamma|\) is not smaller than \(r_1\).

9.7.9 Admissible multifunctions

Definition 9.7.25 Let \(\Omega\) be an open subset of \(\mathbb{C}^n\) and \(\gamma: B \rightarrow Gr_{n-1}(\mathbb{C}^n)\) a multifunction defined on a subset \(B\) of the boundary of \(\Omega\) and with values in the Grassmann manifold of all hyperplanes through the origin. Consider the following three conditions on \(\gamma\).

\[\gamma(b) \subset \Gamma_{\Omega}(b) \text{ for all } b \in B;\]  \hspace{1cm} (9.78)

the graph of \(\gamma\) is closed; and 

\[\gamma(b) \text{ is nonempty and connected for every } b \in B.\]  \hspace{1cm} (9.79)

We shall say that \(\gamma\) is admissible if these conditions are satisfied.

It follows that \(\text{graph}(\gamma)\) is connected if \(\partial \Omega\) is connected; see Lemma 9.7.27 below.

An example of an admissible multifunction is \(\Gamma_{\Omega}: \partial \Omega \rightarrow Gr_{n-1}(\mathbb{C}^n)\) provided \(\Gamma_{\Omega}(b)\) is connected for every \(b \in \partial \Omega\). (In particular, this is the case if the boundary is of class \(C^1\).) The graph is then automatically closed in view of Proposition 9.7.15. It is easy to see that in Examples 9.7.18 and 9.7.20, there is no admissible multifunction \(\gamma\) in any neighborhood of the points \((1, 1)\) and \((0, 2)\), respectively.

If \(\Omega\) is tangentially lineally convex, a candidate for \(\gamma\) might be the closure of \(\Theta_{\Omega, \mathbb{C}}\). Then property (9.78) holds by hypothesis, (9.79) by construction, and (9.80) may hold if the boundary of \(\Omega\) is sufficiently regular.

Example 9.7.26 Let \(\Omega\) be a convex open set in \(\mathbb{C}^n\). If \(\Omega\) is empty or equal to the whole space, then its boundary is empty. If \(\Omega\) is a slice, then its boundary has two components.

In all other cases, \(\partial \Omega\) is connected, and we know that the set of all real hyperplanes passing through a fixed boundary point \(b\) and not intersecting \(\Omega\) is connected. Then also the set of all complex hyperplanes containing \(b\) and contained in such a real hyperplane is connected—the mapping \(Y \mapsto Y_{[b]}\) is continuous as we noted in Section 9.7.2. Thus \(\Gamma_{\Omega}\) is an admissible multifunction except in the first-mentioned cases, even if the boundary is not of class \(C^1\).
If a lineally convex open set $\Omega$ has a $C^1$ boundary, $\Gamma_\Omega(b)$, a singleton set, depends continuously on $b$. When $\Gamma_\Omega(b)$ is no longer a singleton, the following result will serve instead of the continuity.

**Lemma 9.7.27** Let $\Omega$ be an open set in $C^n$ and $\gamma: A \supset \text{Gr}_{n-1}(C^n)$ an admissible multifunction on a subset $A$ of $\partial \Omega$. Then the graph of $\gamma$ over $B$,

$$\text{Graph}_B(\gamma) = \{(b, Z); \ b \in B \text{ and } Z \in \gamma(b)\},$$

is connected for every connected subset $B$ of $A$. In particular the graph of $\gamma$ is connected if the boundary of $\Omega$ is connected and $\gamma$ is defined on all of it.

**Proof** Assume that $\text{Graph}_B(\gamma) = V_0 \cup V_1$, where the $V_j$ are disjoint and closed relative to $\text{Graph}_B(\gamma)$. Define $B_j$ as the set of all points $b$ such that some hyperplane in $\gamma(B)$ belongs to $V_j$, $j = 0, 1$. Then $B_0$ and $B_1$ are disjoint, since by hypothesis every $\gamma(b)$ is connected. Moreover $B_0$ and $B_1$ are closed relative to $B$, since the graph of $\gamma$ is closed and the manifold $\text{Gr}_{n-1}(C^n)$ is compact. By hypothesis $B$ is connected, so either $B_0$ or $B_1$ must be empty. Hence $V_0$ or $V_1$ is empty, proving that the graph of $\gamma$ over $B$ is connected.

We note that $\gamma_*(B)$ is connected as a continuous image of the graph (it is the projection of the graph on the target space $\text{Gr}_{n-1}(C^n)$).

**Proposition 9.7.28** Let $\Omega$ be an open subset of $C^n$ and $F$ an affine subspace of $C^n$. Denote by $\Omega_F$ the set $\Omega \cap F$ considered as an open subset of $F$. Every complex hyperplane $Z$ in $C^n$ which does not contain $F$ gives rise to a complex hyperplane $\psi(Z) = Z \cap F$ in $F$. Let an admissible multifunction $\gamma: B \supset \text{Gr}_{n-1}(C^n)$ be given and define a multifunction $\gamma_F$ on $B \cap \partial \Omega_F$ by $\gamma_F(b) = \{\psi(Z); \ Z \in \gamma(b)\}$. Then $\gamma_F$ is an admissible multifunction on $B \cap \partial \Omega_F$.

**Proof** Since $\psi(Z) \subset Z$, it is clear that $\gamma_F(b) \subset \Gamma_{\Omega_F}(b)$; thus (9.78) in Definition 9.7.25 holds. The graph of $\gamma$ over any compact subset of $\partial \Omega$ is compact; hence the graph of $\gamma_F$ over any compact subset of $\partial \Omega_F$ is compact, thus closed: property (9.79) holds. Finally (9.80) follows since $\psi$ is continuous and thus maps connected subsets onto connected subsets.

The proposition can in particular be applied to $\Gamma_\Omega$ if $\Gamma_\Omega(b)$ is connected for all $b \in \partial \Omega$.

**9.7.10 Links to ordinary convexity**

**Theorem 9.7.29** Let $R$ be a continuous real-valued function defined on $C^n$ and define $\Omega$ as in Definition 9.4.2. Assume that $\Omega$ is connected and that its boundary is of class $C^1$ (at least in a neighborhood of $M_{\sup R}$). Then the set $M_{\sup R}$ where $R$ attains its maximum,

$$M_{\sup R} = \{z \in C^n; \ R(z) = \sup R\}, \quad (9.81)$$

is convex.
Generalized Convexity

Proof A set is convex if and only if its intersection with every one-dimensional complex affine subspace is convex. Therefore it is enough to prove the theorem for \( n = 1 \).

So let \( n = 1 \) and let \( a \) belong to the boundary of \( M_{\text{sup} R} \). We shall prove that there is an open half plane with \( a \) on its boundary which does not meet \( M_{\text{sup} R} \), proving the convexity of that set.

We have \((\text{grad } R)(a) = 0\), and near \( a \) there are points \( c \) with \((\text{grad } R)(c)\) nonzero and arbitrarily small. In view of Corollary 9.7.23 this means that there is a disk of arbitrarily large radius with \( c \) on its boundary. The disk is of the form \( D_{\leq}(c_{\beta}, r) \), where \( r = |c - c_{\beta}|, c_{\beta} = c - 1/\beta \) being the point where the line \( t/R(c) = 1 + \beta(z - c) \) hits the plane \( t = 0 \). The normalized vectors \((c - c_{\beta})/(c - c_{\beta})\) have an accumulation point, and this proves that the union of all the disks \( D_{\leq}(c_{\beta}, r) \) when \( c \) varies in an arbitrarily small neighborhood of \( a \) contains an open half plane with \( a \) on its boundary. We are done.

The assumption that the boundary be of class \( C^1 \) can be weakened, as we shall now show.

Theorem 9.7.30 Let \( R \) be a continuous real-valued function defined on \( \mathbb{C}^n \) and define \( \Omega \) by (9.42). Assume that \( \Omega \) is bounded and connected and that there exists an admissible multifunction \( \gamma \) defined at all points \((z^0, t^0)\) with \(|t^0| = R(z^0) > 0\), thus on the boundary over the base of \( \Omega \) (see Definition 9.7.25). Then the set \( M_{\text{sup} R} \) where \( R \) attains its maximum is convex.

For a Hartogs domain \( \Omega \) we always have \( \Gamma_{\Omega}(b) \subset \Theta_{\Omega, C}(b) \) when \( b = (z^0, t^0) \) with \(|t^0| = R(z^0) > 0\) (Theorem 9.7.13); if the domain is tangentially lineally convex, we have \( \Gamma_{\Omega}(b) = \Theta_{\Omega, C}(b) \). For such domains we therefore have an admissible multifunction \( \gamma = \Gamma_{\Omega} = \Theta_{\Omega, C} \): (9.78) is obvious; (9.79) follows from Proposition 9.7.15; (9.80) follows from Proposition 9.7.7.

We note that the hypothesis is satisfied in particular if \( \Omega \) is lineally convex and \( R \) is of class \( C^1 \). In (Kiselman 1996, Theorem 4.8) the result was proved under this hypothesis, and even under the weaker one that \( R \) can be approximated from below by \( C^1 \) functions.

In view of Zelinsky’s characterization of \( C \)-convex sets mentioned near the end of Section 9.7.2, the hypotheses are satisfied for \( C \)-convex sets, again taking \( \gamma = \Gamma_{\Omega} \). There are easy examples which show that \( M_{\text{sup} R} \) need not be convex if we drop the hypothesis of connectedness; see Example 9.4.8.

Proof of Theorem 9.7.30. Again, the set \( M_{\text{sup} R} \) is convex if its intersection with every one-dimensional complex affine subspace is convex. Proposition 9.7.28 shows that if we have an admissible multifunction on a subset of \( \partial \Omega \), then there is one also on a corresponding subset of \( \partial \Omega_F \), \( F \) being any affine subspace of \( \mathbb{C}^n \times \mathbb{C} \). Therefore, taking \( F \) as the Cartesian product of a complex line in \( \mathbb{C}^n \) and the line \( z = 0 \), we see that it is enough to prove the theorem for \( n = 1 \).
So let \( n = 1 \). To prove that \( M_{\text{sup } R} \) is convex means to prove that the segment \([s_0, s_1]\) is contained in \( M_{\text{sup } R} \) if \( s_0, s_1 \in M_{\text{sup } R} \). There is no loss in generality if we assume that \( s_0 = -1 \) and \( s_1 = 1 \).

A non-vertical and non-horizontal complex line through \((a, t^0)\) with \( t^0 \neq 0 \) has the equation
\[
\frac{t}{\overline{t}} = 1 + \beta(z - a) = \beta(z - a_{\beta}), \quad z \in \mathbb{C},
\]
where \( a_{\beta} = a - 1/\beta \) is the point where the line hits the plane \( t = 0 \). We define
\[
q(a, \beta) = \begin{cases} 
  a - 1/\beta & \text{if } \beta \neq 0, \\
  \infty & \text{if } \beta = 0. 
\end{cases}
\]
In case \( R \) is differentiable at the point \( a, \beta \) is uniquely determined if we require that the line be in \( \Gamma_{\Omega((a, t^0))} \).

We denote as before by \( \omega \) the set of all points \( z \in \mathbb{C} \) such that \( R(z) > 0 \). In general the external tangent is not unique and we shall denote by \( Q(a) \) the set of all points \( a - 1/\beta \) that can be obtained from complex lines in \( \gamma((a, t^0)) \), thus
\[
Q(a) = \{ q(a, \beta); \beta \in \gamma(a, R(a)) \} \subset S^2 = \mathbb{C} \cup \{ \infty \}, \quad a \in \omega. \tag{9.82}
\]
We define \( Q(a) = \{ a \} \) when \( a \notin \omega \). Thus \( Q \) is a multifunction, \( Q: S^2 \rightrightarrows S^2 \setminus \omega \); its images \( Q(a) \) are compact and connected.

The radius can always be estimated by
\[
R(z) \leq R(a)|\beta| \cdot |z - a_{\beta}|, \quad z \in \mathbb{C}, \quad a \in \omega, \quad \beta \in \gamma(a, R(a)), \quad a_{\beta} = q(a, \beta),
\]
with equality for \( z = a \), assuming \( \beta \neq 0 \). In particular, if \( w \in M_{\text{sup } R} \), then
\[
R(a)|\beta| \cdot |a - q(a, \beta)| = R(a) \leq R(w) \leq R(a)|\beta| \cdot |w - q(a, \beta)|.
\]
If \( a_{\beta} \in Q(a) \setminus \{ \infty \} \), then necessarily \( \beta \neq 0 \), so that
\[
|a - a_{\beta}| \leq |w - a_{\beta}|, \quad a \in \omega, \quad w \in M_{\text{sup } R}, \quad a_{\beta} \in Q(a) \setminus \{ \infty \}. \tag{9.83}
\]
Assume that \(-1 \) and \( 1 \) belong to \( M_{\text{sup } R} \); we shall then prove that any point \( c \in [-1, 1] \) belongs to \( M_{\text{sup } R} \). Consider \( Q(c + iy) \) for real \( y \). We know from Lemma 9.7.27 that the set \( Q(c + iR) \) is connected. If \( \omega \) is bounded and \( y \) or \(-y \) is very large, then \( Q(c + iy) = \{ c + iy \} \). In general we can prove that \( \text{Im} \ a > 1 \) implies that \( \text{Im} b > 0 \) for all \( b \in Q(a) \), and similarly \( \text{Im} a < -1 \) implies \( \text{Im} b < 0 \) for all \( b \in Q(a) \). This follows from the following lemma.

**Lemma 9.7.31** If \( \Omega \) is a complete Hartogs domain in \( \mathbb{C}^2 \) with radius function \( R \) and if \( \pm 1 \in M_{\text{sup } R} \), then for all \( b \in \mathbb{C} \) with \( |\text{Re} a| \leq 1 \) and all \( b \in Q(a) \setminus \{ \infty \} \) we have
\[
\text{Im} a \geq 1 \quad \text{implies} \quad \text{Im} b \geq \frac{1}{2}(\text{Im} a - 1) \quad \text{and}
\]
\[
\text{Im} a \leq -1 \quad \text{implies} \quad \text{Im} b \leq \frac{1}{2}(\text{Im} a + 1).
\]
Generalized Convexity

Proof We know from (9.83) that \(|a - b| \leq |1 - b|\). Expanding \(|1 - b|^2 - |a - b|^2 \geq 0\), we get

\[
2(\text{Re } b)(\text{Re } a + 1) + 1 - (\text{Im } a)(\text{Im } a - 2\text{Im } b) \geq (\text{Re } a)^2 \geq 0,
\]

from which we deduce that \(1 \geq (\text{Im } a)(\text{Im } a - 2\text{Im } b)\), an inequality which implies those in the lemma.

Proof of Theorem 9.7.30, cont’d. So \(Q(c + iy)\) must pass from the upper half plane to the lower half plane when \(y\) goes from large positive values to large negative values, \(c\) being fixed. But it can never pass the real axis at points with \(x \geq 1\) or \(x \leq -1\). Indeed, if \(b\) is real and larger than or equal to 1, we get from (9.83), taking \(a = c + iy\),

\[
|a - b| \leq |1 - b| = b - 1,
\]

implying \(\text{Re } a \geq 1\), so that \(c \geq 1\) contrary to assumption. Likewise, \(Q(c + iy)\) cannot pass the real axis at a point with \(x \leq -1\).

However, \(Q(c + iy)\) cannot pass from numbers with arbitrarily large positive imaginary part to numbers with large negative imaginary part in the strip \(-1 < \text{Re } z < 1\) either. In fact, \(\omega\) is connected, so there exists a curve contained in \(\omega\) connecting \(-1\) to \(1\), and \(Q(c + iy)\) cannot cross that curve.

Hence it is impossible for \(Q(c + iy)\) to pass from the upper half plane to the lower half plane if it has only finite values. So it must have an infinite value, which means that \(c + iy^0 \in M_{\sup R}\) for at least one \(y^0\).

We thus know that there is a \(y^0\) such that \(c + iy^0 \in M_{\sup R}\); without loss of generality we may assume that it is nonnegative. Choose \(y^0\) as small as possible. If \(y^0 = 0\) we are done: \(c \in M_{\sup R}\). Let us assume that \(y^0 > 0\) and try to reach a contradiction.

By (9.83) any point \(b \in Q(a) \setminus \{\infty\}\) must lie in each of the three half planes

\[
|a - b| \leq |1 - b|, \quad |a - b| \leq |-1 - b|, \quad |a - b| \leq |c + iy^0 - b|.
\]

The intersection of these three half planes is a triangle, and the union of these triangles when \(a = c + iy\) with \(y \in [\frac{1}{2}y^0, y^0]\) is bounded. Thus the possible finite values for \(b\) when \(a\) varies as indicated is bounded, and for \(a = c + iy\) with \(\frac{1}{2}y^0 \leq y < y^0\) the point \(b\) cannot be infinity. On the other hand, when \(a = c + iy^0 \in M_{\sup R}\), then \(Q(a)\) must contain \(\infty\). This means that the set of all points \(b \in Q(a)\) originating from points \(a = c + iy\) with \(y \in [\frac{1}{2}y^0, y^0]\) consists of \(\infty\) and a nonempty bounded set; it is not connected, in contradiction to Lemma 9.7.27. This contradiction shows that we must have \(c \in M_{\sup R}\) and proves the theorem. \(\square\)

It is easy to modify Theorem 9.7.30 using Möbius mappings, at least if \(n = 1\).

In fact, any mapping

\[
\mathbb{C} \times \mathbb{C} \ni (z, t) \mapsto \left(\frac{a + bz}{c + dz}, \frac{t}{c + dz}\right) = (z', t') \in \mathbb{C} \times \mathbb{C}
\]
preserves lineal convexity, as was shown in (Kiselman 1996: Lemma 8.1). Denote by $a_{\beta}$ the point where a line $t/t^0 = 1 + \beta(z - z^0)$ intersects the $z$-plane. The line can be mapped by a Möbius mapping to a line $t' = \text{constant}$. This mapping takes the point $a_{\beta}$ to infinity, and all circles in the $z$-plane which pass through $a_{\beta}$ are mapped onto straight lines. Convex sets are transformed accordingly:

**Definition 9.7.32** Let $b$ be a complex number or $\infty$. Let us say that a subset $A$ of the Riemann sphere $\mathbb{C} \cup \{\infty\}$ is $b$-convex if

$$(10.4.1) \quad b \not\in A; \quad \text{and}$$

$$(10.4.2) \quad \varphi^*(A) \text{ is convex if } \varphi \text{ is a Möbius mapping which maps } b \text{ to infinity.}$$

**Corollary 9.7.33 (to Theorem 9.7.30)** Let $\Omega$ and $\gamma$ be as in Theorem 9.7.30, assume that $n = 1$ and let $\pi$ denote the projection defined by $\pi(z, t) = z$. Consider a line $Z \in \gamma(a)$, where $a = (z^0, t^0)$, $|t^0| = R(z^0) > 0$, and let $b$ be the point such that $(b, 0) \in a + Z$. Then the set $\pi^*((a + Z) \cap \Omega)$ is $b$-convex. □
9.8 Duality of Functions Defined in Lineally Convex Sets

Abstract of this section

The term duality represents a collection of ideas where two sets of mathematical objects confront each other. A most successful duality is that between the space $\mathcal{D}(\Omega)$ of test functions (smooth functions of compact support) and its dual $\mathcal{D}'(\Omega)$ of distributions.

Similarly, the theory of analytic functionals, developed by André Martineau in his doctoral thesis (1963)—also in (Œuvre de André Martineau 1977:47–210)—is based on a duality, now between the Fréchet space of holomorphic functions $\mathcal{O}(\Omega)$ in an open set $\Omega$ and its dual $\mathcal{O}'(\Omega)$. In many, but not all respects, it is analogous to distribution theory.

In complex geometry, lineal concavity and lineal convexity can be treated successfully using concepts of duality.

9.8.1 Introduction to this section

Lineal convexity, a kind of complex convexity intermediate between usual convexity and pseudoconvexity, appears naturally in the study of Fantappiè transforms of analytic functionals. A set is called lineally convex if its complement is a union of complex hyperplanes. This property can be most conveniently defined in terms of the notion of dual complement: the dual complement of a set in $\mathbb{C}^n$ is the set of all hyperplanes that do not intersect the set. It is natural to add a hyperplane at infinity and consider $\mathbb{C}^n$ as an open subset of $\mathbb{P}^n$, complex projective space of dimension $n$. The definition of dual complement is then the same, and somewhat more natural: the set of all hyperplanes is again a projective space. In this setting, the dual complement is often called the projective complement. Indeed, Martineau (1966) called it le complémentaire projectif; the term dual complement used here was introduced by Andersson, Passare and Sigurdsson in a preprint from 1991 of their forthcoming book (2004).

We can now simply define a lineally convex set as a set which is the dual complement of its dual complement (here it becomes obvious that we should identify the hyperplanes in the space of all hyperplanes with the points in the original space). So this duality works well for sets. What about functions?

In convexity theory, a convenient dual object of a set is its support function as defined in Section 9.2. For functions, we have the Fenchel transformation, defined as well in Section 9.2.

Is there a duality for functions that generalizes the duality for sets defined by the dual complement? In this section we shall study such a duality. We call it the logarithmic transformation. It has many properties in common with the Fenchel transformation. However, there are some striking differences. The effective domain, defined by formula (9.2.5), of a Fenchel transform is always
convex, but the effective domain of a logarithmic transform need not be lineally convex (Example 9.8.16). This is connected with the fact that the union of an increasing sequence of lineally convex sets is not necessarily lineally convex (Example 9.8.17). However, the interior of the effective domain of a logarithmic transform is always lineally convex (Theorem 9.8.14), and the transform is plurisubharmonic there (Theorem 9.8.18).

Working with functions defined on $\mathbb{P}^n$ is the same as working with functions defined on $\mathbb{C}^{1+n} \setminus \{0\}$ which are constant on complex lines, i.e., homogeneous of degree zero. For instance a plurisubharmonic function on an open subset of $\mathbb{P}^n$ can be pulled back to an open cone in $\mathbb{C}^{1+n} \setminus \{0\}$ and the pullback is plurisubharmonic for the $1+n$ coordinates there. However, I cannot define a duality for such functions. I have been led to consider instead functions defined on subsets of $\mathbb{C}^{1+n} \setminus \{0\}$ which are homogeneous in another sense: they satisfy $f(tz) = -\log|t| + f(z)$. Such functions are not pullbacks of functions on projective space, but the duality works for them. In a coordinate patch like $z_0 = 1$ we can identify them with functions on a subset of $\mathbb{P}^n$.

Given any function $F$ on $\mathbb{C}^n$, we can define a function $f$ on $\mathbb{C}^{1+n} \setminus \{0\}$ by $f(z) = F(z_1/z_0, \ldots, z_n/z_0) + c \log|z_0|$ when $z_0 \neq 0$ and $f(z) = +\infty$ when $z_0 = 0$, where $c$ is an arbitrary real constant; this function is homogeneous in the sense that $f(tz) = c \log|t| + f(z)$, so we can choose any type of homogeneity. In other words, locally all kinds of homogeneity are equivalent, and there is no restriction in imposing the homogeneity we have here, viz. $c = -1$.

As mentioned in Section 9.2, there are several other notions related to lineal convexity. The property called Planarkonvexität in German (see Behnke & Peschl 1935), or weak lineal convexity is weaker than lineal convexity: an open connected set is called weakly lineally convex if through any boundary point there passes a complex hyperplane which does not intersect the set. Aizenberg (1967) proved that these domains are precisely the components of $\Omega^{* *}$ (for notation see Subsection 9.8.2 below).

Strong lineal convexity was defined by Martineau (1966: Definition 2.2) as a topological property of the space of holomorphic functions in a domain. Martineau (1966: Theorem 2.2) and Aizenberg (1966) proved independently that convex sets are strongly lineally convex. This property was given a geometric characterization by Znamenskij (1979). This geometric property is now called $C$-convexity. Its relation to lineal convexity has been studied by Zelinskij (1988) and others. For these two properties we refer also to the survey by Andersson, Passare and Sigurdsson (2004) and the monograph by Hörmander (1994).

Another generalization is the notion of $m$-lineal convexity to be studied in the next section.
Duality in Lineally Convex Sets

9.8.2 Notation

Let $A$ be a subset of $\mathbb{C}^{1+n} \setminus \{0\}$, where $n \geq 1$. We shall say that $A$ is homogeneous if $tz \in A$ as soon as $z \in A$ and $t \in \mathbb{C} \setminus \{0\}$. To any homogeneous subset $A$ of $\mathbb{C}^{1+n} \setminus \{0\}$ we define its dual complement $A^*$ as the set of all hyperplanes passing through the origin which do not intersect $A$. Since any such hyperplane has an equation $\zeta \cdot z = \zeta_0 z_0 + \cdots + \zeta_n z_n = 0$ for some $\zeta \in \mathbb{C}^{1+n} \setminus \{0\}$, we can define

$$A^* = \{ \zeta \in \mathbb{C}^{1+n} \setminus \{0\}; \zeta \cdot z \neq 0 \text{ for every } z \in A \}.$$  

(9.84)

Strictly speaking, we should have two copies of $\mathbb{C}^{1+n} \setminus \{0\}$ (a Greek and a Latin one), and consider $A^*$ as a subset of the dual (i.e., the Greek) space. A homogeneous set is called lineally convex if $\mathbb{C}^{1+n} \setminus A$ is a union of complex hyperplanes passing through the origin. A dual complement $A^*$ is always lineally convex, and we always have $A^{**} \supset A$. The set $A^{**}$ is called the lineally convex hull of $A$. A set $A$ is lineally convex if and only if $A = A^{**}$.

The operation of taking the dual complement is an example of a Galois correspondence, and the operation of taking the lineally convex hull defines a cleistomorphism in the ordered set of all subsets of $\mathbb{C}^{1+n} \setminus \{0\}$. For the general definitions of these concepts, see Section 9.3.

We shall write $z = (z_0, z') = (z_0, z_1, \ldots, z_n)$ for points in $\mathbb{C}^{1+n} \setminus \{0\}$, with $z_0 \in \mathbb{C}$ and $z' = (z_1, \ldots, z_n) \in \mathbb{C}^n$. Homogeneous sets in $\mathbb{C}^{1+n} \setminus \{0\}$ correspond to subsets of projective $n$-space $\mathbb{P}^n$, and we can transfer the notions of dual complement and lineal convexity to $\mathbb{P}^n$. In the open set where $z_0 \neq 0$ we can use $z'$ as coordinates in $\mathbb{P}^n$.

We shall denote by

$$Y_\zeta = \{ z \in \mathbb{C}^{1+n} \setminus \{0\}; \zeta \cdot z = 0 \}, \quad \zeta \in \mathbb{C}^{1+n} \setminus \{0\},$$

(9.85)

the hyperplane defined by $\zeta$. Then the dual complement can be conveniently defined as

$$A^* = \{ \zeta; Y_\zeta \cap A = \emptyset \},$$

(9.86)

and its set-theoretical complement in $\mathbb{C}^{1+n} \setminus \{0\}$ is

$$\mathcal{L}A^* = (\mathbb{C}^{1+n} \setminus \{0\}) \setminus A^* = \{ \zeta; Y_\zeta \cap A \neq \emptyset \}.$$  

(9.87)

The complement of the lineally convex hull $A^{**}$ can be written as

$$\mathcal{L}A^{**} = \bigcup_{\alpha \in A^*} Y_\alpha.$$  

We shall use this idea in the following lemma.

Lemma 9.8.1 For any subset $\Gamma$ of $\mathbb{C}^{1+n} \setminus \{0\}$ we define

$$A = \mathcal{L} \left( \bigcup_{\gamma \in \Gamma} Y_\gamma \right) = \bigcap_{\gamma \in \Gamma} \mathcal{L}Y_\gamma.$$  

Then $A$ is lineally convex. Moreover $A^{**} = A = \Gamma^*$ and $A^* = \Gamma^{**} \supset \Gamma$. 

Proof Clearly $A$ as the complement of a union of hyperplanes is lineally convex, so $A^{**} = A$. The statement $a \in A$ is equivalent to $\gamma \cdot a \neq 0$ for all $\gamma \in \Gamma$, which by definition means that $a \in \Gamma^*$; thus $A = \Gamma^*$. As a consequence, $A^* = \Gamma^{**}$.

How does the operation of taking the dual complement intertwine with the topological operations of taking the interior and closure? The answer is the following (we write $A^\circ$ for the interior and $\overline{A}$ for the closure of a set $A$).

**Proposition 9.8.2** For any homogeneous subset $A$ of $\mathbb{C}^{1+n} \setminus \{0\}$ we have

(A) If $A$ is open, then $A^*$ is closed.

(B) If $A$ is closed, then $A^*$ is open.

(C) $\overline{A}^* \subset A^{**}$.

(D) $A^{**} = \overline{A}^*$.

(E) If $A$ is the closure of an open set, then $A^*$ is the interior of a closed set.

**Proof** (A) and (B). To see that $A^*$ is closed if $A$ is open we only have to look at (9.86). The same formula shows that $A^*$ is open if $A$ is closed.

(C). Since $A^\circ$ is open, $A^{**}$ is closed according to (A), so that $A^{**} = \overline{A}^* \supset \overline{A}^\circ$.

(D). Since $\overline{A}^*$ is open according to (B), and since $\overline{A}^* \subset A^*$, we get $\overline{A}^* \subset A^{**}$. To prove the inclusion $A^{**} \subset \overline{A}^*$ we argue as follows. If $\zeta \in A^{**}$, then $Y_\theta \cap A = \emptyset$ for all $\theta$ near $\zeta$. The union of these hyperplanes $Y_\theta$ is a neighborhood of $Y_\zeta$, so $\zeta \in \overline{A}^*$.

(E). If $A = B^\circ$ with $B$ open, then according to (D), $A^* = B^{**} = B^{**}$, the interior of the closed set $B^*$.

This proves the proposition.

If $A$ is the interior of a closed set $C$, we get $A^* = C^{**} \supset C^\circ$, possibly strictly.

**Corollary 9.8.3** If a subset $A$ of $\mathbb{C}^{1+n} \setminus \{0\}$ is strongly contained in a set $B$ in the sense that $\overline{A} \subset B^\circ$, then $B^*$ is strongly contained in $A^*$.

**Proof** Using (C) and (D) in Proposition 9.8.2 we see that $\overline{A} \subset B^\circ$ implies $B^\circ \subset B^{**} \subset \overline{A}^* = A^{**}$.

**Corollary 9.8.4** If a subset $A$ of $\mathbb{C}^{1+n} \setminus \{0\}$ is lineally convex, then its interior $A^\circ$ is also lineally convex.

**Proof** If $A = B^*$, then $A^\circ = B^{**} = B^*$ by (D) in Proposition 9.8.2, which shows that $A^\circ$ is lineally convex.

By way of contrast, the closure of a lineally convex set is not necessarily lineally convex if $n \geq 2$. It turns out that the lineal convexity of the closure is connected with the question whether we have equality in (C) in Proposition 9.8.2, as shown by the following result.
Duality in Lineally Convex Sets

**Corollary 9.8.5** Let $B$ be any lineally convex subset of $\mathbb{C}^{1+n} \setminus \{0\}$. Then its closure $\overline{B}$ is lineally convex if and only if its dual complement $A = B^*$ satisfies (C) in Proposition 9.8.2 with equality.

**Proof** Using the lineal convexity of $B$ and then (C) and (D) in Proposition 9.8.2, we get

$$\overline{B} = \overline{A^*} \subset A^{o*} = B^{o*} = \overline{B}^*.$$

Thus equality in (C) is equivalent to $\overline{B}$ being lineally convex.

The inclusion (C) in Proposition 9.8.2 can be strict simply for dimensionality reasons. This will be clear from the following result, where we use the interior relative to a subspace instead of the interior with respect to the whole space.

**Proposition 9.8.6** Let $A$ be a homogeneous set in $\mathbb{C}^{1+n} \setminus \{0\}$ which is contained in a complex subspace $F$ of $\mathbb{C}^{1+n}$. Let $A_F = \text{relint}(A)$ denote the relative interior of $A$, i.e., the interior taken with respect to $F$. Then $A_F^* \subset (A_F)^* \cup F^\circ$, where $F^\circ$ is the set

$$F^\circ = \{ \zeta \in \mathbb{C}^{1+n} \setminus \{0\}; \; F \setminus \{0\} \subset Y_\zeta \}.$$

If $A$ is open in $F$, then $A^* \cup F^\circ$ is closed.

Note that when $F = \mathbb{C}^{1+n}$, then $F^\circ$ is empty and we are reduced to Proposition 9.8.2.

**Proof** Take a point $\zeta \notin (A_F)^* \cup F^\circ$. Then there is a point $a \in A_F \cap Y_\zeta$ and a non-zero vector $b \in F \setminus Y_\zeta$. If $\theta$ is close to $\zeta$, then the hyperplane $Y_\theta$ cuts the complex line $\{a + tb; \; t \in \mathbb{C}\}$ in a unique point $a(\theta, t)$ close to $a$, and since $a$ is in the relative interior of $A$, $a(\theta, t)$ belongs to $A$ as soon as $\theta$ is close enough to $\zeta$. Therefore $\theta \notin A^*$ for all these $\theta$, which means that $\zeta \notin A_F^*$.

Finally, if $A$ is open in $F$, then $A^* \cup F^\circ = A^* \cup F^\circ \cup Y_\zeta \subset (A_F)^* \cup F^\circ = A^* \cup F^\circ$, since $A_F = A$ and $F^\circ$ is closed.

**Example 9.8.7** It is now obvious that the inclusion in (C) can be strict. Take a nonempty relatively open set $A \subset F \neq \mathbb{C}^{1+n}$. Then $A^o = \emptyset$, and $A^{o*} = \mathbb{C}^{1+n} \setminus \{0\}$. But $A^* = (A^o)^* \cup F^\circ = A^* \cup F^\circ \neq \mathbb{C}^{1+n} \setminus \{0\}$. □

**Example 9.8.8** Also the inclusion in Proposition 9.8.6 can be strict. There are sets $A$ such that $A^o = \emptyset$, $A^o = B \neq \emptyset$, and $B^* = A^*$. Thus $A^{o*} = \mathbb{C}^{1+n} \setminus \{0\}$ and $A^* = B^* = B^* \neq \mathbb{C}^{1+n} \setminus \{0\}$. Such a set is the set $A$ of all $z \in \mathbb{C}^{1+2}$ with $|z_1|^2 + |z_2|^2 < |z_0|^2$ and either $z_1$ is a complex rational or $z_2 = 0$. (Here the only choice for $F$ is the whole space, so that $F^\circ$ is empty.) □

**Example 9.8.9** A simple example of a lineally convex set whose closure is not lineally convex is the following. Define

$$A = \{ z \in \mathbb{C}^{1+2} \setminus \{0\}; \; |z_1| < |z_2| \}; \quad \overline{A} = \{ z \in \mathbb{C}^{1+2} \setminus \{0\}; \; |z_1| \leq |z_2| \}$$
Then $A$ is lineally convex. Any hyperplane which avoids $A$ must pass through the point $(z_0, z_1, z_2) = (1, 0, 0)$. But this point belongs to $\overline{A}$. This shows that $\overline{A}$ is not lineally convex.

More generally, let $\Gamma$ be a lineally convex subset of the Greek copy of $\mathbb{C}^{1+n} \setminus \{0\}$ and define $A$ as in Lemma 9.8.1. We can easily choose $\Gamma$ without interior points but still such that the union

$$
\bigcup_{\gamma \in \Gamma} Y_{\gamma} = \mathbb{C}A
$$

has interior points. Thus $\Gamma^\circ = \emptyset$, $\mathbb{C}\overline{A} = (\mathbb{C}A)^\circ \neq \emptyset$. Then $(\overline{A})^{**} = A^{**} = \Gamma^{**} = \mathbb{C}^{1+n} \setminus \{0\}$ (see Lemma 9.8.1 and (D) in Proposition 9.8.2), but $\overline{A} = \mathbb{C}((\mathbb{C}A)^\circ) \neq \mathbb{C}^{1+n} \setminus \{0\}$. This shows that $\overline{A}$ cannot be lineally convex. \hfill \Box

### 9.8.3 Duality for functions

A function $f : \mathbb{C}^{1+n} \setminus \{0\} \to \mathbb{R}$ with values in the extended real line will be called $(-1)$-homogeneous if

$$
(f(tz) = -\log |t| + f(z), \quad z \in \mathbb{C}^{1+n} \setminus \{0\}, \quad t \in \mathbb{C} \setminus \{0\}). \quad (9.88)
$$

For such functions we define the logarithmic transform $\mathcal{L}f$:

$$
(\mathcal{L}f)(\zeta) = \sup_{z \in \text{dom}(f)} (-\log |\zeta \cdot z| - f(z)), \quad \zeta \in \mathbb{C}^{1+n} \setminus \{0\}. \quad (9.89)
$$

We define $\log 0 = -\infty$. The difference $-\log |\zeta \cdot z| - f(z)$ is well-defined if $f(z) < +\infty$; another way to formulate the definition is to use lower addition $+$:

$$
(\mathcal{L}f)(\zeta) = \sup_{z} ((-\log |\zeta \cdot z|) + (-f(z))), \quad \zeta \in \mathbb{C}^{1+n} \setminus \{0\}. \quad (9.90)
$$

Lower and upper addition are defined in Section 9.2 on page 253.

**Proposition 9.8.10** For any homogeneous function $f : \mathbb{C}^{1+n} \setminus \{0\} \to \mathbb{R}$ its logarithmic transform $\mathcal{L}f$ is a homogeneous function with

$$
\text{dom}(\mathcal{L}f) \subset (\text{dom}(f))^*, \quad (9.91)
$$

where $\text{dom}(f)$ denotes the effective domain of $f$.

**Proof** The homogeneity of $\mathcal{L}f$ is obvious from its definition (9.89). To prove (9.91), we note that $\zeta \notin (\text{dom}(f))^*$ means by definition that the hyperplane $Y_{\zeta}$ and the effective domain $\text{dom}(f)$ have a common point $z$ (cf. (9.87)), so that $(\mathcal{L}f)(\zeta) \geq -\log |\zeta \cdot z| - f(z) = +\infty$, thus $\zeta \notin \text{dom}(\mathcal{L}f)$. The inclusion (9.91) may be strict as will be shown below: see Example 9.8.16 and Remark 9.8.21.
The analogue of Fenchel’s inequality holds:

\[- \log |z| \leq f(z) + (Lf)(\zeta), \quad \zeta, z \in \mathbb{C}^{1+n} \setminus \{0\}. \tag{9.92}\]

Moreover the usual rules for a Galois correspondence hold: \( f \leq g \) implies \( Lf \geq Lg \), and we always have \( L(Lf) \leq f \). As a consequence of these two properties, \( L \circ L \circ L = L \). A function \( f \) will be called \( \mathcal{L} \)-closed if \( L(Lf) = f \) (equivalently, if it belongs to the range of \( L \)). Some simple examples follow.

**Example 9.8.11** If \( f \) assumes the value \(-\infty\), then \( Lf \) is \(+\infty\) identically. The same is true if \( f \) never takes the value \(+\infty\) \((n \geq 1)\). If \( f \) is \(+\infty\) identically, then \( Lf \) is \(-\infty\) identically. If \( f(z) = -\log |z| \) when \( z = ta \) for a fixed \( a \in \mathbb{C}^{1+n} \setminus \{0\} \) and \(+\infty\) otherwise, then \((Lf)(\zeta) = -\log |\zeta|\). If

\[ f(z) = -\log |\alpha \cdot z| \]

for some \( \alpha \), then \((Lf)(\zeta) = -\log |t|\) when \( \zeta = \alpha a \) and \(+\infty\) otherwise. All these functions are \( \mathcal{L} \)-closed. \( \square \)

As a consequence of (9.90), we note that \( \sup_j Lf_j = L(\inf_j f_j) \) for any indexed family \((f_j)\) of functions. Indeed this follows from the rule \( \sup_j (c + a_j) = c + \sup_j a_j \), which is valid also for any constant \( c \in \mathbb{R} \). This implies that any supremum of \( \mathcal{L} \)-closed functions is \( \mathcal{L} \)-closed; in fact, we have

\[ \sup_j f_j = \sup_j (L(\inf_j f_j)) = L(\inf_j f_j) \tag{9.93} \]

if the \( f_j \) are \( \mathcal{L} \)-closed.

Homogeneous functions appear rather naturally in complex analysis. Let \( \mu \) be an analytic functional in an open subset \( \omega \) of \( \mathbb{C}^n \), \( \mu \in \mathcal{O}'(\omega) \). Its Fantappiè transform is

\[ (F\mu)(\zeta) = \mu(z) = (\zeta_0 + \zeta_1 z_1 + \cdots + \zeta_n z_n)^{-1}, \]

which is a holomorphic function of \( \zeta \in \Omega^* \), where \( \Omega \) is the set of all \( z \in \mathbb{C}^{1+n} \setminus \{0\} \) such that \( z_0 \neq 0 \) and \((z_1/z_0, \ldots, z_n/z_0) \in \omega \). This implies that \( \log |F\mu| \) is plurisubharmonic in \( \Omega^* \), and it is moreover homogeneous in the sense of (9.88). (We define it as \(+\infty\) outside \( \Omega^* \).)

Given \( f \) defined in \( \mathbb{C}^{1+n} \setminus \{0\} \), we can define a function \( F \) in \( \mathbb{C}^n \) by putting \( F(z') = f(1, z_1, \ldots, z_n) \), \( z' \in \mathbb{C}^n \). Conversely, if \( F \) is defined in \( \mathbb{C}^n \), we can define a homogeneous function \( f \) in \( \mathbb{C}^{1+n} \setminus \{0\} \) by

\[ f(z) = \begin{cases} F(z_1/z_0, \ldots, z_n/z_0) - \log |z_0|, & z \in \mathbb{C}^{1+n} \setminus \{0\}, \quad z_0 \neq 0; \\ +\infty, & z \in \mathbb{C}^{1+n} \setminus \{0\}, \quad z_0 = 0. \end{cases} \]

The transform (9.89) then takes the form

\[ (LF)(\zeta') = \sup_{F(z') < +\infty} \left( -\log |1 + \zeta' \cdot z'| - F(z') \right), \quad \zeta' \in \mathbb{C}^n. \tag{9.94} \]
In particular, if \( F \) is radial (i.e., a function of \( \|z\|_2 = r \)), then the transform becomes
\[
(\mathcal{L}F)(\rho) = \sup_{F(r) < +\infty} \left( -\log(1 - \rho r) - F(r) \right), \quad \rho = \|\zeta\|_2 \geq 0. \tag{9.95}
\]

**Example 9.8.12** Take \( F(r) = 0 \) when \( r \leq R \) and \( F(r) = +\infty \) otherwise in (9.95). Then \( (\mathcal{L}F)(\rho) = -\log(1 - \rho R), \rho < 1/R, \) and \( (\mathcal{L}F)(\rho) = +\infty, \rho \geq 1/R. \) The second transform is \( \mathcal{L}(\mathcal{L}F) = F, \) so that \( F \) is \( \mathcal{L} \)-closed. \( \square \)

**Example 9.8.13** The radial function
\[
F(r) = \frac{1}{2} \log(1 - r^2)
\]
is selfdual, i.e., \( (\mathcal{L}F)(\rho) = -\frac{1}{2} \log(1 - \rho^2). \) Going back to \( C^{1+n} \setminus \{0\}, \) we see that the function
\[
f(z) = \begin{cases} 
-\frac{1}{2} \log(|z_0|^2 - \|z'|_2^2), & z \in C^{1+n} \setminus \{0\}, \ |z_0| > \|z'|_2; \\
+\infty, & z \in C^{1+n} \setminus \{0\}, \ |z_0| \leq \|z'|_2
\end{cases}
\]
has this property. This function therefore plays the same role as the convex function \( f(x) = \frac{1}{2} \|x\|_2^2, \ x \in R^n, \) for usual convexity. \( \square \)

Now let \( A \) be a homogeneous set in \( C^{1+n} \setminus \{0\}. \) We define a function \( d_A, \) the distance to the complement of \( A \) relative to \( C^{1+n} \setminus \{0\}, \) as
\[
d_A(z) = \inf \{ \|z - w\|_2; \ w \in (C^{1+n} \setminus \{0\}) \setminus A \}, \quad z \in C^{1+n} \setminus \{0\}. \tag{9.96}
\]
The function \( -\log d_A \) is homogeneous, and it is less than \( +\infty \) precisely in the interior of \( A. \) Analogously we define a function \( d_A^* \) by
\[
d_A^*(\zeta) = \inf \{ \|\zeta - \theta\|; \ \theta \in (C^{1+n} \setminus \{0\}) \setminus A^* \}, \quad \zeta \in C^{1+n} \setminus \{0\}. \tag{9.97}
\]
where \( A^* \) is the dual complement of \( A \) defined by (9.84). If \( A \) is empty, then \( d_A = 0 \) identically, whereas \( d_A^* = +\infty \) identically.

**Theorem 9.8.14** Let \( f: C^{1+n} \setminus \{0\} \to R, \) be any homogeneous function. Then
\[
C - \log \|\zeta\|_2 \leq (\mathcal{L}f)(\zeta) \leq C - \log d_{A^*}(\zeta), \quad \zeta \in C^{1+n} \setminus \{0\}, \tag{9.98}
\]
where \( d_{A^*} \) is defined by (9.97) taking \( A = \text{dom}(f), \) and \( C = -\inf_{\|z\|_2 = 1} f(z) \leq +\infty. \) We have \( C = -\infty \) if and only if \( f \) is \( +\infty \) identically; in this case \( \mathcal{L}f \) is \( -\infty \) identically. We have \( C = +\infty \) if and only if \( f \) is unbounded from below on the unit sphere \( S; \) then \( \mathcal{L}f \) is \( +\infty \) identically. If \( f \) is bounded from below on \( S, \) then \( C < +\infty \) and (9.98) shows that \( \mathcal{L}f \) has at most logarithmic growth at the boundary of \( (\text{dom}(f))^*; \) moreover
\[
(\text{dom}(f))^* = (\text{dom}(f))^\circ = (\text{dom}(\mathcal{L}f))^\circ \subset \text{dom}(\mathcal{L}f) \subset (\text{dom}(f))^*.
\tag{9.99}
\]
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and
\[ \text{dom}(L^f) \subseteq (\text{dom}(f))^* \subseteq (\text{dom}(f))^{**}. \] (9.100)

In particular \( \text{dom}(L^f) = (\text{dom}(f))^* \) if \( \text{dom}(f) \) is closed.

**Lemma 9.8.15** For any subset \( A \) of \( C^{1+n} \setminus \{0\} \) we have
\[ |\zeta \cdot z| \geq d_{A^*}(\zeta)\|z\|_2, \quad \zeta \in C^{1+n} \setminus \{0\}, \quad z \in A, \] (9.101)
and
\[ |\zeta \cdot z| \geq \|\zeta\|_2 d_A(z), \quad \zeta \in A^*, \quad z \in C^{1+n} \setminus \{0\}. \] (9.102)

**Proof** Given \( \zeta \in C^{1+n} \setminus \{0\} \) and \( z \in A \) we define \( \alpha = \zeta + t\zeta \) where \( t = -\|z\|_2^2(\zeta \cdot z) \). Then \( \alpha \cdot z = 0 \), which, if \( \alpha \neq 0 \), means that \( \alpha \in \mathcal{C}A^* \) since \( z \in A \). Therefore \( d_{A^*}(\zeta) \leq \|\zeta - \alpha\|_2 = \|\zeta \cdot z\|/\|z\|_2 \), which proves the first inequality except when \( \zeta = \|z\|_2^2(\zeta \cdot z) \). Since \( d_{A^*} \) is continuous, this restriction can be removed. If we now interchange the role of \( \zeta \) and \( z \), we get \( |\zeta \cdot z| \geq \|\zeta\|_2 d_{A^*}(z) \). But \( A^* \supset A \), so \( d_{A^*}(z) \geq d_A(z) \). This proves the lemma. (Interchanging \( z \) and \( \zeta \) once more, we see that (9.101) holds even for all \( z \in A^* \).)

**Proof of Theorem 9.8.14.** By the Schwarz inequality and (9.101) applied to \( A = \text{dom}(f) \) we get
\[ -\log \|\zeta\|_2 \leq -\log |\zeta \cdot z| \leq -\log d_{A^*}(\zeta), \quad \zeta \in C^{1+n} \setminus \{0\}, \quad z \in A \cap S. \]

Thus
\[ (L^f)(\zeta) = \sup\limits_{z \in A \cap S} \left( -\log |\zeta \cdot z| - f(z) \right) \begin{cases} \leq (-\log d_{A^*}(\zeta)) + \sup_{A \cap S}(-f); & \text{if} \sup_{A \cap S}(-f) = \sup_{S}(-f) = -\inf_{S} f. \\ \geq -\log \|\zeta\|_2 + \sup_{A \cap S}(-f). & \end{cases} \]

The cases where \( + \) and \( \sup \) give different results never occur, so we can replace \( \sup \) by usual addition. This proves (9.98); note that \( \sup_{A \cap S}(-f) = \inf_{S}(-f) \).

We already know that \( \text{dom}(L^f) \subseteq (\text{dom}(f))^* \); see (9.91). If \( \zeta \in (\text{dom}(f))^* \) and \( C < +\infty \), then \( d_{A^*}(\zeta) > 0 \) and \( (L^f)(\zeta) \leq C - \log d_{A^*}(\zeta) < +\infty \), so that \( \zeta \in \text{dom}(L^f) \). This proves that \( (\text{dom}(f))^* \subseteq \text{dom}(L^f) \subseteq (\text{dom}(f))^* \). Taking the interior of these sets we get (9.99); taking the closure we get (9.100) (cf. Proposition 9.8.2).

**Example 9.8.16** The effective domain of \( L^f \) may fail to be lineally convex, although it is squeezed in between the two lineally convex sets \( (\text{dom}(f))^{**} = \text{dom}(f)^* \) and \( (\text{dom}(f))^* \); see (9.99). Indeed, let \( w^k = (k^{-2}, k^{-1}, 1) \in C^{1+2} \) and define \( f(w^k) = \log k, \quad k = 1, 2, 3, \ldots, \) and \( f(z) = +\infty \) when \( z \notin Cw^k \). Then
\[ (L^f)(\zeta) = \sup_{k}(-\log |\zeta_0/k + \zeta_1 + k\zeta_2|), \quad \zeta \in C^{1+2} \setminus \{0\}. \]
Put \( \alpha = (1, 0, 0) \) and \( \beta = (1, 1, 0) \). Then
\[
(\mathcal{L} f)(\alpha) = \sup_k (- \log |k^{-1}|) = +\infty,
\]
so that \( \alpha \notin \text{dom}(\mathcal{L} f) \), whereas
\[
(\mathcal{L} f)(\beta) = \sup_k (- \log |k^{-1} + 1|) = 0,
\]
showing that \( \beta \in \text{dom}(\mathcal{L} f) \). The points \( w^k \) define hyperplanes
\[
Y_{w^k} = \{ \zeta; \zeta_0k^{-2} + \zeta_1k^{-1} + \zeta_2 = 0 \},
\]
which converge to a hyperplane \( Y_w = \{ \zeta; \zeta_2 = 0 \} \) with \( w = \lim w^k = (0, 0, 1) \).
By (9.99),
\[
(\text{dom}(f))^* = \mathcal{C}(Y_w \cup \bigcup Y_{w^k}) \subset \text{dom}(\mathcal{L} f) \subset (\text{dom}(f))^* = \mathcal{C}(\bigcup Y_{w^k}).
\]
Both \( \alpha \) and \( \beta \) belong to \( Y_w \), but as \( k \to +\infty \), the hyperplanes \( Y_{w^k} \) approach \( \alpha \) more rapidly than \( \beta \) (note that \( \alpha \cdot w^k = 1/k^2 \), while \( \beta \cdot w^k = 1/k + 1/k^2 \)). This explains why \( \alpha \notin \text{dom}(\mathcal{L} f) \) while \( \beta \in \text{dom}(\mathcal{L} f) \). A hyperplane which avoids \( \text{dom}(\mathcal{L} f) \) must be either one of the hyperplanes \( Y_{w^k} \), or (possibly) their limit \( Y_w \). However, the hyperplanes \( Y_{w^k} \) do not contain \( \alpha \), and the hyperplane \( Y_w \) intersects \( \text{dom}(\mathcal{L} f) \) in \( \beta \). Therefore there is no hyperplane which passes through \( \alpha \) and avoids \( \text{dom}(\mathcal{L} f) \). This shows that \( \text{dom}(\mathcal{L} f) \) is not lineally convex. In particular we must have \((\text{dom}(f))^* \neq \text{dom}(\mathcal{L} f) \neq (\text{dom}(f))^* \); cf. (9.99).

**Example 9.8.17** A fundamental property of convexity is that the union of an increasing sequence of convex sets is convex. (More generally, this is true for the union of a directed family.) This is not so with lineal convexity. Let \( A_k \) be the set of all \( \zeta \) such that \( (\mathcal{L} f)(\zeta) \leq k \). It is easy to see that this is a lineally convex set; indeed,
\[
A_k = \bigcap_{z \in \text{dom}(f)} \{ \zeta \in \mathcal{C}^{1+n} \setminus \{ 0 \}; - \log |\zeta \cdot z| - f(z) \leq k \}.
\]
The union of the \( A_k \) is \( \text{dom}(\mathcal{L} f) \). If we let \( f \) be the function constructed in Example 9.8.16 we get an example where the \( A_k \) are lineally convex but their union is not.

**Theorem 9.8.18** Let \( f \) be a function on \( \mathcal{C}^{1+n} \setminus \{ 0 \} \) which is bounded from below on the unit sphere and let \( \mathcal{L} f \) be its transform defined by (9.89). Then \( \mathcal{L} f \) is plurisubharmonic in the interior of \( \text{dom}(\mathcal{L} f) \), which is a lineally convex set. Moreover \( \mathcal{L} f \) is locally Lipschitz continuous in \((\text{dom}(\mathcal{L} f))^* \); more precisely
\[
\limsup_{t \to 0^+} \frac{(\mathcal{L} f)(\zeta + \theta t) - (\mathcal{L} f)(\zeta)}{t} \leq \frac{\| \theta \|_2}{d_{A^*}(\zeta)}, \quad \zeta \in (\text{dom}(\mathcal{L} f))^*, \theta \in \mathcal{C}^{1+n},
\]
where \( d_{A^*} \) is the distance to the complement of \( \text{dom}(\mathcal{L} f) \).
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Proof Consider the function \( g(\zeta) = -\log|\zeta \cdot z| \). Its gradient has length \( \|z\|_2/|\zeta \cdot z| \). At the point \( \alpha = \zeta + t\zeta \), where \( t = -\|z\|_2^2(\zeta \cdot z) \), \( g \) takes the value \( +\infty \), so

\[
d_{\text{dom}(g)}(\zeta) \leq \|\alpha - \zeta\|_2 \leq \frac{|\zeta \cdot z|}{\|z\|_2} = \frac{1}{\|\nabla g(\zeta)\|_2}.
\]

Now \( \mathcal{L}f \) is a supremum of functions of the form \( g \) plus a constant for various choices of \( z \). All competing functions must satisfy \( \text{dom}(g) \supseteq \text{dom}(\mathcal{L}f) \), so that \( d_{\text{dom}(g)} \geq d_{\mathcal{L}f} \). Therefore they have a gradient whose length is at most \( 1/d_{\mathcal{L}f}(\zeta) \), which implies that \( \mathcal{L}f \) is Lipschitz continuous as indicated. That \( \mathcal{L}f \) is plurisubharmonic now follows from standard properties of such functions: \( f \) is a continuous supremum of plurisubharmonic functions.

Finally (9.99) shows that \( (\text{dom}(\mathcal{L}f))^* \) is lineally convex: it is equal to the dual complement of the closure of \( \text{dom}(f) \).

9.8.4 Examples of functions in duality

In this subsection we shall make a detailed study of the functions

\[
f_c(z) = \begin{cases} 
-(1-c) \log \|z\|_2 - c \log d_A(z), & z \in A; \\
+\infty, & z \in (C^{1+n} \setminus \{0\}) \setminus A,
\end{cases}
\]

and

\[
\varphi_c(\zeta) = \begin{cases} 
-(1-c) \log \|\zeta\|_2 - c \log d_{A^*}(\zeta), & \zeta \in A^*; \\
+\infty, & \zeta \in (C^{1+n} \setminus \{0\}) \setminus A^*,
\end{cases}
\]

where \( 0 \leq c \leq 1 \), \( A \) is any homogeneous subset of \( C^{1+n} \setminus \{0\} \), \( A^* \) its dual complement, and where \( d_{A} \) and \( d_{A^*} \) are defined by (9.96) and (9.97), respectively.

We shall call \( f_0 = I_A \) the logarithmic indicator function of the set \( A \). Its restriction to the unit sphere is the indicator function in the usual sense. And \( \mathcal{L}f_0 = \mathcal{L}I_A \) is analogous to the support function of \( A \), thus preserving the situation from convex analysis where the support function is the Fenchel transform of the indicator function. We shall determine this function explicitly: it is \( \varphi_1 = -\log d_{A^*} \).

More generally, it turns out that the function \( \varphi_{1-c} \) is essentially dual to \( f_c \). It might seem strange to consider functions like \( f_0 \) which are not plurisubharmonic. We must have \( \mathcal{L}(\mathcal{L}f_0) < f_0 \) in the interior of \( A \). From this point of view it is more natural to consider

\[
g_c(z) = \begin{cases} 
-(1-c) \log |z_0| - c \log d_A(z), & z \in A; \\
+\infty, & z \in (C^{1+n} \setminus \{0\}) \setminus A,
\end{cases}
\]
and
\[ \psi_c(\zeta) = \begin{cases} 
- (1 - c) \log |\zeta_0| - c \log d_{A^*}(\zeta), & \zeta \in A^*; \\
+\infty, & \zeta \in (\mathbb{C}^{1+n} \setminus \{0\}) \setminus A^*. 
\end{cases} \]  
(9.106)

If \( A \) is contained in a coordinate patch \( z_0 \neq 0 \) and if moreover \( \|z\|_2/|z_0| \) is bounded when \( z \in A \), then \( f_c \) and \( g_c \) are finite in the same set and differ there by a bounded function. If moreover \( (1, 0, \ldots, 0) \) is an interior point of \( A \), then \( \zeta_0 \neq 0 \) when \( \zeta \in A^* \) and \( \|\zeta\|_2/|\zeta_0| \) is bounded there, so \( \varphi_c \) and \( \psi_c \) are finite in the same set and their difference is bounded there. Therefore our results on \( f_c \) and \( \varphi_c \) can easily be translated into inequalities for \( g_c \) and \( \psi_c \).

The first result is a simple inequality.

**Proposition 9.8.19** With \( f_c \) and \( \varphi_c \) defined by (9.103) and (9.104) we have \( L f_c \leq \varphi_{1-c} \) for \( 0 \leq c \leq 1 \).

**Proof** If \( \zeta \notin A^* \), then \( \varphi_{1-c}(\zeta) = +\infty \), so the inequality certainly holds. If on the other hand \( \zeta \in A^* \), we can estimate \( (L f_c)(\zeta) \) using Lemma 9.8.15:

\[
(L f_c)(\zeta) = \sup_{z \in A(c)} \left[ - \log |\zeta \cdot z| + (1 - c) \log \|z\|_2 + c \log d_A(z) \right] 
\]

\[
= \sup_{z \in A(c)} \left[ - (1 - c) \log |\zeta \cdot z| + (1 - c) \log \|z\|_2 - c \log |\zeta \cdot z| + c \log d_A(z) \right] 
\]

\[
\leq \sup_{z \in A(c)} \left[ - (1 - c) \log(d_{A^*}(\zeta)\|z\|_2) + (1 - c) \log \|z\|_2 - c \log \|z\|_2 + c \log d_A(z) \right] 
\]

\[
\leq -(1 - c) \log d_{A^*}(\zeta) - c \log \|z\|_2 = \varphi_{1-c}(\zeta). 
\]

The supremum is over the set \( A(c) \) of all \( z \) such that \( f_c(z) < +\infty \), that is

\[
A(c) = \begin{cases} 
A & \text{for } c = 0; \\
A^c & \text{for } 0 < c < 1. 
\end{cases} 
\]  
(9.107)

(\( A(c) \) can be empty; in that case \( L f_c \) is \(-\infty\) identically.)

We now study inequalities in the other direction. The cases \( c = 0 \) and \( c = 1 \) are easy and will be considered first.

**Proposition 9.8.20** For any homogeneous subset \( A \) of \( \mathbb{C}^{1+n} \setminus \{0\} \) we have \( L I_A = L f_0 = \varphi_1 = -\log d_A^* \). (The analogue of the support function of \( A \).)

**Remark 9.8.21** Note that here \( \text{dom}(L f_0) = A^* = \overline{A}^* = (\text{dom}(f_0))^* \) is open and lineally convex, whereas \( (\text{dom}(f_0))^* = A^* \); again we see that the inclusion \( \text{dom}(L f) \subset (\text{dom}(f))^\ast \) may be strict (cf. (D) in Proposition 9.8.2). □
Lemma 9.8.22 Assume that $A$ is homogeneous and not empty. For every $\zeta \in A^*$ there is a point $z \in \partial A$, $z \neq 0$, such that $|\zeta \cdot z| \leq d_A^*(\zeta)\|z\|_2$.

Proof For every $\zeta \in A^*$ there is a point $\alpha \in \partial A^*$, $\alpha \neq 0$, such that $\|\alpha - \zeta\|_2 = d_{A^*}^*(\zeta)$. Thus $\alpha \notin A^\circ = \overline{A}^*$ (cf. (D) in Proposition 9.8.2). Now $\alpha \notin \overline{A}^*$ means that $Y_\alpha \cap \overline{A} \neq \emptyset$ (see (9.85) and (9.87)). On the other hand $\alpha \in \partial A^* \subset \overline{A}^* \subset A^\circ$ (cf. (C) in Proposition 9.8.2), so that $Y_\alpha \cap A^\circ = \emptyset$. Therefore $Y_\alpha$ meets the boundary of $A$, and we can choose $z \in S \cap \partial A$ such that $|\alpha \cdot z| = 0$. Then $|\zeta \cdot z| = |(\zeta - \alpha) \cdot z| \leq \|\zeta - \alpha\|_2 \cdot \|z\|_2 = d_{A^*}^*(\zeta)\|z\|_2$.

Proof of Proposition 9.8.20. If $A = \emptyset$, we have $\mathcal{L}I_A = - \log d_{A^*} = -\infty$. Otherwise the lemma provides us, given any $\zeta \in A^*$, with a point $z \in \partial A \cap S$ such that

$$(\mathcal{L}f_0)(\zeta) = \sup_{w \in A} (- \log |\zeta \cdot w| + \log \|w\|_2) \geq - \log |\zeta \cdot z| + \log \|z\|_2 \geq - \log d_{A^*}^*(\zeta) = \varphi_1(\zeta).$$

For $\zeta \in (\mathbb{C}^{1+n} \setminus \{0\}) \setminus A^*$ both $\mathcal{L}f_0$ and $\varphi_1$ take the value $+\infty$. Thus $\mathcal{L}f_0 \geq \varphi_1$ everywhere. The inequality $\mathcal{L}f_0 \leq \varphi_1$ was proved already in Proposition 9.8.19.

Proposition 9.8.23 Assume $A$ that is open and not empty. Then there is a constant $M$, which depends on the geometry of $A$, such that

$$\varphi_0 = I_{A^*} \geq \mathcal{L}f_1 = \mathcal{L}(- \log d_A) \geq I_{A^*} - M.$$ 

In fact $M$ can be taken as $\inf_S f_1 = \inf_S (- \log d_A)$, where as before $S$ is the unit sphere.

Here $\text{dom}(\mathcal{L}f_1) = A^* = (\text{dom}(f_1))^*$ is closed and lineally convex; cf. (9.99).

Lemma 9.8.24 Assume $A$ has a nonempty interior and take any point $z \in A^\circ$. Then there is a constant $C$ such that $|\zeta \cdot z| \leq C\|\zeta\|_2 d_A(z)$ for all $\zeta$.

Proof Given $z \in A^\circ$ define $C = \|z\|_2/d_A(z)$. We have

$$|\zeta \cdot z| \leq \|\zeta\|_2 \|z\|_2 = C\|\zeta\|_2 d_A(z).$$

The best choice is a point $z \in S$ such that $d_A(z) = \sup_S d_A$, so that $C = 1/\sup_S d_A$.

Proof of Proposition 9.8.23 Using the lemma above we get for any $\zeta \in A^*$,

$$(\mathcal{L}f_1)(\zeta) = \sup_{w \in A} (- \log |\zeta \cdot w| + \log d_A(w)) \geq - \log |\zeta \cdot z| + \log d_A(z) \geq - \log C - \log \|\zeta\|_2 = \varphi_0(\zeta) - M.$$ 

When $\zeta \notin A^*$, there is a point $z \in A$ such that $\zeta \cdot z = 0$, and since $A$ is open, $f_1(z) < +\infty$, so that $(\mathcal{L}f_1)(\zeta) = +\infty$. Thus we have $\mathcal{L}f_1 \geq \varphi_0 - M$ everywhere. The inequality $I_{A^*} \geq \mathcal{L}f_1$ was already proved in Proposition 9.8.19.
Theorem 9.8.25 Let $A$ be an open homogeneous set. Then $A$ is lineally convex if and only if $-\log d_A$ is $\mathcal{L}$-closed.

Proof If $A = A^*$, then $\mathcal{L} I_B = -\log d_B^* = -\log d_A$ by Proposition 9.8.20, so $-\log d_A$ is $\mathcal{L}$-closed. Conversely, Proposition 9.8.23 shows that $\mathcal{L}(-\log d_A) \supset I_A - M$, which implies $\mathcal{L}(-\log d_A) \leq -\log d_A^{**} + M$. Therefore, if $z$ belongs to the open set $A^{**}$ (cf. Proposition 9.8.2), then $\mathcal{L}(-\log d_A(z))$ is finite. If $-\log d_A$ is $\mathcal{L}$-closed, this is equivalent to $-\log d_A(z)$ being finite, which implies $z \in A$. Thus $A^{**} \subset A$; this inclusion means that $A$ is lineally convex.

Theorem 9.8.26 A closed lineally convex set $A$ can be recovered from $\mathcal{L} I_A$. Indeed, if $A$ is a set with these properties different from $C^{1+n} \setminus \{0\}$, then $I_A \supset \mathcal{L} I_A \supset I_A - M$, so that $A$ is the set where $\mathcal{L} I_A$ is finite. If $A$ is equal to $C^{1+n} \setminus \{0\}$, then $\mathcal{L} I_A$ is $-\infty$ identically. If $A$ is a closed and lineally convex set such that $\|z'\|_2 \leq R|z_0|$ for all $z \in A$, then $\mathcal{L} I_A \supset I_A - \log \sqrt{1+R^2}$.

This theorem is thus analogous to the result in convexity theory which states that a closed convex set can be recovered from its support function. By way of contrast, an open set $A$ can be recovered from $\mathcal{L} I_A$ only under special conditions, since $\mathcal{L} I_A = \mathcal{L} I_A$. If $A$ is open and equal to the interior of its closure, and if its closure is lineally convex, then $A$ is the interior of the set where $\mathcal{L} I_A$ is finite. But an open lineally convex set can always be recovered from $\mathcal{L}(-\log d_A)$; see Proposition 9.8.23.

Proof If $A$ is closed, lineally convex and not equal to all of $C^{1+n} \setminus \{0\}$, then $A^*$ is open and nonempty, so we can apply Proposition 9.8.23 to $A^*$ and obtain

$$I_{A^{**}} \supset \mathcal{L}(-\log d_{A^*}) \supset I_{A^{**}} - M.$$ 

From Proposition 9.8.20 we have $\mathcal{L} I_A = -\log d_{A^*}$. Combining this information we deduce that

$$I_{A^{**}} \supset \mathcal{L}(-\log d_{A^*}) = \mathcal{L} I_A \supset I_{A^{**}} - M.$$ 

Since $A$ is lineally convex, $A = A^{**}$, and we see that $\mathcal{L} I_A$ and $I_A$ are finite in the same set (and differ there at most by a bounded function).

The last statement follows if we keep track of the constant in Lemma 9.8.24. Alternatively we can compare $I_A$ with the function $F(z) = -\log |z_0|$ when $\|z'\|_2 \leq R|z_0|$, defining $F(z) = +\infty$ otherwise. This gives $F(z) \geq I_A(z) \geq F + \log |z_0| - \log \|z\|_2$ for $z \in A$, so that $I_A \supset F - \log \sqrt{1+R^2}$ everywhere, implying that $\mathcal{L} I_A \supset \mathcal{L} F - \log \sqrt{1+R^2}$. Since $\mathcal{L} F = F$ (cf. Example 9.8.12), we can conclude that $\mathcal{L} I_A \supset F - \log \sqrt{1+R^2} \supset I_A - \log \sqrt{1+R^2}$ in $A$.

Finally we shall deduce an estimate from below for $\mathcal{L} f_c$ when $0 < c < 1$. We shall need a definition.
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Definition 9.8.27 We shall say that $A$ satisfies the homogeneous interior cone condition if there exist positive numbers $\gamma$ and $R$ such that for every $b \in \partial A$ and every $r \leq R$, the inequality

$$\sup_z \left( d_A(z); \|z - b\|_2 \leq r\|b\|_2 \right) \geq \gamma r\|a\|_2$$

holds for every $b \in \partial A$ and every $r \leq R$. \hfill \Box

Proposition 9.8.28 Assume that $A$ is open, nonempty, and satisfies the homogeneous interior cone condition just defined. Then there is a constant $M$ such that $\varphi_{1-c} \geq \mathcal{L}f_c \geq \varphi_{1-c} - M$ for every $c \in [0,1]$.

Here $\text{dom}(\mathcal{L}f_c) = (\text{dom}(f_c))^*$ for $0 \leq c < 1$, whereas it is closed for $c = 1$ as already noted.

In particular a set with Lipschitz boundary satisfies the homogeneous interior cone condition. To prove this proposition we shall need a lemma which combines Lemmas 9.8.22 and 9.8.24. The requirements concerning the point $z$ are somewhat contradictory, since $z \in \partial A$ in the first and $z \in A^0$ in the second. Nevertheless, we can find a compromise:

Lemma 9.8.29 With $A$ as in Proposition 9.8.28, there exists a constant $C$ such that for every $\zeta \in A^*$ there is a point $z = z_\zeta \in A$ such that

$$|\zeta \cdot z| \leq C\|\zeta\|_2 d_A(z) \quad \text{and} \quad |\zeta \cdot z| \leq C d_A^*(\zeta)\|z\|_2.$$  

Proof First pick any point $w \in A$. It will serve as the point $z_\zeta$ for all $\zeta$ such that $d_A^*(\zeta) \geq R\|\zeta\|_2$:

$$|\zeta \cdot w| \leq \|\zeta\|_2 w_2 = \frac{\|w\|_2}{d_A(w)} \|\zeta\|_2 d_A(w) \leq C\|\zeta\|_2 d_A(w)$$

and

$$|\zeta \cdot w| \leq \|\zeta\|_2 w_2 = \frac{\|\zeta\|_2}{d_A^*(\zeta)} d_A^*(\zeta)\|w\|_2 \leq \frac{1}{R} d_A^*(\zeta)\|w\|_2 \leq C d_A^*(\zeta)\|w\|_2$$

for a constant $C \geq \max(R^{-1}, \|w\|_2/d_A(w))$.

The case $d_A^*(\zeta) \leq R\|\zeta\|_2$ remains to be considered. To a given $\zeta \in A^*$ we choose $\alpha \in \partial A^*, \alpha \neq 0$, such that $\|\alpha - \zeta\|_2 = d_A^*(\zeta) = r\|\zeta\|_2, r \leq R$. Since $Y_\alpha$ meets $\partial A$ (cf. the proof of Lemma 9.8.22), we can choose $a \in \partial A, a \neq 0$, such that $\alpha \cdot a = 0$. The homogeneous interior cone condition now implies the existence of a point $z = z_\zeta \in A$ such that $d_A(z) \geq \gamma r\|a\|_2$ and $\|z - a\|_2 \leq r\|a\|_2 \leq \mathcal{R}\|a\|_2$. Then $|\zeta \cdot a| = |(\zeta - \alpha) \cdot a| \leq \|\zeta - \alpha\|_2\|a\|_2 = d_A^*(\zeta)\|a\|_2$ and $\|z - a\|_2 \leq \mathcal{R}\|a\|_2 = \|a\|_2 d_A^*(\zeta)/\|\zeta\|_2$, so that

$$|\zeta \cdot z| = |\zeta \cdot a + \zeta(z - a)| \leq |\zeta \cdot a| + \|\zeta\|_2|z - a| \leq 2d_A^*(\zeta)\|a\|_2 \leq \frac{2}{1 - R} d_A^*(\zeta)\|z\|_2.$$  

Here the last inequality follows from $\|z - a\|_2 \leq \mathcal{R}\|a\|_2$; it is no restriction to
assume that $R < 1$. On the other hand $d_A(z) \geq \gamma r \|a\|_2 = \gamma \|a\|_2 d_{A^*}(\zeta)/\|\zeta\|_2$ so that
\[
|\zeta \cdot z| \leq 2d_{A^*}(\zeta)\|a\|_2 \leq \frac{2}{\gamma}d_A(z)\|\zeta\|_2.
\]
With the constant
\[
C = \max \left[\frac{1}{R}, \frac{\|w\|_2}{d_A(w)}, \frac{1}{1 - R}, \frac{2}{\gamma}\right]
\]
this proves the lemma.

**Proof of Proposition 9.8.28.** Since $A$ is open, $\text{dom}(f_c) = A$ for all $c$; cf. (9.107). Using the lemma we get for any $\zeta \in A^*$, taking $z \in A$ from Lemma 9.8.29,
\[
(L f_c)(\zeta) = \sup_{w \in A} \left( -\log |\zeta \cdot w| + (1 - c) \log \|w\|_2 + c \log d_A(w) \right)
\geq -\log |\zeta \cdot z| + (1 - c) \log \|z\|_2 + c \log d_A(z)
= (1 - c)(\log \|z\|_2 - \log |\zeta \cdot z|) + c(\log d_A(z) - \log |\zeta \cdot z|)
\geq (1 - c)(\log \|z\|_2 - \log(Cd_{A^*}(\zeta)\|z\|_2))
+ c(\log d_A(z) - \log(C\|\zeta\|_2 d_A(z))
= -(1 - c) \log d_{A^*}(\zeta) - c \log \|\zeta\|_2 - \log C = \varphi_{1-c}(\zeta) - M.
\]
If $\zeta \notin A^*$, then $(L f_c)(\zeta) = \varphi_{1-c}(\zeta) = +\infty$.

### 9.8.5 The support function of a convex set

Let $A_0$ be a subset of $\mathbb{C}^n$. Its support function is
\[
H_{A_0}(\zeta') = \sup_{z' \in A_0} \text{Re}(\zeta' \cdot z'), \quad \zeta' = (\zeta_1, \ldots, \zeta_n) \in \mathbb{C}^n. \quad (9.108)
\]
If $A_0$ is closed and convex, then $H_{A_0}$ determines $A_0$ in fact, $A_0$ is the set of all $z' = (z_1, \ldots, z_n)$ such that $\text{Re}(\zeta' \cdot z') \leq H_{A_0}(\zeta')$ for all $\zeta'$. Now let $A$ be the homogeneous set of all $z \in \mathbb{C}^{1+n} \setminus \{0\}$ such that $z_0 \neq 0$ and $z'/z_0 \in A_0$. With the set $A$ we associate the function $L I_A$ defined by
\[
(L I_A)(\zeta) = -\log d_{A^*}(\zeta) = \sup_{z \in A} \left( -\log |\zeta \cdot z| + \log \|z\|_2 \right), \quad \zeta \in \mathbb{C}^{1+n} \setminus \{0\},
\]
the projective analogue of the support function. What is the relation between these two support functions? To answer this question we first modify $I_A$ a little and define
\[
h_A(\zeta) = \sup_{z \in A} \left( -\log |\zeta \cdot z| + \log |z_0| \right), \quad \zeta \in \mathbb{C}^{1+n} \setminus \{0\}. \quad (9.110)
\]
If $A_0$ is bounded, then $h_A$ and $L I_A$ are finite in the same set and differ there by a bounded function.
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We shall express \( h_A \) in terms of \( H_{A_0} \). We first formulate an auxiliary result, which we shall need for convex sets in the complex plane only, but which is also valid in \( \mathbb{R}^n \). We shall therefore use the real support function

\[ H_A(\zeta) = \sup_{x \in A} \zeta \cdot x, \quad \zeta \in \mathbb{R}^n. \]  

(9.111)

Lemma 9.8.30 Let \( A \) be a convex set in \( \mathbb{R}^n \). Then

\[ - \inf_{x \in A} \|x\|_2 \leq \inf_{\|\xi\|_2 = 1} \|H_A(\xi)\| \leq \inf_{x \notin A} \|x\|_2 \]  

(9.112)

with equality on the left if \( 0 \notin A^c \), and on the right if \( 0 \in \overline{A} \).

Proof For any set \( A \) we have, writing \( S \) for the unit sphere,

\[ \inf_S H_A = \inf_{\xi \in S} \sup_{x \in A} \zeta \cdot x = \sup_{x \in A} (\inf_{\|\xi\|_2 = 1} \|\zeta\|) = - \inf_{x \in A} \|x\|_2. \]

If \( A \) is convex and \( x \notin A \), then there is a \( \xi \in S \) such that \( \zeta \cdot x \geq H_A(\xi) \); thus \( \|x\|_2 \geq \xi \cdot x \geq H_A(\xi) \), so that \( \|x\|_2 \geq \inf_S H_A \). This shows that \( \inf_{x \notin A} \|x\|_2 \geq \inf_S H_A \) and proves (9.112) for all convex sets.

Now assume that \( 0 \notin A^c \). Then \( A \) must be contained in a half-space \( \{x; \zeta \cdot x \geq c\} \) for some \( \xi \in S \) and \( c \geq 0 \), which shows that \( H_A(\xi) = c \leq 0 \). If \( A \) is empty, (9.112) has the form \( -\infty \leq -\infty \leq 0 \), so the result is true. If \( A \) is not empty, then we can choose \( c = \inf_{x \in A} \|x\|_2 \), so that \( \inf_S H_A \leq -c = -\inf_{x \notin A} \|x\|_2 \); we have proved equality on the left in (9.112).

On the other hand, if \( 0 \in \overline{A} \) and \( H_A(\xi) < c \) for some \( \xi \in S \) and some \( c \), then necessarily \( c > 0 \) and the vector \( c\xi \) cannot belong to \( A \), so that \( \inf_{x \notin A} \|x\|_2 \leq \|c\xi\|_2 = c \). Thus \( \inf_{x \notin A} \|x\|_2 \leq \inf_S H_A \); we have proved equality on the right in (9.112).

Lemma 9.8.31 For any convex set \( A \) in \( \mathbb{R}^n \) we have

\[ \inf_{x \in A} \|x\|_2 = \sup_{\|\xi\|_2 = 1} H_A(\xi)^-; \]  

(9.113)

\[ \inf_{x \notin A} \|x\|_2 = \inf_{\|\xi\|_2 = 1} H_A(\xi)^+; \]  

(9.114)

where \( t^+ = \max(t, 0) \), \( t^- = \max(-t, 0) \) for \( t \in \mathbb{R} \).

Proof If \( 0 \notin A \), then (9.113) is just the first part of (9.112) with equality, and (9.114) reduces to \( 0 = 0 \). If \( 0 \in A \), then (9.113) reduces to \( 0 = 0 \) while (9.114) is the second part of (9.112) with equality.

Proposition 9.8.32 Let \( A_0 \) be any convex set in \( C^n \), and \( A \) the homogeneous set of all \( z \in C^{1+n} \setminus \{0\} \) such that \( z_0 \neq 0 \) and \( z' / z_0 \in A_0 \). Define \( H_{A_0} \) and \( h_A \) by (9.108) and (9.110), respectively. Then

\[ h_A(\zeta) = \inf_{\|\xi\|_2 = 1} \log (H_{A_0}(t\zeta') + \text{Re}(t\zeta_0))^-, \quad \zeta \in C^{1+n} \setminus \{0\}. \]  

(9.115)
Proof If \( A \) is empty, (9.115) certainly holds, because both sides are equal to \(-\infty\). Fix \( \zeta \in \mathbb{C}^{1+n} \setminus \{0\} \) and denote by \( L \) the linear mapping \( z' \mapsto \zeta' \cdot z' \). If \( A \) is not empty, then
\[
e^{-h_A(\zeta)} = \inf_{z' \in A_0} |\zeta_0 + \zeta' \cdot z'| = \inf_{s \in L(A_0)} |\zeta_0 + s| = |\zeta_0 + a(\zeta_0)|, \quad (9.116)
\]
where \( a(\zeta_0) \) denotes the point in the closure of \( L(A_0) \) which is closest to \(-\zeta_0\).

Here the first equality holds because, in view of the homogeneity, it is enough to let \( z \) vary with \( z_0 = 1 \) in the definition of \( h_A \).

We now note that
\[
H_{A_0}(t\zeta') = \sup_{z' \in A_0} \Re(t\zeta' \cdot z') = \sup_{z' \in A_0} \Re tL(z') = \sup_{s \in L(A_0)} \Re ts = H_{L(A_0)}(t),
\]
which shows that the support function of the set \( M = L(A_0) + \zeta_0 \) is
\[
H_M(t) = H_{A_0}(t\zeta') + \Re(t\zeta_0), \quad t \in \mathbb{C}.
\]

We can apply (9.113) to the convex set \( M \). This yields
\[
e^{-h_A(\zeta)} = |\zeta_0 + a(\zeta_0)| = \sup_{|t|=1} H_M(t) = \sup_{|t|=1} \left( H_{A_0}(t\zeta') + \Re(t\zeta_0) \right),
\]
where the first equality is that of (9.116), and thus proves (9.115).

Conversely, we can express \( H_{A_0} \) in terms of \( h_A \).

**Proposition 9.8.33** Let \( A_0 \) be a bounded but not necessarily convex set in \( \mathbb{C}^n \), and \( A \) the set of all \( z \in \mathbb{C}^{1+n} \setminus \{0\} \) such that \( z_0 \neq 0 \) and \( z'/z_0 \in A_0 \). Define \( H_{A_0} \) and \( h_A \) by (9.108) and (9.110), respectively. Then
\[
H_{A_0}(\zeta') = \lim_{\zeta_0 \to \infty} \left( -\zeta_0 - a^{-h_A(\zeta)} \right), \quad \zeta' \in \mathbb{C}^n. \quad (9.117)
\]

**Proof** We can still use (9.116) even though \( A_0 \) now is perhaps not convex, if we let \( a(\zeta_0) \) denote one of the closest points to \(-\zeta_0\) in the closure of \( L(A_0) \). Let \( \zeta_0 \) be real and tend to \(-\infty\). Then
\[
e^{-h_A(\zeta)} + \zeta_0 = |\zeta_0 + a(\zeta_0)| + \zeta_0 \to -\Re a(-\infty),
\]
where \( a(-\infty) \) is an accumulation point of \( a(\zeta_0) \) as \( \zeta_0 \in \mathbb{R} \) and \( \zeta_0 \to -\infty \). It is a point in the closure of \( L(A_0) \) which satisfies
\[
H_{A_0}(\zeta') = \sup_{z' \in A_0} \Re(\zeta' \cdot z') = \sup_{s \in L(A_0)} \Re s = \Re a(-\infty).
\]
This implies (9.117).
9.8.6 The dual functions expressed as a dual complement

In convexity theory, the Fenchel transform generalizes the support function: the support function \( (9.111) \) is just the Fenchel transform of the indicator function. Conversely, we can express the Fenchel transform \( \tilde{f} \) of a function \( f \) in terms of the support function if we add one dimension: by definition we have \( \tilde{f}(\xi) = \sup_z (\xi \cdot x - f(x)) \), and we see that \( f(\xi) = H_{\text{epi}} f(\xi, -1) \), where \( H_{\text{epi}} f \) is the support function of the finite epigraph of \( f \), i.e.,

\[
H_{\text{epi}} f(\xi, \eta) = \sup_{(x,y) \in \mathbb{R}^n \times \mathbb{R}} (\xi \cdot x + \eta y; f(x) \leq y), \quad (\xi, \eta) \in \mathbb{R}^n \times \mathbb{R}.
\]

We already know that the dual complement \( A^* \) of a closed set \( A \) can be expressed in terms of the dual function (indeed, \( A^* \) is the set where \( \mathcal{Z} I_A \) is finite; see Proposition 9.8.20). Conversely, we shall see here that we can express the dual function in terms of a dual complement if we go up one step in dimension (cf. (9.130) below). The functions will then give rise to Hartogs sets, which we proceed to discuss.

Hartogs domains and complete Hartogs domains were defined in Section 9.4 above. We shall generalize this in two ways: first we shall need to study sets that are not necessarily open; second, it is natural to add a hyperplane at infinity and look at subsets of projective space. Thus we consider sets \( A \subseteq (\mathbb{C}^{1+n} \setminus \{0\}) \times \mathbb{C} \) that are homogeneous in the sense of Subsection 9.8.3, i.e., such that \((sz,st) \in A \) if \((z,t) \in A \) and \(s \in \mathbb{C} \setminus \{0\} \). We shall say that \( A \) is a complete Hartogs set if \((z,t') \in A \) belongs to \( A \) as soon as \((z,t) \in A \) and \(|t'| \leq |t| \). Such a set is therefore defined by an inequality \(|t| < R(z) \) or \(|t| \leq R(z) \) for some function \( R \) with \( 0 \leq R \leq +\infty \). We shall however use \( f = -\log R \) to indicate the radius of the disks.

**Definition 9.8.34** Let \( f : \mathbb{C}^{1+n} \setminus \{0\} \to \mathbb{R} \) be a homogeneous function and \( X \) a homogeneous subset of \( \mathbb{C}^{1+n} \setminus \{0\} \). We associate to \( f \) and \( X \) a homogeneous complete Hartogs set \( E(X; f) \) in \( \mathbb{C}^{1+n+1} \): it is the set of all \((z,t) \in (\mathbb{C}^{1+n} \setminus \{0\}) \times \mathbb{C} \) such that \(|t| \leq e^{-f(z)} \) when \( z \in X \), and \(|t| < e^{-f(z)} \) when \( z \notin X \).

The fiber of \( E(X; f) \) over \( z \) is thus the whole \( t \)-plane if \( f(z) = -\infty \); it is a closed disk of finite positive radius if \( z \in \text{dom}(f) \cap X \) and \( f(z) > -\infty \); it is an open disk of finite positive radius if \( z \in \text{dom}(f) \) and \( f(z) > -\infty \); it is the origin if \( z \in X \setminus \text{dom}(f) \); finally, the fiber is empty if \( z \notin X \cup \text{dom}(f) \).

If \( X_1 \subseteq X_2 \) and \( f_1 \geq f_2 \), we have an obvious inclusion \( E(X_1; f_1) \subseteq E(X_2; f_2) \).

Every complete Hartogs set \( A \) is of the form \( E(X; f) \) for some \( X \) and some \( f \); we can take \( X \) as the set of all \( z \) such that the fiber is not open and define \( f(z) \) as the infimum of all real numbers \( c \) such that \((z, e^{-c}) \) belongs to \( A \). A complete Hartogs set defines the set \( X \cap \{ z; f(z) > -\infty \} \) uniquely: if \( f(z) > -\infty \), then \( z \in X \) if and only if the fiber over \( z \) is closed and nonempty. On the other hand the choice of \( X \cap \{ z; f(z) = -\infty \} \) is immaterial in the definition of \( E(X; f) \).\(^8\)

\(^8\)To get uniqueness, one could for example require that \( X \) always contain \( \{ z; f(z) = -\infty \} \).
Theorem 9.8.35 Consider the dual complement of $E(X; f)$,

$$E(X; f)^* = \{(\zeta, \tau) \in \mathbb{C}^{1+n+1} \setminus \{0\}; \; \zeta \cdot z + \tau t \neq 0 \text{ for all } (z, t) \in E(X; f)\}.$$  

(9.118)

If both $X$ and $\text{dom}(f)$ are empty, then also $E(X; f)$ is empty and $E(X; f)^*$ is equal to the whole set $\mathbb{C}^{1+n+1} \setminus \{0\}$. If on the other hand $X \cup \text{dom} f \neq \emptyset$, then $E(X; f)$ is nonempty and its dual complement $E(X; f)^*$ is a subset of $(\mathbb{C}^{1+n} \setminus \{0\}) \times \mathbb{C}$ and a complete Hartogs set, thus

$$E(X; f)^* = E(\Xi; \varphi)$$

for some set $\Xi$ and some function $\varphi$. Here the function $\varphi$ is uniquely determined:

$$\varphi(\zeta) = \begin{cases} (\mathcal{L} f)(\zeta) & \text{when } \zeta \in (X \cup \text{dom}(f))^*, \text{ and} \\ +\infty & \text{when } \zeta \in (\mathbb{C}^{1+n} \setminus \{0\}) \setminus (X \cup \text{dom} f)^*; \end{cases}$$  

(9.119)

thus $\varphi = \mathcal{L} f$ as soon as $\text{dom}(\mathcal{L} f) \subset (X \cup \text{dom}(f))^*$, in particular if $X \subset \text{dom}(f)$.

We define the set $\Xi$ as follows. We let $\xi \in \Xi$ if and only if $\zeta \in (X \cup \text{dom}(f))^*$ and either $f$ takes the value $-\infty$ or

$$\inf_{z \in \text{dom}(f) \cap X} |\zeta \cdot z| e^{f(z)}, \; \text{or else}$$

$$\inf_{w \in \text{dom}(f) \setminus X} |\zeta \cdot w| e^{f(w)}$$  

(9.120)

for all $z^0 \in X \cap \text{dom}(f)$ we have $|\zeta \cdot z^0| e^{f(z^0)} > \inf_{z \in \text{dom}(f) \cap X} |\zeta \cdot z| e^{f(z)}$.  

(9.121)

If $f$ is not $+\infty$ identically, then $\Xi$ is uniquely determined, so that this is the only set which satisfies $E(X; f)^* = E(\Xi; \varphi)$. Moreover, we always have

$$E(X; f)^* \cap ((\mathbb{C}^{1+n} \setminus \{0\}) \times \{0\}) = (X \cup \text{dom}(f))^* \times \{0\},$$  

(9.122)

which proves that $\Xi \cup \text{dom}(\varphi) = (X \cup \text{dom}(f))^*$. The particular cases when $X$ is empty or equal to $\text{dom}(f)$ are of interest. If $\text{dom} f \neq \emptyset$ we have

$$E(0; f)^* = E((\text{dom}(f))^*; \mathcal{L} f) \cup E(\text{dom}(\mathcal{L} f); \mathcal{L} f).$$  

(9.123)

If $\text{dom}(f)$ is closed and nonempty and $f$ is lower semicontinuous and never takes the value $-\infty$, then

$$E(\text{dom}(f); f)^* = E(0; \mathcal{L} f).$$  

(9.124)

or that these two sets be disjoint, or else take the Riemann sphere as the fiber over points in $X$ such that $f(z) = -\infty$, but we shall refrain from doing so.
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Remark 9.8.36 If \( X \cup \text{dom}(f) \neq \emptyset \), then \( E(X; f)^* \) contains
\[
(\mathcal{O} \cap (\text{dom}(f))^* \times (\mathbb{C} \setminus \{0\})) \cup ((X \cup \text{dom}(f))^* \times \{0\})
\]
and is contained in
\[
E((\text{dom}(\mathcal{L} f)) \cup (X \cup \text{dom}(f))^*; \mathcal{L} f).
\]
If \( X \) is a subset of \( \text{dom}(f) \), then \( \varphi = \mathcal{L} f \) and these inclusions simplify to:
\[
E(\mathcal{O}; \mathcal{L} f) \cup ((\text{dom}(f))^* \times \{0\}) \subset E(X; f)^* \subset E((\text{dom} f)^*; \mathcal{L} f).
\]
(9.125)

We also note the following two special cases. If \( f = +\infty \) identically, then
\[
E(X; +\infty) = X \times \{0\} \text{ and } E(X; +\infty)^* = X^*.
\]
In this case the definition of \( \Xi \) in the theorem yields \( \Xi = X^* \).
We have
\[
E(X; f)^* = (X \cup \text{dom}(f))^* \times \{0\} = E((X \cup \text{dom}(f))^*; \mathcal{L} f),
\]
if \( f \) assumes the value \(-\infty\). \( \square \)

Proof of Theorem 9.8.35. If \( X \cup \text{dom}(f) \) is empty, then \( E(X; f) \) is empty, and its dual complement is the whole space except the origin. If \( X \cup \text{dom}(f) \) is not empty, then the hyperplane \((C^{1+n} \setminus \{0\}) \times \{0\}\) cuts \( E(X; f) \), so that no point \((0, \tau)\) belongs to \( E(X; f)^* \), which therefore is contained in \((C^{1+n} \setminus \{0\}) \times \mathbb{C}\).

We need to find the conditions for \((\zeta, \tau)\) to belong to \( E(X; f)^* \). This happens precisely when \( \zeta \cdot z + \tau \epsilon \) is non-zero for all \((z, t) \in E(X; f)\). The case \( \tau = 0 \) is easy: we find that \((\zeta, 0) \in E(X; f)^* \) if and only if \( \zeta \cdot z \neq 0 \) for all \( z \in X \cup \text{dom}(f) \), thus if and only if \( \zeta \in (X \cup \text{dom}(f))^* \). This proves (9.122). Now let \( \tau \neq 0 \). Then we see that \((\zeta, \tau) \in E(X; f)^* \) precisely when the following three conditions hold:
\[
\begin{align*}
|\tau| &< |\zeta \cdot z| e^{\epsilon\tau(z)} \text{ for all } z \in (\text{dom}(f)) \cap X; \\
|\tau| &< |\zeta \cdot w| e^{\epsilon\tau(w)} \text{ for all } w \in (\text{dom} f) \setminus X; \\
|\zeta \cdot z| &\neq 0 \text{ for all } z \in X \setminus \text{dom}(f).
\end{align*}
\]
(9.127) (9.128) (9.129)

Fix \( \zeta \in (X \cup \text{dom}(f))^* \). We see that the three formulas (9.127)–(9.129) imply that
\[
|\tau| \leq \exp(-L(f)(\zeta)),
\]
and that they are implied by \( |\tau| < \exp(-L(f)(\zeta)) \). This shows that \( \varphi \) is as described in (9.119), and it only remains to be seen when the inequality \( |\tau| \leq \exp(-L(f)(\zeta)) \) is strict. The condition on \( \tau \) means that it shall belong to all open disks of radius \( |\zeta \cdot z| e^{\epsilon\tau(z)} \) for \( z \in (\text{dom}(f)) \cap X \), and all closed disks of radius \( |\zeta \cdot w| e^{\epsilon\tau(w)} \) for \( w \in \text{dom}(f) \setminus X \). Now an intersection of a family of
closed disks is always closed, and an intersection of nonempty concentric open
disks with finite radii is closed exactly when it contains, along with any disk,
also a disk of strictly smaller radius. This is what is expressed by conditions
(9.120) and (9.121). Finally (9.123) and (9.124) follow from an analysis of
(9.120)–(9.121) in the special cases \( X = \emptyset \) and \( X = \text{dom}(f) \).

**Corollary 9.8.37** The logarithmic transform \( \mathcal{L}f \) of any function \( f \) can be
obtained from the dual complement of \( E(\emptyset; f) \); it is minus the logarithm of a
certain distance, viz. the distance from \((\zeta, 0)\) to the complement of \( E(\emptyset; f)^* \)
in the direction \((0, \ldots, 0, 1)\):

\[
(\mathcal{L}f)(\zeta) = -\log \left( \inf_{\tau} |\tau|; \ (\zeta, \tau) \in \mathbb{C}^{1+n+1} \setminus E(\emptyset; f)^* \right).
\]  

(9.130)

**Proof** This follows from (9.123). The result also explains why we cannot
expect these functions to be pullbacks of functions on projective space.

### 9.8.7 Lineally convex Hartogs sets

Intuitively, it seems that \( E(\emptyset; f) \) and \( E(\text{dom}(f); f) \) ought to be lineally con-
vex simultaneously. This is not quite true. We shall note three results in the
positive direction, Propositions 9.8.38–9.8.40 below, and one result in the negative
direction, Example 9.8.41. Then we shall establish conditions under which
it is true that \( f \) is \( \mathcal{L} \)-closed if and only if \( E(\text{dom}(f); f) \) is lineally convex
(Corollary 9.8.43), as well as conditions which guarantee that \( f \) is \( \mathcal{L} \)-closed
if and only if \( E(\emptyset; f) \) is lineally convex (Theorem 9.8.49).

**Proposition 9.8.38** If \( E(X; f) \) is lineally convex, then also \( X \cup \text{dom}(f) \)
and \( E(X \cup \text{dom}(f); f) \) are lineally convex. In particular, if \( E(\emptyset; f) \) is lineally convex,
then so are \( \text{dom}(f) \) and \( E(\text{dom}(f); f) \).

**Proof** Suppose that \( E(X; f) \) is lineally convex. That \( X \cup \text{dom}(f) \) is lineally convex
then follows from the easily proved result that the intersection of a
lineally convex set and a complex subspace is lineally convex as a subset
of the latter. If \( E(X; f) \) is lineally convex, then also \( E(X; f + a) \) is lineally convex
for any real number \( a \). Any intersection of lineally convex sets has the
same property, so we only need to note that \( E(X \cup \text{dom}(f); f) \) is equal the
intersection of all \( E(X; f - a), \ a > 0 \).

**Proposition 9.8.39** If \( f \) is upper semicontinuous and there exists a set \( X \)
such that \( E(X; f) \) is lineally convex, then \( E(\emptyset; f) \) is lineally convex.

**Proof** We know from Corollary 9.8.4 that \( E(X; f)^o \) is lineally convex if
\( E(X; f) \) is lineally convex. Now \( E(X; f)^o = E(\emptyset; f) \) if \( f \) is upper semicontin-
uous, hence the result.
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However, the semicontinuity of \( f \) is not important—it is the fact that the effective domain is open which is relevant. This is shown by the following result.

**Proposition 9.8.40** If \( X \cup \text{dom}(f) \) is open and \( E(X; f) \) is lineally convex, then \( E(X \setminus \text{dom}(f); f) \) is lineally convex. In particular, \( E(\emptyset; f) \) is lineally convex if \( \text{dom}(f) \) is open and \( E(X; f) \) is lineally convex for some subset \( X \) of \( \text{dom} f \).

**Proof** Assume that \( E(X; f) \) is lineally convex. Then also \( E(X; f + a) \) is lineally convex for any real \( a \), and we shall prove that the union of all \( E(X; f + 1/k), k = 1, 2, \ldots \) which equals \( E(X \setminus \text{dom}(f); f) \), is lineally convex.

The hyperplanes in \( C^{1+n+1} \) will be denoted by \( Y(\zeta, \tau) \) in analogy with (9.85), thus

\[
Y(\zeta, \tau) = \{(z, t) \in (C^{1+n} \times C) \setminus \{0\} : \zeta \cdot z + \tau t = 0\}. \tag{9.131}
\]

Let \((z, t) \notin E(X \setminus \text{dom}(f); f)\) be given with \( z \in X \cup \text{dom}(f) \). For any \( k \) there is a hyperplane \( Y(\zeta_k, \tau_k) \) that contains the point \((z, t)\) and which does not meet the set \( E(X; f + 1/k) \). We may assume that \( \|\zeta_k\|^2 + |\tau_k|^2 = 1 \). Take an accumulation point \((\zeta, \tau)\) of the sequence \((\zeta_k, \tau_k)\). Since \( X \cup \text{dom}(f) \) is open, we can be sure that \( \tau \neq 0 \). The hyperplane \( Y(\zeta, \tau) \) passes through \((z, t)\) and does not meet \( E(X \setminus \text{dom}(f); f) \) since \( \tau \neq 0 \). If on the other hand \( z \notin X \cup \text{dom}(f) \), there is a hyperplane \( Y(\zeta, 0) \) which passes through \((z, t)\) and does not cut \( E(X; f) \).

The openness in Proposition 9.8.40 cannot be dispensed with as we shall see now.

**Example 9.8.41** There is a function \( f \) such that \( E(\text{dom}(f); f) \) is lineally convex while \( E(\emptyset; f) \) is not. Define

\[
R(z) = \inf_{k \in \mathbb{N} \setminus \{0\}} |(k + 1)z_1 - z_0 - z_0/k|, \quad z = (z_0, z_1) \in C^{1+1} \setminus \{0\},
\]

and let \( f = -\log R \). Then \( \text{dom}(f) \) consists of the complement of the hyperplanes \( Y(\zeta, \zeta = 1, -k) \), and

\[
E(\text{dom}(f); f) = \bigcap_{k=1}^{\infty} \{(z, t) : z \notin Y(\zeta_k, \zeta = 1, -k) \text{ and } |t| \leq |(k + 1)z_1 - z_0 - z_0/k|\}
\]

is lineally convex. (Note, however, that \( E(X; f) \) is not lineally convex if \( X \) contains \( \text{dom}(f) \) strictly; cf. Example 9.8.9.) The function \( f \) is \( \mathcal{L} \)-closed; cf. (9.93) and Theorem 9.8.42 below. To prove that \( E(\emptyset; f) \) is not lineally convex, let us note that \( (1, 0, 1) \notin E(\emptyset; f) \), for \( R(1, 0) = 1 \). Suppose there exists a hyperplane \( Y(\zeta, \tau) \) which passes through the point \((1, 0, 1)\) but does not
cut $E(O; f)$. Then $\zeta_0 + \tau = 0$. We must also have $\tau \neq 0$ since $(1, 0, 0) \in E(O; f)$ as well as $\zeta_1 \neq 0$ since $(1, 2, 1) \in E(O; f)$. Moreover

$$\frac{|\zeta \cdot z|}{|\tau|} = \frac{|\zeta \cdot z|}{|\zeta z_0|} \geq \inf_k |(k+1)z_1 - z_0 - z_0/k| \quad \text{for all z.}$$

Taking $z = (\zeta_1, -\zeta_0)$ we see that there is a number $m$ such that $\zeta_1 = -m\zeta_0$, and we can conclude that, taking $z_0 = 1$,

$$\frac{|\zeta \cdot z|}{|\zeta z_0|} = m|z_1 - 1/m| \geq \inf_k |(k+1)z_1 - 1 - 1/k|.$$

However, for $z_1$ close to $1/m$ we must have

$$\inf_k |(k+1)z_1 - 1 - 1/k| = |(m+1)z_1 - 1 - 1/m|,$$

so that

$$m|z_1 - 1/m| > (m+1)|z_1 - 1/m|$$

for all $z_1$ close to $1/m$. This is impossible, which shows that there is no such hyperplane. \hfill \square

**Theorem 9.8.42** If $f = L f$ in $(X \cup \text{dom}(f))^{**}$, then $E((X \cup \text{dom}(f))^{**}; f)$ is lineally convex. In particular, if we assume $\text{dom} f$ to be lineally convex and $X \subset \text{dom}(f)$, then $L f = f$ in $\text{dom}(f)$ implies that $E(\text{dom}(f); f)$ is lineally convex.

Conversely, if $E(X; f)$ is lineally convex, then $f = L L f$ in $X \cup \text{dom}(f)$, which is a lineally convex set. If $f$ is bounded from below on the unit sphere, then $f = L L f = +\infty$ outside $\text{dom}(f)^{**}$. Thus in this case $L L f = f$ everywhere if $X \supset \text{dom}(f)^{**} \setminus \text{dom}(f)$.

**Proof** Suppose that $f = L L f$ in $(X \cup \text{dom}(f))^{**}$. Take

$$(z^0, t^0) \notin E((X \cup \text{dom}(f))^{**}; f).$$

We shall then prove that there is a hyperplane $Y_{(\zeta, \tau)}$ (see (9.131)) which contains $(z^0, t^0)$ and does not cut $E((X \cup \text{dom} f)^{**}; f)$. Consider first the case

$$z^0 \in (X \cup \text{dom}(f))^{**}.$$

We know that $|t^0| > e^{\tau f(z^0)}$. By the definition of $L f$ and since $(L L f)(z^0) = f(z^0) > -\log |t^0|$, we can choose $\zeta$ such that

$$-\log |\zeta \cdot z^0| - (L f)(\zeta) > -\log |t^0|.$$

Then we take $\tau = -\zeta \cdot z^0/t^0$, so that $(z^0, t^0) \in Y_{(\zeta, \tau)}$ and $-(L L f)(\zeta) > \log |\tau|$. Moreover, for any $(z, t) \in Y_{(\zeta, \tau)}$ we have

$$f(z) \geq (L L f)(z) \geq -\log |\zeta \cdot z| - (L f)(\zeta) = -\log |\tau t| - (L f)(\zeta) > -\log |t|,$$
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which shows that \((z, t) \notin E((X \cup \text{dom}(f))^{**}; f)\). The case \(z^0 \notin (X \cup \text{dom}(f))^{**}\) remains to be considered. In this case there is a hyperplane \(Y_\zeta\) that contains \(z^0\) and does not meet \((X \cup \text{dom}(f))^{**}\), so the hyperplane \(Y_{(\zeta, 0)}\) does not cut \(E((X \cup \text{dom}(f))^{**}; f)\).

Now assume that \(E(X; f)\) is lineally convex. We already know that \(X \cup \text{dom}(f)\) is lineally convex (cf. Proposition 9.8.38). If \(f\) assumes the value \(-\infty\), then \(E(X; f) = (X \cup \text{dom}(f)) \times C\), and \(f = L L f = -\infty\) in \(X \cup \text{dom}(f)\).

If \(f > -\infty\), let \(z^0\) be any point in \(X \cup \text{dom}(f)\) and take \(t^0\) such that \(|t^0| > e^{-f(z^0)}\), thus \((z^0, t^0) \notin E(X; f)\). By hypothesis there is a hyperplane \(Y_{(\zeta, \tau)}\) which passes through \((z^0, t^0)\) and does not meet \(E(X; f)\). Since \((z^0, 0) \in E(X, f)\), we must have \(\tau \neq 0\), so we obtain a minorant of \(f\) of the form 

\[-\log |\zeta| \cdot |z| + \log |\tau| \leq f(z),\]

where the left-hand side takes the value \(-\log |t^0| < f(z^0)\) at the point \(z^0\) and moreover can be chosen larger than any number less than \(f(z^0)\). Thus \((L L f)(z^0) \geq f(z^0)\) and we conclude that \(L L f = f\) in all of \(X \cup \text{dom}(f)\).

Finally, assume that \(f \geq -C\) on the unit sphere without any further assumption. Thus, putting \(A = \text{dom}(f)\), we have \(f \geq g = I_A - C\), so that \(L f \leq L g = C - \log d_A\), by Proposition 9.8.20. We now note that \(d_A = d_A^{**}\) and take the transformation once again, this time using Proposition 9.8.23.

We get \(L L f \geq L L g \geq I_A^{**} - M - C\). In particular \(L f(z) = +\infty\) if \(z \notin A^{**} = \overline{\text{dom}(f)}^{**}\) (cf. \(\text{D}\) in Proposition 9.8.2). This finishes the proof.

**Corollary 9.8.43** Assume that \(f\) is bounded from below on the unit sphere and that \(\text{dom}(f)\) is closed and lineally convex. Then \(f\) is \(L\)-closed if and only if \(E(\text{dom}(f); f)\) is lineally convex.

We now proceed to study the case when \(\text{dom}(f)\) is open and \(f\) tends to \(+\infty\) at the boundary. Propositions 9.8.44 and 9.8.47 below are applicable when \(f\) tends rather fast to \(+\infty\), and Theorem 9.8.49 in a more general situation.

**Proposition 9.8.44** Let \(f\) be a homogeneous function on \(C^1 + n \setminus \{0\}\) which tends to \(+\infty\) at the boundary of \(\text{dom}(f) = A\) in the strong sense that \(f \geq -C - \log d_A\) for some constant \(C\), where \(d_A\) is defined by (9.96). Then \(f\) is \(L\)-closed if and only if \(E(0; f)\) is lineally convex.

**Proof** If \(f\) is \(L\)-closed, then its effective domain \(A = A^\circ\) must be lineally convex by Theorem 9.8.14, for \(A = (\text{dom}(L f))^\circ = \overline{\text{dom}(L f)}^*\) in view of (9.99) applied to \(L f\) (this function is bounded from below on the unit sphere unless \(f\) is \(+\infty\) identically, a trivial case). Theorem 9.8.42 now shows that \(E(A; f)\) is lineally convex and Proposition 9.8.40 implies that \(E(0; f)\) is lineally convex.

Conversely, assume that \(E(0; f)\) is lineally convex. In view of Theorem 9.8.42 it only remains to be proved that \(L L f = f = +\infty\) outside \(A = \text{dom}(f)\). Now if \(f \geq -C - \log d_A\), then \(L f \leq C + L(-\log d_A) \leq C + I_A^*\).
by Proposition 9.8.23. We take the transformation again and obtain $\mathcal{L} \mathcal{L} f \geq -C + I_{A^*} = -C - \log d_{A'}$, using Proposition 9.8.20. But $\text{dom}(f)$ is lineally convex, so $A'' = A$. Hence $\mathcal{L} \mathcal{L} f = +\infty$ in the complement of $A$.

Functions with bounded logarithmic transforms exhibit the behavior studied in Proposition 9.8.4:

**Proposition 9.8.45** Let $f$ be a homogeneous function on $\mathbb{C}^{1+n} \setminus \{0\}$ such that $\text{dom}(f) = A$ equals the interior of its closure and such that $\overline{\text{dom}(f)}$ is lineally convex. Assume that $f$ is bounded from below on the unit sphere and that $\mathcal{L} f$ is bounded from above in $S \cap \text{dom}(\mathcal{L} f)$. Then $f \geq -C - \log d_A$, where $C$ is a constant.

**Proof** We have $\mathcal{L} f \leq C + I_B$, where $B = \text{dom}(\mathcal{L} f)$. Therefore $f \geq \mathcal{L} (\mathcal{L} f) \geq -C + I_B = -C - \log d_{B'} = -C - \log d_{B''}$. The next lemma shows that $B'' = A$.

**Lemma 9.8.46** Let $f$ be a homogeneous function on $\mathbb{C}^{1+n} \setminus \{0\}$ such that $\overline{\text{dom}(f)} = \text{dom}(f)$.

Assume that $f$ is bounded from below on the unit sphere and that $\overline{\text{dom}(f)}$ is lineally convex. Then $(\overline{\text{dom}(\mathcal{L} f)})^\circ = \text{dom}(f)$.

**Proof** From (9.99) we deduce, recalling that $f$ is bounded from below on $S$, that $\overline{\text{dom}(f)} = (\overline{\text{dom}(\mathcal{L} f)})^\circ = (\text{dom}(f))^\circ$. The next lemma shows that $B'' = A$.

Under a regularity assumption we can let $f$ tend to infinity at a slower pace:

**Proposition 9.8.47** Let $f$ be a homogeneous function on $\mathbb{C}^{1+n} \setminus \{0\}$ which tends to $+\infty$ at the boundary of $\text{dom}(f) = A$ in the sense that $f \geq -C - c \log d_A$ on the unit sphere for some constants $C$ and $c$ with $0 < c \leq 1$. Assume that $A^*$ satisfies the homogeneous interior cone condition (Definition 9.8.27). Then $f$ is $\mathcal{L}$-closed if and only if $E(0; f)$ is lineally convex.

**Remark 9.8.48** It can be easily proved that if $A$ is lineally convex and its dual complement $A^*$ satisfies the homogeneous interior cone condition, then so does its set-theoretic complement $\overline{\mathbb{C} \setminus A}$.

**Proof** In view of Theorem 9.8.42 and the proof of Proposition 9.8.44, it only remains to be proved that $\mathcal{L} \mathcal{L} f = f = +\infty$ outside $\text{dom}(f)$ if $E(0; f)$ is lineally convex. Now if $f \geq -C - c \log d_A$ on the unit sphere $S$, then we obtain $f \geq -C + f_c$ everywhere, introducing the function $f_c$ of (9.103). We
take the logarithmic transformation once to obtain $\mathcal{L}f \leq C + \mathcal{L}f_c \leq C + \varphi_{1-c}$ (Proposition 9.8.19), and then again to get $\mathcal{L}^2f \geq -C + \mathcal{L}\varphi_{1-c} \geq -C - M + f_c$, this time applying Proposition 9.8.28 to the function $\varphi_{1-c}$ and using the homogeneous interior cone condition (Definition 9.8.27) on $A^*$. This shows that $\mathcal{L}^2f$ equals $+\infty$ in the complement of $A$.

We finally come to the general case of a function which tends to infinity at the boundary.

**Theorem 9.8.49** Let $f$ be a homogeneous function on $C^{1+n} \setminus \{0\}$. Assume that $f$ is bounded from below on the unit sphere and tends to $+\infty$ at the boundary of $A = \text{dom}(f)$ in the sense that $A_s = \{z \in S : f(z) < s\}$ is strongly contained in $\text{dom}(f)$ for all numbers $s$, i.e., the closure of $A_s$ is contained in the interior of $A$. (This implies that $A$ is open.) Assume moreover that $A^*$ satisfies the homogeneous interior cone condition. Then $E(0; f)$ is lineally convex if and only if $f$ is $\mathcal{L}$-closed.

**Proof** For the proof we shall need the functions $f_{A,r}$, where $A$ is a homogeneous set and $r$ is a positive number, defined as $f_{A,r} = -\log r + I_A$, thus $f_{A,r} = -\log r - \log ||z||_2$ when $z \in A$ and $f_{A,r} = +\infty$ otherwise. We note that $\mathcal{L}f_{A,r} = \log r - \log d_A$. (Proposition 9.8.20, so that $(\zeta, \tau)$ belongs to $E(A^*; \mathcal{L}f_{A,r})$ if and only if $\zeta \in A^*$ and $r|\tau| \leq d_{A^*}(\zeta)$.

What remains to be done, considering Theorem 9.8.42 and the proof of Proposition 9.8.44, is the following, assuming $E(0; f)$ to be lineally convex. Given any $z^0 \notin \text{dom}(f) = A$ it is required to find a hyperplane $Y_{(\zeta, \tau)}$ with $\tau \neq 0$ which does not cut $E(0; f)$ and passes through $(z^0, t^0)$ with $|t^0|$ arbitrarily small; the problem is to avoid the vertical hyperplanes, those with $\tau = 0$. Since $f$ is bounded from below on $S$, there is a number $R$ such that $E(0; f)$ is contained in $E(0; f_{A,R})$. On the other hand, given any $\varepsilon > 0$, there is a homogeneous set $K$ which is strongly contained in $A$ and such that $f \geq -\log \varepsilon$ on $S \setminus K$; this means that $E(0; f)$ is contained in $E(0; f_{A,\varepsilon}) \cup E(0; f_{K,R})$. We shall find a hyperplane which does not cut the latter set for a suitable choice of $\varepsilon$. This amounts to finding $(\zeta, \tau)$ in $E(0; f_{A,\varepsilon})^* \cap E(0; f_{K,R})^*$, equivalently in $E(A^*; \mathcal{L}f_{A,\varepsilon}) \cap E(K^*; \mathcal{L}f_{K,R})$; cf. (9.123). The hyperplane shall also contain a point $(z^0, t^0)$ with $|t^0| = \delta$ positive but arbitrarily small.

We shall thus find $\zeta$ and $\tau$ such that

$$0 \neq |\tau| \leq \frac{1}{\varepsilon} d_{A^*}(\zeta) \text{ and } |\tau| \leq \frac{1}{R} d_{K^*}(\zeta).$$

We take $0 \neq \tau = - (\zeta \cdot z^0)/t^0$ to ensure that $(z^0, t^0)$ belongs to $Y_{(\zeta, \tau)}$, and then the problem is reduced to finding $\zeta$ such that

$$0 \neq |\zeta \cdot z^0| \leq \frac{\delta}{\varepsilon} d_{A^*}(\zeta) \text{ and } |\zeta \cdot z^0| \leq \frac{\delta}{R} d_{K^*}(\zeta). \quad (9.132)$$

Since by hypothesis $z^0 \notin A = A^{**}$, there is a point $\zeta^0 \in A^*$ such that $\zeta^0 \cdot z^0 = 0$. If $\zeta^0$ is in the interior of $A^*$, then finding $\zeta$ is easy: we have $d_{K^*}(\zeta^0) > 0$. Otherwise, let $\zeta^0$ be a point in the boundary of $A^*$, and then $\zeta^0 \cdot z^0 = 0$.
\( d_A \cdot (\zeta^0) > 0 = |\zeta^0 \cdot z^0| \) and can take \( \zeta \) close to \( \zeta^0 \). If on the other hand \( \zeta^0 \in \partial A^* \), we argue as follows. By the homogeneous interior cone condition,

\[
\sup_{\|\zeta - \zeta^0\|_2 \leq s\|\zeta^0\|_2} d_A \cdot (\zeta) \geq \gamma s\|\zeta^0\|_2
\]

for some positive constant \( \gamma \). On the other hand

\[
\sup_{\|\zeta - \zeta^0\|_2 \leq s\|\zeta^0\|_2} |\zeta \cdot z^0| = s\|\zeta^0\|_2\|z^0\|_2.
\]

Given any positive \( \delta \), it is thus enough to choose \( \varepsilon \) such that \( \varepsilon|z^0| \leq \delta \gamma \) to satisfy the first inequality in (9.132) for some \( \zeta \) close to \( \zeta^0 \), more precisely satisfying \( |\zeta - \zeta^0| \leq s|\zeta^0| \) for any given sufficiently small positive \( s \). The second is then satisfied strictly when \( \zeta = \zeta^0 \), because \( A^* \) is strongly contained in \( K^* \) by Corollary 9.8.3, so that \( d_K \cdot (\zeta^0) > d_A \cdot (\zeta^0) = 0 \), and it must therefore also be satisfied for all \( \zeta \) satisfying \( |\zeta - \zeta^0| \leq s|\zeta^0| \) for all sufficiently small positive \( s \). This completes the proof.

### 9.8.8 A necessary differential condition for \( L \)-closed functions

It is well known that convex functions as well as plurisubharmonic functions of class \( C^2 \) can be characterized by differential conditions. Is the same true for \( L \)-closed functions? We shall first establish a necessary differential condition.

**Proposition 9.8.50** Suppose that \( f \) is an \( L \)-closed function of class \( C^2 \) in some open set \( \Omega \) of \( \mathbb{C}^{1+n} \setminus \{0\} \). Then

\[
|\sum (f_{z_j z_k} - 2f_{z_j} f_{z_k}) b_j b_k| \leq \sum f_{z_j z_k} b_j b_k, \quad \text{in } \Omega \text{ for all } b \in \mathbb{C}^{1+n}.
\]

In particular, if \( n = 1 \) and we define \( F(z) = f(1, z), z \in \mathbb{C} \), then

\[
|F_{z z} - 2F^2_z| \leq F_{z z}.
\]

**Proof** Define \( g(z) = -\log |\beta \cdot z| \). For every point \( a \) where \( f(a) \) is finite there is a vector \( \beta \) such that \( \text{grad} g(a) = \text{grad} f(a) \). Indeed, let us first note that by homogeneity \( \sum a_j f_{z_j}(a) = -1/2 \) for all \( a \). If we choose \( \beta_j = f_{z_j}(a) \), then \( \beta \cdot a = -1/2 \) and

\[
\frac{\partial g}{\partial z_j}(z) = -\frac{1}{2} \frac{\beta_j}{\beta \cdot z}
\]

takes the value

\[
\frac{\partial g}{\partial z_j}(a) = -\frac{1}{2} \frac{\beta_j}{\beta \cdot a} = \beta_j
\]

at \( z = a \). Then by \( L \)-closedness \( f(z) \geq f(a) + g(z) - g(a) \) for all \( z \), for the definition of the \( L \)-transformation uses precisely the functions \( g \) plus a
constant. Take a curve \( t \mapsto \gamma(t) \) such that \( \gamma(0) = a \) and compare the two functions \( \varphi = f \circ \gamma \) and \( \psi = f(a) + g \circ \gamma - g(a) \). We have \( \varphi(0) = \psi(0) \) and \( \varphi'(0) = \psi'(0) \) and must therefore have \( \varphi''(0) \geq \psi''(0) \). We calculate \( \varphi'' \):

\[
\frac{1}{2} \varphi''(t) = \text{Re} \sum f_{z_j z_k}(\gamma(t)) \gamma_j'(t) \gamma_k'(t) + \text{Re} \sum f_{z_j}(\gamma(t)) \gamma_j''(t).
\]

The corresponding formula for \( \psi \) simplifies to

\[
\frac{1}{2} \psi''(t) = \text{Re} \sum g_{z_j z_k}(\gamma(t)) \gamma_j'(t) \gamma_k'(t) + \text{Re} \sum g_{z_j}(\gamma(t)) \gamma_j''(t),
\]

since \( g_{z_j} \) is holomorphic, i.e., \( g_{z_j} z_k = 0 \). Also

\[
g_{z_j z_k}(z) = \frac{1}{2} \frac{\beta_j \beta_k}{(\beta \cdot z)^2} = 2 g_{z_j}(z) g_{z_k}(z).
\]

Moreover \( g_{z_j} = f_{z_j} \) at \( z = a \), so that

\[
\frac{1}{2} \psi''(0) = 2 \text{Re} \sum f_{z_j}(a) f_{z_k}(a) \gamma_j'(0) \gamma_k'(0) + \text{Re} \sum f_{z_j}(a) \gamma_j''(0).
\]

The inequality \( \varphi''(0) \geq \psi''(0) \) then means that

\[
\text{Re} \sum f_{z_j z_k}(a) \gamma_j'(0) \gamma_k'(0) + \text{Re} \sum f_{z_j}(a) \gamma_j''(0) \geq 2 \text{Re} \sum f_{z_j}(a) f_{z_k}(a) \gamma_j'(0) \gamma_k'(0) + \text{Re} \sum f_{z_j}(a) \gamma_j''(0).
\]

If we now let the direction \( \gamma'(0) \) vary, this means that (9.133) holds.

We shall now prove that the differential condition (9.134) is not sufficient for \( \mathcal{L} \)-closedness in a simply connected domain which is not a disk.

**Lemma 9.8.51** Let \( f \) be an \( \mathcal{L} \)-closed function which is of class \( C^1 \) in a neighborhood of a point \( a \in \mathbb{C}^{1+1} \setminus \{0\} \). We consider its restriction to \( z_0 = 1 \), and write \( z \) for the coordinate there, thus \( F(z) = f(1, z) \). Let \( G(z) = \log |1 + \beta z|, \) \( z \in \mathbb{C} \). We now choose \( \beta \) such that \( \partial F(a)/\partial z = \partial G(a)/\partial z = -\beta/(1 + \beta a) \).

With this value of \( \beta \) we have \( F(z) \geq F(a) + G(z) - G(a) \) for all \( z \). In particular \( F(z) \geq F(a) \) at all points on the circle \( |1 + \beta z| = |1 + \beta a| \).

The proof is easy.

**Proposition 9.8.52** Let \( \omega \) be a connected open subset of the Riemann sphere \( S^2 = \mathbb{C} \cup \{\infty\} \) such that \( S^2 \setminus \omega \) has at least one component which is not a disk.

Then there exists a function \( F \) which is \( +\infty \) in \( S^2 \setminus \omega \), \( C^\infty \) in \( \omega \), and satisfies the differential condition (9.134) in \( \omega \) such that \( \mathcal{L} \mathcal{L} f \neq f \) at some point in \( \omega \) for the corresponding function \( f \) defined by \( f(z_0, z_1) = F(z_1/z_0) - \log |z_0| \).

**Proof** Let \( K \) be the complement of a component of \( S^2 \setminus \omega \) which is not a disk; thus \( K \) contains \( \overline{\omega} \). Moreover the complement of \( K \) is connected and \( \partial K \subset \partial \omega \). Let \( a, b, c, d \) be the four points whose existence is guaranteed by
Lemma 9.4.35; recall that \( b, d \in K \) and \( a, c \notin K \). Since they are on a circle, we can move them by a Möbius transformation: we can move \( a \) to 0 and \( c \) to \( \infty \), and \( b, d \) to some points on the real axis, \( d < 0 < b \). Also \( b, d \in \partial \omega \) since \( b, d \in \partial K \). Now define

\[
F(z) = \begin{cases} 
\exp(-1/(\text{Im } z + \varepsilon)) & \text{when } z \in \omega, \text{ Re } z > 0, \text{ Im } z > -\varepsilon, \\
0 & \text{at other points in } \omega, \\
+\infty & \text{otherwise}.
\end{cases}
\]

Here we choose \( \varepsilon > 0 \) so small that the disk \( \|z\| \leq \varepsilon \) does not intersect \( K \); this means that the function is continuous, even identically zero in a neighborhood of the intersection of \( \omega \) and the imaginary axis. Moreover we choose \( \varepsilon \) so small that \( F \) satisfies the differential condition \((\partial F/\partial y)^2 \leq \partial^2 F/\partial y^2\), and such that the point \( b_1 = b - i\varepsilon \) belongs to \( \omega \). This means that \( F \) satisfies (9.134). Now at a point \( b_2 \) near \( b_1 \), \( F \) takes arbitrarily small positive values. If there is a function \( G \) tangent to \( F \) at such a point, it forces \( F \) to be positive at some point with negative real part. Indeed, the circle \( |1 + \beta z| = |1 + \beta b_2| \) passes through points arbitrarily close to the line \( \text{Im } z = \text{Im } b_2 \) (see Lemma 9.8.51). This is a contradiction, since we defined \( F \) to be identically zero at all points in \( \omega \) in the left half plane, and there are such points by construction.
9.9 Lineal Convexity in Infinite Dimension

Abstract of this section

The purpose of this section is to study some problems on lineal convexity in spaces of infinite dimension, in particular to prove that a pseudoconvex or lineally convex set which is open in a subspace can be extended to an open set in the whole space which is pseudoconvex or lineally convex, respectively.

Given a pseudoconvex open set \( \omega \) in \( C^k \) regarded as a subspace \( C^k \times \{0\} \) of \( C^k \times C^{n-k} = C^n \), we can fatten it into pseudoconvex set which is open in the whole space by taking a Cartesian product \( \omega \times \omega' \) for a suitable set \( \omega' \) in \( C^{n-k} \). How can this be done in an infinite-dimensional space? It is also of interest to construct a set which tapers off at the boundary of \( \omega \).

In an infinite-dimensional Hausdorff topological vector space the subspaces of finite dimension (which are always closed) and those subspaces of finite codimension which are closed are of particular interest. Here the codimension of a subspace \( F \) of \( E \) is the dimension of the quotient space \( E/F \).

9.9.1 Generalizing lineal convexity

We start by generalizing the concept of lineally convex set to higher codimensions:

**Definition 9.9.1** Let \( E \) be a topological vector space and \( A \) a subset of \( E \). We shall say that \( A \) is \( m \)-lineally convex if its complement is a union of closed affine subspaces of codimension \( m \).

Thus 1-lineal convexity is just lineal convexity as defined in Definition 9.4.1 on page 279. An \( m \)-lineally convex set is also \( k \)-lineally convex for \( k \geq m \). An \( m \)-lineally convex open set in \( C^n \) is \((n-m)\)-pseudoconvex in the sense of Rothstein (1955:130).

9.9.2 Inverse images of \( m \)-lineally convex sets

The intersection of an \( m \)-lineally convex set with a subspace is also \( m \)-lineally convex in that subspace—more generally, we have the following result.

**Proposition 9.9.2** Let \( f : E \to F \) be a continuous linear mapping of a topological vector space \( E \) into another one, \( F \). If \( Y \) is an \( m \)-lineally convex subset of \( F \), then its inverse image \( A = f^*(Y) \) under \( f \) is also \( m \)-lineally convex.

**Proof** Take \( x \in E \setminus X \). Then \( y = f(x) \) belongs to \( F \setminus Y \) and by hypothesis we can find a linear subspace \( K \subset F \) of codimension \( m \) such that \( y + K \) does
not meet $Y$. The inverse image $H = f^*(K)$ is a vector subspace of $E$ and $E/H$ is of finite dimension. For the codimensions we obtain
\[ \text{codim}_E(H) = \dim E/H = \dim f_*(E)/K \leq \dim F/K = \text{codim}_F K = m. \]
So $E \setminus X$ is a union of closed affine subspaces of codimension at most $m$, thus $X$ is $m$-lineally convex.

9.9.3 Constructing thicker sets

**Theorem 9.9.3** Let $E$ be a normed complex vector space, $V$ an open cone in $E$ which is $m$-lineally convex, and $F$ a vector subspace of $E$ of dimension $m$. Assume that $V \cap F = F \setminus \{0\}$. If $\omega$ is a pseudoconvex open set in $F$, then
\[ \Omega = \bigcap_{x \in F \setminus \omega} (x + V) \]
is open and pseudoconvex in $E$, and $\Omega \cap F = \omega$.

**Proof** We have
\[ E \setminus \Omega = (F \setminus \omega) + (E \setminus V), \]so that $E \setminus \Omega$ is the vector sum of two closed sets in $E$. Let $B$ be the unit ball in $E$. Since $F$ is of finite dimension and $V$ contains $F \setminus \{0\}$, there is a positive number $s$ such that
\[ (F \setminus B) + sB \subset V; \]
in view of the homogeneity of $F$ and $V$ we have for all $r > 0$
\[ [F \setminus (rs^{-1}B)] + rB \subset V. \]Let now $x \in E \setminus \Omega$ with $\|x\| \leq r/s$. There is, according to (9.135) a representation $x = y + z$ of $x$ with $y \in F \setminus \omega$ and $z \in E \setminus V$. It follows that $\|y\| \leq r/s$, for if we have $\|y\| > r/s$, it would follow from (9.136) that $y \in F \setminus (rs^{-1}B)$. This proves that
\[ rB \setminus \Omega \subset [(rs^{-1}) \cap (F \setminus \omega)] + (E \setminus V). \]
Since $(rs^{-1}B) \cap (F \setminus \omega)$ is compact and the vector sum of a compact set and a closed set is closed, we have finally proved that $(rB) \setminus \Omega$ is closed; consequently, since $r$ is arbitrary, that $\Omega$ is open.

In order to prove that $\Omega$ is pseudoconvex, we shall find, given any point $y \in E \setminus \Omega$, a continuous linear projection $\pi: E \to F$ such that $\pi(\omega)$ contains $\Omega$ but not $y$.

So let $y$ belong to $E \setminus \Omega$. We then have $y \notin x + V$ for some $x \in F \setminus \omega$. By hypothesis there exists a closed vector subspace $H$ of $E$ of codimension $m$ and such that $y - x + H$ does not meet $V$. Lemma 9.9.4 below shows that
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\( y - x + H = H \) and since \( F \) is of codimension \( m \) and \( F \cap H = \{ 0 \} \), we have \( E = F + H \), a direct sum. The projection \( \pi: E \to F \) with kernel \( H \) and image \( F \) is continuous. Since \( y - x \in H \), we have \( \pi(y) = \pi(x) = x \in F \); that is, \( y \notin \pi^*(\omega) \). On the other hand, \( \pi_*(V) = F \setminus \{ 0 \} \), which after a translation proves that \( \pi_*(x + V) = F \setminus \{ x \} \), hence

\[
\pi_*(\Omega) \subset \bigcap_x (F \setminus \{ x \}; \ x \in F \setminus \omega) = \omega.
\]

Summing up, \( y \notin \pi^*(\omega) \) and \( \pi^*(\omega) \supset \pi^*(\pi_*(\Omega)) = \Omega \), the conclusion we wanted.

During the proof above we needed the following lemma.

**Lemma 9.9.4** Let \( V \) be an \( m \)-lineally convex open cone in a topological vector space \( E \). Assume that \( V \) contains \( F \setminus \{ 0 \} \), where \( F \) is a vector subspace of \( E \) of dimension \( m \). Then \( \Lambda \setminus V \) is a union of closed vector subspaces of dimension \( m \); in fact, every closed affine vector subspace of \( E \) which is contained in \( E \setminus V \) and of codimension \( m \) contains the origin and is a linear subspace.

**Proof** Let \( H \) be a subspace of \( E \) of codimension \( m \) and such that \( (a + H) \cap V = 0 \). We shall thus prove that \( a + H = H \). If we had \( F + H \neq E \), there would exist \( x \in F \cap H, x \neq 0 \). Then \( a + rx \in a + H \subset E \setminus V \). But \( a + rx \) belongs to \( V \) for all sufficiently large \( r \) since \( x \in F \setminus \{ 0 \} \subset V \). This contradiction shows that \( F + H = E \); it follows that \( a \) can be represented as \( a = y + z \), where \( y \in F, z \in H \) and \( y = a - z \in a + H \). Hence

\[
y \in (y + H) \cap F \subset (a + H) \cap (V \cup \{ 0 \}) \subset \{ 0 \};
\]

in other words that \( 0 \in a + H \), which is a vector subspace of \( E \). This proves the lemma and so completes the proof of Theorem 9.9.3.

Let us note that we also have the next result, which is closely related to the theorem just proved.

**Theorem 9.9.5** Let \( E, F, V \) and \( \Omega \) be as in Theorem 9.9.3. Assume that that \( \omega \) is \( m \)-lineally convex in \( F \). Then \( \Omega \) is also \( m \)-lineally convex.

**Proof** Let \( y \in E \setminus \Omega \). As in the proof of Theorem 9.9.3, there is an \( x \in F \setminus \omega \) and a projection \( \pi: E \to F \) such that \( \pi(y) = \pi(x) = x \). According to Proposition 9.9.2, \( \pi^*(\omega) \) is \( m \)-lineally convex and it follows that \( \Omega \), the intersection of all the sets \( \pi^*(\omega) \) for the various choices of \( \pi \), is also \( m \)-lineally convex.

### 9.9.4 Constructing convex cones

Finally we shall indicate how it is possible to construct cones \( V \) that can serve in Theorems 9.9.3 and 9.9.5.
Proposition 9.9.6 Let $E$ be a topological vector space and let $A_j$, $j = 1, \ldots, p$, be $m_j$-lineally convex subsets of $E$. Then the intersection $A = A_1 \cup \cdots \cup A_p$ is $m$-lineally convex, where $m = m_1 + \cdots + m_p$.

Proof Take any point $a \in E \setminus A$. Then $a \in E \setminus A_j$ for every $j$ and there exists a closed subspace $F$ of $E$ of codimension $m_j$ such that $a + F_j$ does not meet $A_j$. The intersection $F = F_1 \cap \cdots \cap F_p$ is closed and of codimension at most equal to $m$ and does not cut $A$. We are ready.

Let us denote by $T$ the unit circle $T = \{ t \in \mathbb{C}; |t| = 1\}$, and by $TA = \{ tx; x \in A\}$ the circled set generated by a set $A$ in a vector space.

Lemma 9.9.7 If $V$ is an open convex cone in a complex vector space $E$, then $TV$ is 1-lineally convex.

Proof If $TV$ is equal to $E$ there is nothing to prove, so let us suppose that $TV \neq E$. If $a \in E \setminus TV$, let us denote by $F = Ca$ the vector subspace generated by $a$. Then $F$ does not meet $V$ (note that the origin does not belong to $V$), so the Hahn–Banach theorem gives us a closed hyperplane $H$ that contains $F$ and does not meet $V$. Now $TH = H$, so $H$ does not meet $TV$ either, and, since $a \in H$, this proves that $TV$ is 1-lineally convex.

Proposition 9.9.8 Let $V_1, \ldots, V_m$ be open convex cones in a topological vector space $E$. Then

$$V = TV_1 \cup \cdots \cup TV_m$$

is $m$-lineally convex.

Proof We combine Proposition 9.9.2 and Lemma 9.9.7.

Proposition 9.9.9 Let $E$ be a normed complex vector space, $F$ a vector subspace of $E$ and $\xi_1, \ldots, \xi_m$ nonzero linear forms on $E$ such that their restrictions to $F$ have norm at most equal to 1. Then

$$V = \{ x \in E; \| x - y \| < \sup_{j=1, \ldots, m} |\xi_j(y)| \text{ for some } y \in F\}$$

(9.137)

is an open $m$-lineally convex cone in $E$.

Proof Let us define

$$u_j(x) = \inf_{y \in F} (\| x - y \| - \Re \xi_j(y)), \quad x \in E, \quad j = 1, \ldots, m.$$ 

This is a convex function and it never takes the value $-\infty$, since

$$\| x - y \| - \Re \xi_j(y) \geq \| x - y \| - \| y \| \geq -\| x \| > -\infty$$

for every $x \in E$ and every $y \in F$. Hence we have $-\| x \| \leq u_j \leq \| x \|$. The cone

$$V_j = \{ x \in E; u_j(x) < 0\}$$
is convex and open in view of the continuity of \( u_j \). From Proposition 9.9.8 we see that \( V = TV_1 \cup \cdots \cup TV_m \) is \( m \)-lineally convex.

Let us also note that the cone \( V \) defined by (9.137) contains the cone

\[
V_0 = F \setminus \bigcap_{j=1}^m \ker \xi_j.
\]

This cone is open in \( F \) but not in \( E \). If we make it a bit thicker to obtain an open cone in \( E \), it can serve in Theorems 9.9.3 and 9.9.5 provided \( F \) is a subspace of dimension \( m \) and

\[
F \cap \ker \xi_1 \cap \cdots \cap \ker \xi_m = \{0\};
\]

this latter condition means that the restrictions of the \( \xi_j \) to \( F \) form a base in the dual of \( F \).
9.10 References


References


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Author’s address: Uppsala University, Department of Information Technology, P. O. Box 337, SE-75105 Uppsala, Sweden.

Email addresses: kiselman@it.uu.se, christer@kiselman.eu

URL: http://www.cb.uu.se/˜kiselman

ORCiD, Open Researcher and Contributor ID: 0000-0002-0262-8913